SOME RESULTS ON THE EXTREME DISTRIBUTIONS
OF SURPLUS PROCESS WITH NONHOMOGENEOUS
CLAIM OCCURRENCES

Fatih TANK and Altan TUNCEL

Abstract

In this paper, survival (non-ruin) probability after a definite time period of an insurance company is studied in a discrete time model based on non-homogenous claim occurrences. Furthermore, distributions of the minimum and maximum levels of surplus in compound binomial risk model with non-homogeneous claim occurrences are obtained and some of its characteristics are given.

Keywords: Surplus process, Non-homogenous claim occurrences, Extreme distributions, Survival probability.

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1. Introduction

Surplus process (or risk process) is a model of accumulation of insurer’s capital and the premium incomes during the periods. So, the surplus process is one of the most important stochastic process for an insurance company which can be defined as discrete or continuous time in actuarial risk theory. Ruin occurs when surplus is zero or negative value which means that the total claim amounts equal or exceed the surplus at a certain time for insurance companies. Furthermore, the estimation of the surplus at a certain time is essential for the insurance companies due to their future investment strategies and actions to be taken just before ruin. In this regard, it is vital importance for controlling the maximum and minimum level of the surplus and its related quantities.

The compound binomial model has been first proposed by Gerber (1988 a). Distributional properties of some actuarial quantities associated with compound binomial model have been studied in De Vylder and Goovaerts (1984,1988), Shiu (1989), Willmot (1993), Dickson (1994) and De Vylder and Marceau (1996). The compound binomial model, as a discrete time version of the classical compound Poisson model of risk theory has been widely studied in the recent literature (see, e.g. Yuen and Guo (2001), Cossette and Marceau (2000), Cossette et al. (2003, 2004 and 2006), Liu and Zhao (2007), Tuncel and Tank (2014) ).

In classical risk model, the number of the claims is assumed to have a Poisson process \( \{N_t : t \geq 0\} \) with parameter \( \lambda \) and the claim amounts \( Y_1, Y_2, \ldots \) are non-negative, independent and identically distributed random variables with same distribution function. The total claim amounts process \( \{S_t : t \geq 0\} \) is a compound Poisson process with

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parameter $\lambda$ where $S_t = \sum_{i=1}^{N_t} Y_i$ designates the total claim amounts up to time $t$. In this regards, surplus of the insurers at time $t$ can be defined as follows

$$U_t = u + ct - S_t$$

(1.1)

where $U_0 = u$ is the amount of initial reserve, the premium income is $c$ per each period, and $Y_i$ is the eventual claim amount in period $i$. For simplicity, throughout the paper we assume that $c = 1$.

Let $I_i$ be a binary random variable representing the claim occurrence. That is $I_i = 1$ if a claim occurs in period $i$ and $I_i = 0$, otherwise. For $i \geq 1$, define $Y_i = I_i X_i$, where the random variable $I_i$ and the individual claim amount random variable $X_i$ are independent in each time period. The random variable $X_i$ is strictly positive and forms a sequence of iid random variables with probability mass function (p.m.f.) $f(x) = P(X = x)$. Under this assumptions, the process (1.1) can be rewritten as

$$U_t = u + t - \sum_{i=1}^{N_t} I_i X_i,$$

(1.2)

where $u$ is non negative integer, $N_t$ is the number of claims up to time $t$ and $X_i$ is the amount of $i$th claim. It is assumed that $X_i$ random variables are independent and identically distributed (i.i.d.) and independent of the claim number process. Ruin of insurer’s occurs when $U_t \leq 0$ for some $t \geq 1$. The random time to ruin is defined as

$$T = \inf \{ t > 0 : U_t \leq 0 \}.$$ (1.3)

by Gerber (1988). Thus, ultimate ruin probability and survival probability can be defined as

$$\psi(u) = P(T < \infty | U_0 = u)$$

$$\phi(u) = 1 - \psi(u)$$

respectively. Similarly, ruin probability and survival probability in finite time can be defined as

$$\psi(u,n) = P(T \leq n | U_0 = u)$$

$$\phi(u,n) = 1 - \psi(u,n)$$

respectively.

Let the random indicators $I_1, I_2, ...$ be independent with $p = P\{I_i = 1\}$, then the model (1.2) is called the compound binomial model and

$$P\{N_n = k\} = C_n^k p^k (1 - p)^{n-k},\ k = 0, 1, ..., n.$$ 

In here, distribution of $N_n$ is classical binomial distribution. Tuncel and Tank (2014) suggested a recursive formula when the claim occurrences probabilities are non-homogeneous such as $p_i = P\{I_i = 1\}$.

Let $M_n$ and $K_n$ denote respectively the maximum and minimum levels of the surplus process up to period $n$,

$$M_n = \max_{1 \leq t \leq n} U_t , \quad K_n = \min_{1 \leq t \leq n} U_t.$$ 

These quantities may be useful tools on possible future investment or borrowing strategies for their consistent financial statement in an insurance company. Recursive equations are given for both marginal and joint distributions of the $M_n$ and $K_n$ values under the condition that insurance company survives at time $n$ for homogenous case by Eryilmaz et.al. (2012).
The remainder of the present paper is organized as follows: Section 2 presents recursive equations to compute marginal and joint distributions of $M_n$ and $K_n$ under the condition $T > n$. Section 3 gives means and variances of $M_n$ and $K_n$ for zero truncated geometric claim size distribution. Finally, discussions are given in Section 4.

2. Distributions of Extremes Surplus Process

For $u = 1, 2, \ldots$ and $n \geq 0$, define

$$\phi^{(1,n)}(u) = P_u(T > n)$$

$$\phi^{(1,n)}(u; k) = P_u(M_n \leq k, T > n)$$

$$\gamma^{(1,n)}(u; k) = P_u(K_n \geq k, T > n)$$

where

$$P_u(T > n) = \phi^{(1,n)}(u)$$

$$= \left\{ \begin{array}{ll}
1 & , n = 0 \\
\sum_{t=1}^{n} p_t \prod_{i=1}^{t-1} q_i \sum_{x=1}^{u+t-1} f(x) P_{n+t-x}(T^{(t+1,n)} > n-t) + \\
1 - \sum_{t=1}^{n} p_t \prod_{i=1}^{t-1} q_i & , n > 0
\end{array} \right.$$ 

and $k$ is a positive threshold which can be also considered as an upper barrier for surviving of the insurance company. In here, $T^{(t+1,n)}$ and $\phi^{(1,n)}(u)$ represents ruin time after the $t$-th period and non-ruin probability when the claim occurrences have nonhomogeneous probabilities respectively (Tuncel and Tank(2014)).

2.1. Theorem. For $u = 1, 2, \ldots$

$$P_u(M_n \leq k \mid T > n) = \frac{\phi^{(1,n)}(u; k)}{\phi^{(1,n)}(u)} \quad (2.1)$$

where

a. If $k \geq u + n$ and $n \geq 0$ then

$$\phi^{(1,n)}(u; k) = \phi^{(1,n)}(u)$$

b. If $u \leq k < u + n$ and $n \geq 0$ then

$$\phi^{(1,n)}(u; k) = \sum_{t=1}^{k-u+1} p_t \prod_{i=1}^{t-1} q_i \sum_{x=\max(1,u+t-k)}^{u+t-1} f(x)\phi^{(t+1,n-k)}(u+t-x; k) \quad (2.2)$$

Proof. It is clear that $P_u(M_n \leq k \mid T > n) = 1$ for $k \geq u + n$. So $\phi^{(1,n)}(u; k) = \phi^{(1,n)}(u)$ is trivial.

By conditioning on $W_1$, the time of the first claim, for $u \leq k < u + n$ and $n \geq 0$ then

$$P_u(M_n \leq k, T > n) = \sum_{t=1}^{\infty} P_u(U_1 \leq k, \ldots, U_n \leq k \mid W_1 = t) P(W_1 = t)$$

where $P(W_1 = t) = \prod_{i=1}^{t-1} q_i p_{i+1}$.

If $t \leq n$ then

$$P_u(U_1 = u + 1, \ldots, U_{t-1} = u + t-1 \mid W_1 = t) = 1$$

and

$$P_u(U_1 \leq k, \ldots, U_n \leq k \mid W_1 = t) = P_u(U_1 \leq k, \ldots, U_n \leq k, T^{(t+1,n)} > n-t)$$
for \( t \leq k - u + 1 \). Noting that \( U_t > 0 \) for \( t \leq n \) since the ruin occurs after period \( n \) and then by conditioning on the value of the first claim one obtains

\[
P_u (U_t \leq k, ..., U_n \leq k; T > n - t | W_1 = t)
\]

\[
= \sum_{x = 1}^{\infty} P_u \left( u + t - X > 0, X = x, M_{u+t-x}^{(t+1,n)} \leq k, T^{(t+1,n)} > n - t \right)
\]

\[
= \sum_{x = \max(1,u+t-k)}^{u+t-1} f(x)P_{u+t-x} \left( M_{u+t-x}^{(t+1,n)} \leq k, T^{(t+1,n)} > n - t \right)
\]

(2.4)

for \( t \leq k - u + 1 \). For \( t > n \), \( P(M_n = u + n) = 1 \). Thus \( P(M_n \leq k; T > n) = 0 \), if \( k < u + n \) and \( t > n \). Thus, for \( u \leq k < u + n \),

\[
\theta^{(1,n)} (u; k) = \sum_{t=1}^{k-u+1} \sum_{i=1}^{t-1} q_i \prod_{t=1}^{u+t-1} f(x) \theta^{(1,n-1)} (u + t - x; k)
\]

(2.5)
can be obtained by using (2.3) and (2.4). Hence the proof is completed. \( \square \)

Expansion of (2.2), which is recursive formula given in Theorem 2.1, as in follows:

- For \( n = 1 \) and \( u \leq k < u + 1 \)

\[
\theta^{(1,1)} (u; k) = \frac{p_1 \sum_{x = u+1-k}^{u} f(x)}{\theta^{(1,1)} (u)}
\]

- For \( n = 2 \) and \( u \leq k < u + 2 \)

\[
\theta^{(1,2)} (u; k) = \begin{cases} \frac{1}{\phi^{(1,2)} (u)} [A_1], & k = u \\ \frac{1}{\phi^{(1,2)} (u)} [A_2], & k = u + 1 \end{cases}
\]

where

\[
A_1 = p_1 p_2 \sum_{x = \max(1,u+1-k)}^{u} f(x) \sum_{y = \max(1,u+2-k-x)}^{u+1-x} f(y) + p_1 q_2 \sum_{x = \max(1,u+2-k)}^{u} f(x).
\]

\[
A_2 = p_1 p_2 \sum_{x = \max(1,u+1-k)}^{u} f(x) \sum_{y = \max(1,u+2-k-x)}^{u+1-x} f(y) + p_1 q_2 \sum_{x = \max(1,u+2-k)}^{u+1} f(x).
\]

- For \( n = 3 \) and \( u \leq k < u + 3 \)

\[
\theta^{(1,3)} (u; k) = \begin{cases} \frac{1}{\phi^{(1,3)} (u)} [A_3], & k = u \\ \frac{1}{\phi^{(1,3)} (u)} [A_4], & k = u + 1 \\ \frac{1}{\phi^{(1,3)} (u)} [A_5], & k = u + 2 \end{cases}
\]

where
\[ A_3 = \sum_{x=\max(1,u+1-k)}^{u} f(x) \sum_{y=\max(1,u+2-k-x)}^{u+1-x} f(y) \sum_{z=\max(1,u+3-k-x-y)}^{u+2-x-y} f(z) \]
\[ + \sum_{x=\max(1,u+1-k)}^{u} f(x) \sum_{y=\max(1,u+3-k-x-y)}^{u+1-x} f(y) \]
\[ + \sum_{x=\max(1,u+2-k)}^{u} f(x) \sum_{y=\max(1,u+2-k-x)}^{u+2-x} f(y) \]
\[ + \sum_{x=\max(1,u+3-k)}^{u} f(x) \]

\[ A_4 = \sum_{x=\max(1,u+1-k)}^{u} f(x) \sum_{y=\max(1,u+2-k-x)}^{u+1-x} f(y) \sum_{z=\max(1,u+3-k-x-y)}^{u+2-x-y} f(z) \]
\[ + \sum_{x=\max(1,u+1-k)}^{u} f(x) \sum_{y=\max(1,u+3-k-x-y)}^{u+1-x} f(y) \]
\[ + \sum_{x=\max(1,u+2-k)}^{u} f(x) \sum_{y=\max(1,u+2-k-x)}^{u+2-x} f(y) \]
\[ + q_1 p_2 q_3 \sum_{x=\max(1,u+3-k)}^{u+1} f(x) \]

\[ A_5 = \sum_{x=\max(1,u+1-k)}^{u} f(x) \sum_{y=\max(1,u+2-k-x)}^{u+1-x} f(y) \sum_{z=\max(1,u+3-k-x-y)}^{u+2-x-y} f(z) \]
\[ + \sum_{x=\max(1,u+1-k)}^{u} f(x) \sum_{y=\max(1,u+3-k-x-y)}^{u+1-x} f(y) \]
\[ + \sum_{x=\max(1,u+2-k)}^{u} f(x) \sum_{y=\max(1,u+2-k-x)}^{u+2-x} f(y) \]
\[ + q_1 p_2 q_3 \sum_{x=\max(1,u+3-k)}^{u+1} f(x) \]
\[ + p_1 q_2 q_3 \sum_{x=\max(1,u+1-k)}^{u} f(x) + q_1 p_2 q_3 \sum_{x=\max(1,u+2-k)}^{u+1} f(x) \]

2.2. Theorem. For \( u = 1, 2, \ldots \)

\[ P_u (K_n \geq k \mid T > n) = \frac{\gamma^{(1,n)}(u; k)}{\phi^{(1,n)}(u)} \quad (2.7) \]

where
a. If \( k \leq n \) and \( n = 0 \) then
\[
\gamma^{(1,n)}(u;k) = 1
\]
b. If \( k \leq u + 1 \) and \( n \geq 0 \) then,
\[
\gamma^{(1,n)}(u;k) = \sum_{t=\max(1,k-u+1)}^{n} \prod_{i=1}^{t-1} p_i \frac{u+t-k}{q_i} \sum_{x=1}^{u+t-k} f(x) \gamma^{(t+1,n-1)}(u-t-x;k) + \sum_{t=n+1}^{\infty} \prod_{i=1}^{t-1} p_i q_i
\]

**Proof.** The proof is clear for \( k \leq n \) and \( n = 0 \).

By conditioning on the time of first claim, for \( k \leq u + 1 \)
\[
P_u(K_n \geq k | T > n) = \sum_{t=1}^{\infty} P_u(U_1 \geq k, ..., U_n \geq k | W_1 = t) \cdot P(W_1 = t) \quad \text{(2.8)}
\]
where \( P(W_1 = t) = \prod_{i=1}^{t-1} q_i p_i \). If \( t \leq n \) and \( k \leq u + 1 \) then
\[
P_u(U_1 \geq k, ..., U_{t-1} \geq k | W_1 = t) = 1 \quad \text{(2.9)}
\]
and
\[
P_u(U_1 \geq k, ..., U_{t-1} \geq k, T > n | W_1 = t) = P_u(U_t \geq k, ..., U_n \geq k, T^{(t+1,n)} > n-t | W_1 = t)
\]
If \( t > n \) and \( k \leq u + 1 \) then
\[
P_u(U_1 \geq k, ..., U_n \geq k | W_1 = t) = 1. \quad \text{(2.10)}
\]
By conditioning on the time of first claim, for \( t \leq n \) and \( k \leq u + 1 \)
\[
P_u(U_t \geq k, ..., U_n \geq k, T > n-t | W_1 = t) = \sum_{x=1}^{u+t-k} P_{u+t-x}(K_{x-1}^{(t+1,n)} \geq k, T^{(t+1,n)} > n-t)
\]
for \( t \leq n \) and \( k \leq u + 1 \). Hence,
\[
\gamma^{(1,n)}(u;k) = \sum_{t=\max(1,k-u+1)}^{n} \prod_{i=1}^{t-1} p_i \frac{u+t-k}{q_i} \sum_{x=1}^{u+t-k} f(x) \gamma^{(t+1,n-1)}(u-t-x;k) + \sum_{t=n+1}^{\infty} \prod_{i=1}^{t-1} p_i q_i
\]
can be obtained by using (2.9),(2.10) and (2.11). Thus the proof is completed. \( \square \)

**Expansion of (2.7)** for \( n = 1, 2, 3 \), which is also recursive formula given in Theorem 2.2, as in follows:

- **For** \( n = 1 \)
  \[
  \gamma^{(1,1)}(u;k) = \left\{ \begin{array}{ll}
  1 & , k = u + 1 \\
  \prod_{x=1}^{u+k} f(x) & , k < u + 1
  \end{array} \right. \quad \text{(2.12)}
  \]
  where
  \[
  a_1 = p_1 \sum_{x=1}^{u+k} f(x) + q_1
  \]
- **For** \( n = 2 \)
  \[
  \gamma^{(1,2)}(u;k) = \left\{ \begin{array}{ll}
  \prod_{x=1}^{u+k} f(x) & , k = u + 1 \\
  \prod_{x=1}^{u+k} f(x) + q_1 q_2 & , k < u + 1
  \end{array} \right. \quad \text{(2.13)}
  \]
  where
  \[
  a_2 = q_1 q_2 \sum_{x=1}^{u+k} f(x) + q_1 q_2
  \]
and
\[
a_3 = p_1p_2 \sum_{x=1}^{u+1-k} f(x) \sum_{y=1}^{u+2-k} f(y) + p_1q_2 \sum_{x=1}^{u+1-k} f(x) + q_1p_2 \sum_{x=1}^{u+2-k} f(x) + q_1q_2
\]

- For \( n = 3 \)
\[
\gamma^{(1,3)}(u; k) = \begin{cases} 1 \frac{1}{\varphi^{1-\gamma(u)}}[a_4], & k = u + 1 \\ \frac{1}{\varphi^{1-\gamma(u)}}[a_5], & k < u + 1 \end{cases}
\]
where
\[
a_4 = q_1q_2p_3 \sum_{y=1}^{u+3-k-x} f(y) + q_1p_2p_3 \sum_{x=1}^{u+2-k} f(x) \sum_{y=1}^{u+3-k-x} f(y) + q_1q_2q_3 \sum_{y=1}^{u+2-k} f(y)
\]
and
\[
a_5 = p_1p_2p_3 \sum_{x=1}^{u+1-k} f(x) \sum_{y=1}^{u+2-k} f(y) \sum_{z=1}^{u+3-k-x} f(z)
+ p_1q_2q_3 \sum_{x=1}^{u+1-k} f(x) + p_1p_2p_3 \sum_{x=1}^{u+1-k} f(x) \sum_{y=1}^{u+3-k-x} f(y)
+ p_1p_2q_3 \sum_{x=1}^{u+2-k} f(x) \sum_{y=1}^{u+3-k-x} f(y)
+ q_1p_2q_3 \sum_{x=1}^{u+2-k} f(x) + q_1p_2p_3 \sum_{x=1}^{u+2-k} f(x) \sum_{y=1}^{u+3-k-x} f(y)
+ q_1q_2q_3
\]

3. Case study

As mentioned before, insurance company may face nonhomogeneous claim occurrences probabilities in different periods (e.g., month). For this reason, in this section four different cases are considered for different values of \( \alpha \) and \( u \) in finite time model and given in Table 1 where \( P(I_i = 1) = p_i \) for \( i = 1, ..., 12 \).

<table>
<thead>
<tr>
<th>Case1</th>
<th>Case2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p_i = 0.01 \times i, \ i = 1, ..., 12 )</td>
<td>( p_i = 0.02 \times i, \ i = 1, ..., 12 )</td>
</tr>
<tr>
<td>Case3</td>
<td>Case4</td>
</tr>
<tr>
<td>( p_i = 0.03 \times i, \ i = 1, ..., 12 )</td>
<td>( p_i = 0.04 \times i, \ i = 1, ..., 12 )</td>
</tr>
</tbody>
</table>

Let claim size distribution be geometric with the following cdf and pmf
\[
F(x) = 1 - \alpha^x, \ x = 1, 2, ...
\]
(3.1)
\[
f(x) = (1 - \alpha)\alpha^{x-1}, \ x = 1, 2, ...
\]
(3.2)
respectively. It is clear that
\[
E(X) = \frac{1}{1 - \alpha}, \ 0 < \alpha < 1.
\]
(3.3)
According to the cases which are given in Table 1, we obtained expected values and variances of \( M_n \) and \( K_n \) for the cases, which are given in Table 2 where the claim
amount distribution as in (3.2) and $\mu_1 = E(M_n \mid T > n), \sigma_1^2 = Var(M_n \mid T > n), \mu_2 = E(K_n \mid T > n)$ and $\sigma_2^2 = Var(K_n \mid T > n)$.

Table 2. Expected values and variances of $M_n$ and $K_n$ for $\alpha = 9/10$ in cases

<table>
<thead>
<tr>
<th>$u$</th>
<th>Cases</th>
<th>$\mu_1$</th>
<th>$\sigma_1^2$</th>
<th>$\mu_2$</th>
<th>$\sigma_2^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>Case1</td>
<td>14.7363</td>
<td>3.6557</td>
<td>4.9425</td>
<td>0.1632</td>
</tr>
<tr>
<td></td>
<td>Case2</td>
<td>13.7613</td>
<td>5.0934</td>
<td>4.8865</td>
<td>0.3170</td>
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<tr>
<td></td>
<td>Case3</td>
<td>12.9846</td>
<td>5.4676</td>
<td>4.8333</td>
<td>0.4561</td>
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<tr>
<td></td>
<td>Case4</td>
<td>12.3376</td>
<td>5.3540</td>
<td>4.7801</td>
<td>0.5920</td>
</tr>
<tr>
<td>8</td>
<td>Case1</td>
<td>18.3975</td>
<td>5.2155</td>
<td>8.8418</td>
<td>0.8073</td>
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<tr>
<td></td>
<td>Case2</td>
<td>17.2249</td>
<td>6.9138</td>
<td>8.6875</td>
<td>1.5438</td>
</tr>
<tr>
<td></td>
<td>Case3</td>
<td>16.3182</td>
<td>7.1493</td>
<td>8.5432</td>
<td>2.2048</td>
</tr>
<tr>
<td></td>
<td>Case4</td>
<td>15.6258</td>
<td>6.7602</td>
<td>8.4036</td>
<td>2.8110</td>
</tr>
</tbody>
</table>

We sketch the graphics of cumulative distribution function of $M_n$ and $K_n$ for cases in Figure 1 and Figure 2 respectively. In the Figures 1 and 2 solid line represents for $u = 4$ and dashed line represents $u = 8$.

Figure 1. Cumulative Distribution function of $M_n$ given $T > n$

a) Case 1     b) Case 2

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<tr>
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<td>Case 1</td>
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<td>Case 3</td>
<td>Case 4</td>
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</tbody>
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Figure 2. Cumulative Distribution function of $K_n$ given $T > n$

a) Case 1     b) Case 2

<p>| | | | | | |</p>
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<td>Case 4</td>
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</tr>
</tbody>
</table>
4. Conclusions

This study presents some characteristical results and distributions of maximum and minimum levels of surplus in compound binomial risk model with nonhomogeneous claim occurrences by different cases which have critical importance for an insurance company. This study may also lead to future studies with stochastic premium income in continuous time model.

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References