A NOTE ON THE ENDMORPHISM RING OF FINITELY PRESENTED MODULES OF THE PROJECTIVE DIMENSION \( \leq 1 \)

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Abstract. In this paper, we study the behavior of endomorphism rings of a cyclic, finitely presented module of projective dimension \( \leq 1 \). This class of modules extends to arbitrary rings the class of couniformly presented modules over local rings.

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1. Introduction

Throughout this paper, all rings will be associative with identity and modules will be unital right modules. For any ring \( R \), the Jacobson radical of \( R \) will be denoted by \( J(R) \).

Recall that \( M_R \) is couniform if it has dual Goldie dimension one (if and only if it is non-zero and the sum of any two proper submodules of \( M_R \) is a proper submodule of \( M_R \)). It is well know that a projective right module \( P_R \) is couniform if and only if \( \text{End}(P_R) \) is a local ring if and only if there exists an idempotent \( e \in R \) with \( P_R \cong eR \) and \( eRe \) a local ring, if and only if is a finitely generated module with a unique maximal submodule.

In [7], Facchini and Girardi introduced and studied the notion of couniformly presented modules. A module \( M_R \) is called couniformly presented if it is non-zero and there exists an exact sequence

\[
0 \to C_R \xrightarrow{\iota} P_R \to M_R \to 0
\]

with \( P_R \) projective and both \( C_R \) and \( P_R \) couniform modules. In this case, every endomorphism \( f \) of \( M_R \) lifts to an endomorphism \( f_0 \) of its projective cover \( P_R \), and we will denote by \( f_1 \) the restriction to \( C_R \) of \( f_0 \). Hence we have a commutative diagram

\[
\begin{array}{ccc}
0 & \to & C_R & \xrightarrow{\iota} & P_R & \to & M_R & \to & 0 \\
\downarrow f_1 & & \downarrow f_0 & & \downarrow f & & \downarrow f & & \downarrow 0 \\
0 & \to & C_R & \xrightarrow{\iota} & P_R & \to & M_R & \to & 0.
\end{array}
\]

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In [7, Theorem 2.5], Facchini and Girardi proved that:

- Let $0 \to C_R \to P_R \to M_R \to 0$ be a couniform presentation of a couniformly presented module $M_R$. Set $K := \{ f \in \text{End}(M_R) \mid f$ is not surjective $\}$ and $I := \{ f \in \text{End}(M_R) \mid f_1 : C_R \to C_R$ is not surjective $\}$. Then $K$ and $I$ are completely prime two-sided ideals of $\text{End}(M_R)$, and the union $K \cup I$ is the set of all non-invertible elements of $\text{End}(M_R)$. Moreover, one of the following two conditions holds:
  (a) Either $\text{End}(M_R)$ is a local ring, or
  (b) $K$ and $I$ are the two maximal right, maximal left ideals of $\text{End}(M_R)$.

If $M_R$ and $M'_R$ are two couniformly presented modules with couniform presentations $0 \to C_R \to P_R \to M_R \to 0$ and $0 \to C'_R \to P'_R \to M'_R \to 0$, we say that $M_R$ and $M'_R$ have the same lower part, and we write $[M_R]_{\ell} = [M'_R]_{\ell}$, if there are two homomorphisms $f_0 : P_R \to P'_R$ and $f'_0 : P'_R \to P_R$ such that $f_0(C_R) = C'_R$ and $f'_0(C'_R) = C_R$.

Recall that a ring $R$ is semilocal if $R/J(R)$ is semisimple artinian, that is, isomorphic to a finite direct product of rings $M_{n_i}(D_i)$ of $n_i \times n_i$ matrices over division rings $D_i$. A ring $R$ is homogeneous semilocal if $R/J(R)$ is simple artinian, that is, isomorphic to the ring $M_n(D)$ of all $n \times n$ matrices for some positive integer $n$ and some division ring $D$ [2, 4]. Examples of such rings include all local rings and all simple Artinian rings. If $R$ is a homogeneous semilocal ring, then so are the rings $eRe$ and $M_n(R)$, where $e$ is a nonzero idempotent element of $R$ and $M_n(R)$ is the matrix ring over $R$. Also, homogeneous semilocal rings appear in a natural way when one localizes a right Noetherian ring with respect to a right localizable prime ideal.

In [4], Corisello and Facchini showed that:

- a homogeneous semilocal ring has a unique maximal proper two-sided ideal and a unique simple module up to isomorphism. Similarly, as in the case of local rings, a homogeneous semilocal ring has only one indecomposable projective module $P_R$ up to isomorphism, and all projective modules are direct sums of copies of this $P_R$.
- for a module $M$ over any ring $R$, the Krull-Schmidt theorem holds for $M$ provided $\text{End}_R(M)$ is homogeneous semilocal—that is, the direct sum decomposition of $M$ into indecomposable summands is unique up to isomorphism.

In [2], Barioli-Facchini-Raggi proved that:

- The later result fails to extend to modules $M_R$ with finite direct sum decompositions whose indecomposable summands have homogeneous semilocal endomorphism rings.
  - If a module $M$ over a ring $R$ has two decompositions $M = M_1 \oplus \cdots \oplus M_t = N_1 \oplus \cdots \oplus N_s$ where all the summands are indecomposable with homogeneous semilocal endomorphism rings, then these two decompositions are isomorphic.
2. The endomorphism ring

The following results describe the endomorphism ring of a cyclic, finitely presented module of projective dimension \( \leq 1 \) over a local ring. Throughout this paper, we will assume that \( M_R \neq 0 \).

**Theorem 2.1.** Let \( R \) be a local ring and let \( M_R := R_R/I \) be a cyclic, finitely presented module of projective dimension \( \leq 1 \). Suppose \( \text{Ext}^1_R(M_R, R_R) = 0 \).

Assume \( 0 \neq I \neq R \) and let \( E \) be the idealizer of the right ideal \( I \) of \( R \), that is, the set of all \( r \in R \) with \( rI \subseteq I \), so that \( \text{End}(M_R) \cong E/I \). Set \( L := \{ r \in R \mid rI \subseteq IJ(R) \} \) and \( K := E \cap J(R) \). Let \( \psi : E \to \text{End}_R(I/IJ(R)) \) be the ring morphism defined by

\[
\psi(e)(x + IJ(R)) = ex + IJ(R),
\]

for every \( e \in E \) and \( x \in I \). Let \( n \) be the dimension of the right vector space \( I/IJ(R) \) over the division ring \( R/J(R) \). Then:

1. \( L \) and \( K \) are prime two-sided ideals of \( E \) containing \( I \) and \( K \) is a completely prime ideal of \( E \).
2. For every \( e \in E \), the element \( e + I \) of \( E/I \) is invertible in \( E/I \) if and only if \( e + J(R) \) is invertible in \( R/J(R) \) and \( \psi(e) \) is invertible in \( \text{End}_R(I/IJ(R)) \).
3. The quotient ring \( E/L \) is isomorphic to the ring \( M_n(R/J(R)) \) of all \( n \times n \) matrices over the division ring \( R/J(R) \).
4. Exactly one of the following two conditions holds:
   a) Either \( K \subseteq L \), in which case \( E/I \) is a homogeneous semilocal ring with Jacobson radical \( L/I \), or
   b) \( L \) and \( K \) are not comparable.

**Proof.** (1) and (3). Notice that \( L \) is contained in \( E \) and is the kernel of \( \psi \), so that \( L \) is a two-sided ideal of \( E \). Trivially, \( I \) is contained in \( L \). Let us prove that \( \psi \) is onto.

Let \( f : I/IJ(R) \to I/IJ(R) \) be a morphism. Since \( M_R := R_R/I \) is of projective dimension \( \leq 1 \), the ideal \( I \) is projective, so that \( f \) lifts to a morphism \( f' : I_R \to I_R \).

Apply the functor \( \text{Hom}(-, R_R) \) to the exact sequence \( 0 \to I_R \to R_R \to M_R \to 0 \), getting a short exact sequence

\[
0 \to \text{Hom}(M_R, R_R) \to \text{Hom}(R_R, R_R) \to \text{Hom}(I_R, R_R) \to 0
\]

because \( \text{Ext}^1_R(M_R, R_R) = 0 \). Hence \( f' \) can be extended to a morphism \( f'' : R_R \to R_R \) which is necessarily left multiplication by an element \( r \in R \). Since \( f'' \) restricts to the endomorphism \( f' \) of \( I_R \), we get that \( r \in E \), and \( \psi(e) = f \). This proves that \( \psi \) is an onto ring morphism, so that

\[
E/L = E/\ker \psi \cong \text{End}_R(I/IJ(R)) \cong M_n(R/J(R)).
\]

This proves (3).

As \( \text{End}_R(I/IJ(R)) \cong M_n(R/J(R)) \) is a simple ring, it follows that \( L \) is a prime ideal and a maximal two-sided ideal. Similarly, \( K \) is the kernel of the composite morphism \( \varphi : E \to R/J(R) \) of the embedding \( E \to R \) and the canonical projection \( R \to R/J(R) \). Since \( R/J(R) \) is a division ring, we get that \( K \) is a completely prime, two-sided ideal of \( E \) containing \( I \). This concludes the proof of (1).
Since \( \varphi(I) = 0 \) and \( \psi(I) = 0 \), the morphisms \( \varphi \) and \( \psi \) induce morphisms \( \tilde{\varphi} : E/I \rightarrow R/J(R) \) and \( \tilde{\psi} : E/I \rightarrow \text{End}(I/IJ(R)) \), respectively. Hence \( e + I \) invertible implies \( \varphi(e) = e + J(R) \) invertible in \( R/J(R) \) and \( \psi(e) \) is invertible in \( \text{End}_R(I/IJ(R)) \).

(\( \Leftarrow \)) Assume that \( e \in E \) and that \( \varphi(e) \) and \( \psi(e) \) are invertible in \( R/J(R) \) and \( \text{End}_R(I/IJ(R)) \), respectively. Then we have a commutative diagram with exact rows

\[
\begin{array}{cccccc}
0 & \rightarrow & I & \rightarrow & R_R & \rightarrow & R_R/I & \rightarrow & 0 \\
& & e & \downarrow & \mathrm{\pi} & \downarrow & \mathrm{\pi} & & \\
0 & \rightarrow & I & \rightarrow & R_R & \rightarrow & R_R/I & \rightarrow & 0.
\end{array}
\]

Now \( \varphi(e) = e + J(R) \) invertible implies that \( e \in R \setminus J(R) \), and so \( e \) is invertible in \( R \). Hence the middle vertical arrow is an isomorphism. Since \( \psi(e) \) is invertible, it is an automorphism of \( I/IJ(R) \), and so \( e(I/IJ(R)) = I/IJ(R) \), that is, \( eI + IJ(R) = I \). By Nakayama’s Lemma, \( eI = I \). Hence the left vertical arrow is an epimorphism. By the Snake Lemma, the right vertical arrow is a monomorphism, hence an isomorphism. That is, \( e + I \) is invertible in \( E/I \).

(4) We have the three cases (a) \( L \subseteq K \), (b) \( K \subseteq L \), and (c) \( L \nsubseteq K \) and \( K \nsubseteq L \).

Assume \( L \subseteq K \). In this case, \( L \subseteq K \subseteq E \) implies that \( 0 \subseteq K/L \subseteq E/L \), so that \( E/L \cong M_n(R/J(R)) \) has a proper non-zero two-sided ideal. This is impossible, because \( M_n(R/J) \) is a simple ring. Hence this case cannot occur.

Assume \( K \subseteq L \). From (2), it follows that an element \( e + I \) of \( E/I \) is invertible in \( E/I \) if and only if \( e + J(R) \) is invertible in \( R/J(R) \) and \( e + L \) is invertible in \( E/L \). Hence, in order to prove (4) in this case \( K \subseteq L \), it suffices to prove that \( J(E/I) = L/I \).

(\( \subseteq \)) If \( e + I \in J(E/I) \), then \( 1 - xey + I \) is invertible in \( E/I \) for every \( x, y \in E \). Thus \( 1 - xey + L \) is invertible in \( E/L \) for all \( x, y \in E \), so that \( e + L \in J(E/L) \). But \( E/L \cong M_n(R/J(R)) \) has Jacobson radical 0 so that \( e \in L \).

(\( \supseteq \)) Take \( l + I \in L/I \) with \( l \in L \). Then \( 1 - xly + L = 1 + L \) in \( E/L \) for every \( x, y \in E \). Hence \( 1 - xly + L \) is invertible in \( E/L \). In particular, \( 1 - xly \notin K \). Thus \( 1 - xly \notin K \), so that \( 1 - xly \notin J(R) \). As \( R/J(R) \) is a division ring, it follows that \( 1 - xly + J(R) \) is invertible in \( R/J(R) \). Thus \( 1 - xly + I \) is invertible in \( E/I \), and \( l \in J(E/I) \). \( \square \)

It is known that a finitely presented module over a semilocal ring always has a semilocal endomorphism ring. We have the following natural question.

Question 2.2. Characterize \( J(E/I) \). This was done in [1] for cyclically presented modules.

As far as Question 2.2 is concerned, notice that, in the proof of Theorem 2.1(2), we have seen that the mapping

\[
\tilde{\varphi} \times \tilde{\psi} : E/J \rightarrow R/J(R) \times \text{End}(I/IJ(R))
\]

is a local morphism, so that its kernel \( K/I \cap L/I \) is contained in \( J(E/I) \). In particular, when \( K \subseteq L \), we have that \( L/I = J(E/I) \) as we have seen in Theorem 2.1(4)(a). We are not able to describe \( J(E/I) \) when \( K \) and \( K \) are not comparable.
Remark 2.3. Let $R$ be a local right self-injective ring. Let $M_R$ be a cyclic and finitely presented module of projective dimension $\leq 1$. Since $R_R$ is injective, we have that $\text{Ext}^1_R(M_R, R_R) = 0$. Thus, Theorem 2.1 can be applied.

Let $A$ and $B$ be two modules. We say that:

- $A$ and $B$ have the same monogeny class, and write $[A]_m = [B]_m$, if there exist a monomorphism $A \to B$ and a monomorphism $B \to A$ [5];
- $A$ and $B$ have the same epigeny class, and write $[A]_e = [B]_e$, if there exist an epimorphism $A \to B$ and an epimorphism $B \to A$;

It is clear that a module $A$ has the same monogeny (epigeny) class as the zero module if and only if $A = 0$.

- Two cyclically presented modules $R/aR$ and $R/bR$ over a local ring $R$ are said to have the same lower part, denoted $[R/aR]_l = [R/bR]_l$, if there exist $r, s \in R$ such that $raR = bR$ and $sbR = aR$ [1].
- If $M_R$ and $M'_R$ are two countinuously presented modules with countinuous presentations
  
  $0 \to C_R \to P_R \to M_R \to 0$
  
  and
  
  $0 \to C'_R \to P'_R \to M'_R \to 0$,

  we say that $M_R$ and $M'_R$ have the same lower part, and we write $[M_R]_l = [M'_R]_l$, if there are two homomorphisms $f_0: P_R \to P'_R$ and $f'_0: P'_R \to P_R$ such that $f_0(C_R) = C'_R$ and $f'_0(C'_R) = C_R$ [7].

Theorem 2.4. Let $R$ be a semiperfect ring and let $R_R/L$ be a cyclic uniform right $R$-module with $L \neq 0$. Let $E$ be the idealizer of the right ideal $L$ of $R$, that is, the set of all $r \in R$ with $rL \subseteq L$, so that

$$\text{End}(R_R/L) \cong E/L.$$ 

Similarly, let $E'$ be the idealizer of the right ideal $L + J(R)$ of $R$, so that

$$\text{End}(R_R/(L + J(R))) \cong E'/(L + J(R)).$$

Set $I := \{e \in E \mid \text{left multiplication by } e + I \text{ is a non-injective endomorphism of } R_R/L\}$ and $K := E \cap (L + J(R))$. Then:

1. $I$ and $K$ are two two-sided ideals of $E$ containing $L$, and $I$ is completely prime in $E$.
2. For every $e \in E$, the element $e + L$ of $E/L$ is invertible in $E/L$ if and only if $e + L + J(R)$ is invertible in $E'/L + J(R)$ and $e \notin I$.
3. Moreover:
   a. If $I \subseteq K$, then every epimorphism $R_R/L \to R_R/L$ is an automorphism of $R_R/L$,
   b. $K \not\subseteq I$ if and only if $[R_R/L]_m = [L + J(R)/L]_m$.

Proof. (1) We know that $\text{End}(R_R/L) \cong E/L$. Every endomorphism $e + L$ of $R_R/L$ extends to an endomorphism $e_1$ of the injective envelope $E(R_R/L)$. Define a ring morphism

$$\varphi: E \to \text{End}(E(R_R/L))/J(\text{End}(E(R_R/L)))$$
by \( \varphi(e) = e_1 + J(\text{End}(E(R_R/L))) \) for every \( e \in E \). Since \( R_R/L \) is uniform, the injective envelope \( E(R_R/L) \) is indecomposable, the endomorphism ring \( \text{End}(E(R_R/L)) \) is a local ring, and the Jacobson radical \( J(\text{End}(E(R_R/L))) \) consists of all non-injective endomorphisms of \( E(R_R/L) \). It follows that \( I \), which is equal to the kernel of the ring morphism \( \varphi \), whose range is the division ring

\[
\text{End}(E(R_R/L))/J(\text{End}(E(R_R/L))),
\]

must be a completely prime two-sided ideal of \( E \). The remaining part of statement (1) is easily checked.

(2) We have already seen that there is a ring morphism

\[
\varphi : E \to \text{End}(E(R_R/L))/J(\text{End}(E(R_R/L)))
\]

whose kernel is \( I \). Hence if \( e \in E \) and \( e + L \) is invertible in \( E/L \), then \( \varphi(e) \) must be invertible in the division ring \( \text{End}(E(R_R/L))/J(\text{End}(E(R_R/L))) \). Thus \( \varphi(e) \neq 0 \), that is, \( e \notin \ker \varphi = I \). Similarly, we can consider the ring morphism

\[
\psi : E \to \text{End}(R_R/L + J(R))
\]

defined by \( \psi(e)(r + L + J(R)) = er + L + J(R) \) for every \( e \in E \) and every \( r \in R \). Its kernel is \( K \), which contains \( L \). Hence \( e + L \) invertible in \( E/L \) implies \( \psi(e) \) invertible in \( \text{End}(R_R/L + J(R)) \). But

\[
\text{End}(R_R/(L + J(R))) \cong E'/(L + J(R)),
\]

so that \( e + L + J(R) \) must be invertible in \( E'/L + J(R) \).

Conversely, assume \( e \in E \), \( e + L + J(R) \) invertible in \( E'/L + J(R) \) and \( e \notin I \). We want to show that \( e + L \) is invertible in \( E/L \). Since \( E/L \cong \text{End}(R_R/L) \), this is equivalent to showing that left multiplication \( \mu_e : R_R/L \to R_R/L \) by \( e \) is an automorphism of \( R_R/L \). Now \( e \notin I \) is equivalent to \( \mu_e \) is injective by definition of \( I \). In order to show that \( \mu_e \) is onto as well, it suffices to prove that \( \mu_e \) induces an onto endomorphism

\[
(R_R/L)/(R_R/L)J(R) \to (R_R/L)/(R_R/L)J(R)
\]

by Nakayama’s Lemma. But \( (R_R/L)J(R) = L + J(R)/L \), so that

\[
(R_R/L)/(R_R/L)J(R) \cong R_R/L + J(R).
\]

Hence \( e + L + J(R) \) invertible in \( E'/L + J(R) \cong \text{End}(R_R/(L + J(R))) \) means that the endomorphism \( \psi(e) \) of \( R_R/L + J(R) \) induced by \( \mu_e \) is onto, as desired.

(3) (a) Assume \( I \subseteq K \). Let \( e + L : R_R/L \to R_R/L \) be an epimorphism with \( e \in E \). Then the induced morphism \( \varphi(e) : R_R/L + J(R) \to R_R/L + J(R) \) is also an epimorphism, so that it is an automorphism because \( R_R/L + J(R) \) is a semisimple module of finite Goldie dimension. In the isomorphism

\[
\text{End}(R_R/(L + J(R))) \cong E'/(L + J(R)),
\]

we obtain that \( e + L + J(R) \) is invertible in the ring \( E'/L + J(R) \). Thus \( e \notin K \). Hence \( e \notin I \). It follows from (2) that \( e + L \) is invertible, that is, it is an automorphism of \( R_R/L \).

(b) Assume \( K \not\subseteq I \). Then there is an element \( f \in K \), \( f \notin I \). Thus \( f \in E \) induces an endomorphism \( f \) of \( R_R/L \). Now \( f \notin I \) means that \( f \) is injective, and \( f \in K \) means that the image of \( f \) is contained in \( L + J(R)/L \). Hence \( [R_R/L]_m = [L + J(R)/L]_m \). Conversely, if \( [R_R/L]_m = [L + J(R)/L]_m \), then there is a monomorphism
Let every epimorphism
By Theorem 2.4, the only case in which we cannot apply Theorem 2.5 proves that the theorem holds when hypothesis (b) holds.

so that $R$ is isomorphic to a direct summand of $R$. It follows that $R$ is an automorphism of $R$. Assume that either 

1. every monomorphism $R_R/L \rightarrow R_R/L$ is an automorphism of $R_R/L$, or
2. every epimorphism $R_R/L \rightarrow R_R/L$ is an automorphism of $R_R/L$, or
3. $[R_R/L]_m = [L + J(R)/L]_m$.

Then the followings are equivalent.

(a) $R_R/L \cong R_R/L'$
(b) $[R_R/L]_m = [R_R/L']_m$ and $[R_R/L]_e = [R_R/L']_e$.

Proof. Assume $[R_R/L]_m = [R_R/L']_m$ and $[R_R/L]_e = [R_R/L']_e$. Then there are monomorphisms $\alpha: R_R/L \rightarrow R_R/L'$ and $\beta: R_R/L' \rightarrow R_R/L$ and epimorphisms $\alpha': R_R/L \rightarrow R_R/L'$ and $\beta': R_R/L' \rightarrow R_R/L$. Then $\beta\alpha$ is a monomorphism $R_R/L \rightarrow R_R/L$ and $\beta'\alpha'$ is an epimorphism $R_R/L \rightarrow R_R/L$. If hypothesis (a) holds, then $\beta$ is an automorphism of $R_R/L$ that factors through $R_R/L'$, so that $R_R/L$ is isomorphic to a direct summand of $R_R/L'$. But $R_R/L \neq 0$ and $R_R/L'$ is uniform, so that $R_R/L \cong R_R/L'$. This proves our theorem under hypothesis (a). Dually one proves that the theorem holds when hypothesis (b) holds.

Assume now that hypothesis (c) holds, i.e., $[R_R/L]_m = [L + J(R)/L]_m$. Equivalently, there exists a monomorphism $\gamma: R_R/L \rightarrow R_R/L$ whose image is contained in $L + J(R)/L$. Now if either $\alpha$ or $\alpha'$ are isomorphisms, then the existence of $\alpha$ or $\alpha'$ shows that $R_R/L \cong R_R/L'$. This allows us to conclude. Thus we can assume that $\alpha$ is not an epimorphism and $\alpha'$ is not a monomorphism. Then $\alpha' + \alpha\gamma: R_R/L \rightarrow R_R/L'$ is an isomorphism, because:

1. It is injective, because it is the sum of the injective morphism $\alpha\gamma: R_R/L \rightarrow R_R/L'$ and the non-injective morphism $\alpha': R_R/L \rightarrow R_R/L'$, and $R_R/L$ is uniform.
2. The ideal $J(R)$ is superfluous in $R_R$ by Nakayama’s Lemma. Considering the canonical projection $R_R \rightarrow R_R/L$, it follows that $L + J(R)/L$ is superfluous in $R_R/L$. Applying the morphism $\alpha: R_R/L \rightarrow R_R/L'$, we get that the image of $\alpha\gamma$ is contained in $\alpha(L + J(R)/L)$, hence is a superfluous submodule of $R_R/L'$. Thus the sum of $\alpha\gamma$ and the surjective morphism $\alpha': R_R/L \rightarrow R_R/L'$ is a surjective morphism $\alpha' + \alpha\gamma: R_R/L \rightarrow R_R/L'$.

Thus $\alpha' + \alpha\gamma$ is an isomorphism of $R_R/L$ onto $R_R/L'$.

Remark 2.6. By Theorem 2.4, the only case in which we cannot apply Theorem 2.5 is when $K$ is properly contained in $I$. Namely, if $K \subseteq I$, then $[R_R/L]_m = [L + J(R)/L]_m$ and we can apply Theorem 2.5(a); if $K \not\subseteq I$, then either $K$ is properly contained in $I$, which is the case still unknown, or $K = I$, but in the latter case every epimorphism $R_R/L \rightarrow R_R/L$ is an automorphism of $R_R/L$ by Theorem 2.4(1).

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