MORE GENERAL FORMS OF GENERALIZED FUZZY BI-IDEALS IN SEMIGROUPS

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Abstract
More general forms of the paper by O. Kazancı and S. Yamak (Generalized fuzzy bi-ideals of semigroups, Soft Comput. 12, 1119–1124, 2008) are discussed. The notion of $(\in, \in \lor q_k)$-fuzzy bi-ideals in a semigroup $S$ is introduced, and several properties are investigated. Characterizations of an $(\in, \in \lor q_k)$-fuzzy bi-ideal in a semigroup $S$ are discussed.

Keywords: $(\in, \in \lor q)$-fuzzy bi-ideal, $(\in, \in \lor q_k)$-fuzzy bi-ideal.

2000 AMS Classification: 06 F 05, 20 M 12, 08 A 72.

1. Introduction
Kuroki initiated the theory of fuzzy semigroups (see [12, 13, 14]). The monograph by Mordeson et al. [15] dealt with the theory of fuzzy semigroups and their application in fuzzy coding, fuzzy finite state machines and fuzzy languages.

The idea of fuzzy point and its “belongingness” and “quasico incidence” with a fuzzy set were given by Pu and Liu [16]. In [5], Bhakat and Das used this idea to define $(\alpha, \beta)$-fuzzy subgroups. In [1, 2, 3, 4, 5], $(\alpha, \beta)$-fuzzy substructures of algebraic structures are discussed. As a generalization of fuzzy interior ideals of a semigroup, Jun and Song [10] discussed generalized fuzzy interior ideals in semigroups.

Yin et al. [17] discussed the $(\in, \in \lor q)$-fuzzy subsemigroups and ideals of an $(\in, \in \lor q)$-fuzzy semigroup. Jun [7] considered a more general form of the notion of quasi-coincidence of a fuzzy point with a fuzzy set, and generalized the results in the papers [8, 9]. As a generalization of fuzzy bi-ideals of a semigroup, Kazancı and Yamak [11] considered the generalized fuzzy bi-ideals of a semigroup.

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The purpose of this article is to obtain more general forms than in Kazancı and Yamağ’s paper [11]. We introduce the notion of \((\in, \in \lor \forall \kappa)\)-fuzzy bi-ideals, and investigate related properties. We provide characterizations of an \((\in, \in \lor \forall \kappa)\)-fuzzy bi-ideal. The important achievement of the study with an \((\in, \in \lor \forall \kappa)\)-fuzzy bi-ideal is that the notion of an \((\in, \in \lor \kappa)\)-fuzzy bi-ideal is a special case of an \((\in, \in \lor \forall \kappa)\)-fuzzy bi-ideal, and thus the related results obtained in the paper [11] are corollaries of our results obtained in this paper.

2. Preliminaries

Let \(S\) be a semigroup. By a subsemigroup of \(S\) we mean a nonempty subset \(G\) of \(S\) such that \(G^2 \subseteq G\). A subsemigroup \(A\) of a semigroup \(S\) is called a bi-ideal of \(S\) if \(ASA \subseteq A\). A semigroup \(S\) is said to be right (resp., left) zero if \(xy = y\) (resp., \(xy = x\)) for all \(x, y \in S\) (see [13]). A semigroup \(S\) is said to be completely regular if, for every element \(a \in S\), there is an element \(x \in S\) such that \(a = axa\) and \(ax = xa\).

2.1. Definition. [6, 12] A fuzzy set \(\mu\) in a semigroup \(S\) is called a fuzzy bi-ideal of \(S\) if it is a fuzzy subsemigroup of \(S\) and satisfies:

\[ (\forall x, y \in S) (\mu(xy) \geq \min\{\mu(x), \mu(y)\}). \]

It is well known that a fuzzy set \(\mu\) in a semigroup \(S\) is a fuzzy bi-ideal of \(S\) if and only if \(U(\mu; t) := \{x \in S | \mu(x) \geq t\}\) is a bi-ideal of \(S\) for all \(t \in (0, 1]\).

2.2. Proposition. [14] For a non-empty subset \(I\) of a semigroup \(S\) the following assertions are equivalent:

1. \(I\) is a bi-ideal of \(S\).
2. The characteristic function \(\chi_I\) of \(I\) is a fuzzy bi-ideal of \(S\).

A fuzzy set \(\mu\) in a set \(S\) of the form

\[ \mu(y) := \begin{cases} t \in (0, 1] & \text{if } y = x, \\ 0 & \text{if } y \neq x, \end{cases} \]

is said to be a fuzzy point with support \(x\) and value \(t\) and is denoted by \((x, t)\).

For a fuzzy point \((x, t)\) and a fuzzy set \(\mu\) in a set \(S\), Pu and Liu [16] introduced the symbol \((x, t) \alpha \mu\), where \(\alpha \in \{\in, \in \lor \forall \kappa, \in \land \forall q\}\).

For any fuzzy set \(\mu\) in a set \(S\), we say that a fuzzy point \((x, t)\) is

1. contained in \(\mu\), denoted by \((x, t) \in \mu\), if \(\mu(x) \geq t\).
2. quasi-coincident with \(\mu\), denoted by \((x, t) \mu\kappa\), if \(\mu(x) + t > 1\).

For a fuzzy point \((x, t)\) and a fuzzy set \(\mu\) in \(S\), we say that

1. \((x, t) \in \forall q \mu\) if \((x, t) \in \mu\) or \((x, t) \mu\kappa\).
2. \((x, t) \alpha \mu\) if \((x, t) \alpha \mu\) does not hold for \(\alpha \in \{\in, \in \lor \forall q, \in \land \forall q\}\).

3. Generalizations of \((\in, \in \lor q)\)-fuzzy bi-ideals

In what follows let \(S\) denote a semigroup and \(k\) an arbitrary element of \([0, 1]\) unless otherwise specified. For a fuzzy point \((x, t)\) and a fuzzy set \(\mu\) in \(S\), we say that

1. \((x, t) \mu\kappa\) if \(\mu(x) + t + k > 1\).
2. \((x, t) \in \forall q \mu\) if \((x, t) \in \mu\) or \((x, t) \mu\kappa\).
3. \((x, t) \alpha \mu\) if \((x, t) \alpha \mu\) does not hold for \(\alpha \in \{\in, \in \lor \forall q, \in \land \forall q\}\).

3.1. Definition. A fuzzy set \(\mu\) in \(S\) is called an \((\in, \in \lor \forall q)\)-fuzzy bi-ideal of \(S\) if it satisfies:
(i) \((x, t_1) \in \mu, (y, t_2) \in \mu \Rightarrow (xy, \min\{t_1, t_2\}) \in \vee q_k \mu\),
(ii) \((x, t_1) \in \mu, (y, t_2) \in \mu \Rightarrow (xay, \min\{t_1, t_2\}) \in \vee q_k \mu\)
for all \(a, x, y \in S\) and \(t_1, t_2 \in (0, 1]\).

An \((\in, \in \vee \vee q_k)\)-fuzzy bi-ideal of \(S\) with \(k = 0\) is called an \((\in, \in \vee \vee q)\)-fuzzy bi-ideal of \(S\) (see [11, Definition 3.1]).

3.2. Example. Consider a semigroup \(S = \{a, b, c, d\}\) with a multiplication table given by Table 1.

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(1) Let \(\mu\) be a fuzzy set in \(S\) defined by
\[
\mu(x) := \begin{cases} 
0.3 & \text{if } x \in \{a, b\}, \\
0.1 & \text{if } x \in \{c, d\}.
\end{cases}
\]
Then \(\mu\) is an \((\in, \in \vee \vee q_{0.4})\)-fuzzy bi-ideal of \(S\).

(2) Let \(\nu\) be a fuzzy set in \(S\) defined by
\[
\nu(x) := \begin{cases} 
0.24 & \text{if } x \in \{a, c\}, \\
0.1 & \text{if } x \in \{b, d\}.
\end{cases}
\]
Then \(\nu\) is an \((\in, \in \vee \vee q_{0.52})\)-fuzzy bi-ideal of \(S\).

Note that every fuzzy bi-ideal is an \((\in, \in \vee \vee q_k)\)-fuzzy bi-ideal. But the converse is not true as seen in the following example.

3.3. Example. Let \(S = \{a, b, c, d, e\}\) be a set with Cayley table given by Table 2.

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Then \(S\) is a semigroup (see [11]). Define a fuzzy set \(\mu\) in \(S\) by
\[
\mu(x) := \begin{cases} 
0.7 & \text{if } x \in \{a, b, d\}, \\
0.6 & \text{if } x = c, \\
0.4 & \text{if } x = e.
\end{cases}
\]
Then \(\mu\) is an \((\in, \in \vee \vee q_{0.2})\)-fuzzy bi-ideal of \(S\). But it is not a fuzzy bi-ideal of \(S\) since
\[
\mu(dad) = \mu(c) = 0.6 < 0.7 = \min\{\mu(d), \mu(d)\}.
\]
3.4. Theorem. Let $\mu$ be a fuzzy set in $S$. Then $\mu$ is an $(\in, \in \lor q_k)$-fuzzy bi-ideal of $S$ if and only if it satisfies:

1. $(\forall x, y \in S) \ (\mu(xy) \geq \min \{\mu(x), \mu(y), \frac{1-k}{2}\})$,
2. $(\forall x, a, y \in S) \ (\mu(xay) \geq \min \{\mu(x), \mu(y), \frac{1-k}{2}\})$.

Proof. Let $\mu$ be an $(\in, \in \lor q_k)$-fuzzy bi-ideal of $S$. Assume that

$$\mu(ab) < \min \{\mu(a), \mu(b), \frac{1-k}{2}\}$$

for some $a, b \in S$. If $\min \{\mu(a), \mu(b)\} \geq \frac{1-k}{2}$, then $\mu(ab) < \frac{1-k}{2}$. Hence $(a, \frac{1-k}{2}) \in \mu$ and $(b, \frac{1-k}{2}) \in \mu$, but $(ab, \frac{1-k}{2}) \not\in \mu$. Moreover,

$$\mu(ab) + \frac{1-k}{2} < \frac{1-k}{2} + \frac{1-k}{2} = 1 - k,$$

and so $(ab, \frac{1-k}{2}) \nsubseteq \mu$. Thus $(ab, \frac{1-k}{2}) \nsubseteq \lor q_k \mu$. This is a contradiction.

If $\min \{\mu(a), \mu(b)\} < \frac{1-k}{2}$, then $\mu(ab) < \min \{\mu(a), \mu(b)\}$. Hence there exists $t \in (0, 1]$ such that

$$\mu(ab) < t \leq \min \{\mu(a), \mu(b)\}.$$

It follows that $(a, t) \in \mu$ and $(b, t) \in \mu$, but $(ab, t) \not\in \mu$. Also, $\mu(ab) + t < \frac{1-k}{2} + \frac{1-k}{2} = 1 - k$, i.e., $(ab, t) \nsubseteq q_k \mu$. Thus $(ab, t) \nsubseteq \lor q_k \mu$, a contradiction. Consequently, (1) is valid. Now suppose that

$$\mu(xay) < \min \{\mu(x), \mu(y), \frac{1-k}{2}\}$$

for some $x, a, y \in S$. We consider the following two cases:

Case 1. $\min \{\mu(x), \mu(y)\} \geq \frac{1-k}{2}$
Case 2. $\min \{\mu(x), \mu(y)\} < \frac{1-k}{2}$.

Case 1 implies that $(x, \frac{1-k}{2}) \in \mu$, $(y, \frac{1-k}{2}) \in \mu$, $\mu(xay) < \frac{1-k}{2}$, that is, $(xay, \frac{1-k}{2}) \nsubseteq \mu$, and

$$\mu(xay) + \frac{1-k}{2} < \frac{1-k}{2} + \frac{1-k}{2} = 1 - k,$$

i.e., $(xay, \frac{1-k}{2}) \nsubseteq q_k \mu$. Hence $(xay, \frac{1-k}{2}) \nsubseteq \lor q_k \mu$, a contradiction. For Case 2, we have $\mu(xay) < \min \{\mu(x), \mu(y)\}$ and so $\mu(xay) < t \leq \min \{\mu(x), \mu(y)\}$ for some $t \in (0, 1]$. Thus $(x, t) \in \mu$ and $(y, t) \in \mu$, but $(xay, t) \not\in \mu$. Also,

$$\mu(xay) + t < t + t < \frac{1-k}{2} + \frac{1-k}{2} = 1 - k,$$

i.e., $(xay, t) \nsubseteq q_k \mu$. Hence $(xay, t) \nsubseteq \lor q_k \mu$, a contradiction. Therefore (2) is valid.

Conversely, suppose that $\mu$ satisfies conditions (1) and (2). Let $x, y \in S$ and $t_1, t_2 \in (0, 1]$ be such that $(x, t_1) \in \mu$ and $(y, t_2) \in \mu$. Then $\mu(x) \geq t$ and $\mu(y) \geq t$. Using (1), we have

$$\mu(xy) \geq \min \{\mu(x), \mu(y), \frac{1-k}{2}\} \geq \min \{t_1, t_2, \frac{1-k}{2}\}.$$

If $\min \{t_1, t_2\} > \frac{1-k}{2}$, then $\mu(xy) \geq \frac{1-k}{2}$ and so

$$\mu(xy) + \min \{t_1, t_2\} > \frac{1-k}{2} + \frac{1-k}{2} = 1 - k,$$

i.e., $(xy, \min \{t_1, t_2\}) q_k \mu$.

If $\min \{t_1, t_2\} \leq \frac{1-k}{2}$, then $\mu(xy) \geq \min \{t_1, t_2\}$, i.e., $(xy, \min \{t_1, t_2\}) \in \mu$. Hence $(xy, \min \{t_1, t_2\}) \in \lor q_k \mu$. Let $x, a, y \in S$ and $t_1, t_2 \in (0, 1]$ be such that $(x, t_1) \in \mu$ and $(y, t_2) \in \mu$. Then $\mu(x) \geq t_1$ and $\mu(y) \geq t_2$. It follows from (2) that

$$\mu(xay) \geq \min \{\mu(x), \mu(y), \frac{1-k}{2}\} \geq \min \{t_1, t_2, \frac{1-k}{2}\}.$$

If $\min \{t_1, t_2\} > \frac{1-k}{2}$, then $\mu(xay) \geq \frac{1-k}{2}$ and so

$$\mu(xay) + \min \{t_1, t_2\} > \frac{1-k}{2} + \frac{1-k}{2} = 1 - k,$$
Proof. For all $\mu$.

If $\min\{t_1, t_2\} \leq \frac{1-k}{2}$, then $\mu(xay) \geq \min\{t_1, t_2\}$, i.e., $(xay, \min\{t_1, t_2\}) \in S$. Thus $(xay, \min\{t_1, t_2\}) \in \cup k \mu$. Therefore $\mu$ is an $(\epsilon, \in \cup k \mu)$-fuzzy bi-ideal of $S$.

If we take $k = 0$ in Theorem 3.4, then we have the following corollary.

3.5. Corollary. [11] Let $\mu$ be a fuzzy set in $S$. Then $\mu$ is an $(\epsilon, \in \cup q)$-fuzzy bi-ideal of $S$ if and only if it satisfies:

1. $(\forall x, y \in S) (\mu(x) \geq \min\{\mu(x), \mu(y), 0.5\}),$
2. $(\forall x, a, y \in S) (\mu(xay) \geq \min\{\mu(x), \mu(y), 0.5\}).$

The following theorem is a generalization of [11, Theorem 3.5].

3.6. Theorem. A non-empty subset $I$ of $S$ is a bi-ideal of $S$ if and only if the characteristic function $\chi_I$ of $I$ is an $(\epsilon, \in \cup q_k)$-fuzzy bi-ideal of $S$.

Proof. Assume that $\chi_I$ is an $(\epsilon, \in \cup q_k)$-fuzzy bi-ideal of $S$. Let $x, y \in I$ and $a \in S$. Then

$$\chi_I(xy) \geq \min\{\chi_I(x), \chi_I(y), \frac{1-k}{2}\} = \frac{1-k}{2}$$

and

$$\chi_I(xay) \geq \min\{\chi_I(x), \chi_I(y), \frac{1-k}{2}\} = \frac{1-k}{2}$$

by Theorem 3.4. It follows that $\chi_I(xy) = 1$ and $\chi_I(xay) = 1$ so that $xy \in I$ and $xay \in I$. Hence $I$ is a bi-ideal of $S$. Since every fuzzy bi-ideal is an $(\epsilon, \in \cup q_k)$-fuzzy bi-ideal, the necessity follows from Proposition 2.2.

3.7. Theorem. Let $I$ be a bi-ideal of $S$. For every $t \in (0, \frac{1-k}{2}]$, there exists an $(\epsilon, \in \cup q_k)$-fuzzy bi-ideal $\mu$ of $S$ such that $U(\mu; t) = I$.

Proof. Let $\mu$ be a fuzzy set in $S$ defined by

$$\mu(x) = \begin{cases} t & \text{if } x \in I, \\ 0 & \text{otherwise} \end{cases}$$

for all $x \in S$, where $t \in (0, \frac{1-k}{2}]$. Obviously, $U(\mu; t) = I$. Assume that

$$\mu(ab) < \min\{\mu(a), \mu(b), \frac{1-k}{2}\}$$

for some $a, b \in S$. Since $\# \text{Im}(\mu) = 2$, it follows that $\min\{\mu(a), \mu(b), \frac{1-k}{2}\} = t$ and $\mu(ab) = 0$. Hence $\mu(a) = t = \mu(b)$, and so $a, b \in I$. Since $I$ is a bi-ideal of $S$, $ab \in I$. Thus $\mu(ab) = t$, which is a contradiction. Therefore

$$\mu(xy) \geq \min\{\mu(x), \mu(y), \frac{1-k}{2}\}$$

for all $x, y \in S$. Similarly we have

$$\mu(xay) \geq \min\{\mu(x), \mu(y), \frac{1-k}{2}\}$$

for all $x, a, y \in S$. Using Theorem 3.4, we know that $\mu$ is an $(\epsilon, \in \cup q_k)$-fuzzy bi-ideal of $S$.

Taking $k = 0$ in Theorem 3.7, we have the following corollary.

3.8. Corollary. Let $I$ be a bi-ideal of $S$. For every $t \in (0, 0.5]$, there exists an $(\epsilon, \in \cup q)$-fuzzy bi-ideal $\mu$ of $S$ such that $U(\mu; t) = I$.

3.9. Theorem. If $\{\mu_i | i \in \Lambda\}$ is a family of $(\epsilon, \in \cup q_k)$-fuzzy bi-ideals of $S$, then $\bigcap_{i \in \Lambda} \mu_i$ is an $(\epsilon, \in \cup q_k)$-fuzzy bi-ideal of $S$, where $\bigcap_{i \in \Lambda} \mu_i(x) = \inf_{i \in \Lambda} \mu_i(x)$. 

Proof. Let \( x, a, y \in S \). Then
\[
\left( \bigcap_{i \in \Lambda} \mu_i \right) (xy) = \inf_{i \in \Lambda} \mu_i (xy) \\
\geq \inf_{i \in \Lambda} \min \{ \mu_i(x), \mu_i(y), \frac{1-k}{2} \} \\
= \min \{ \inf_{i \in \Lambda} \min \{ \mu_i(x), \mu_i(y) \}, \frac{1-k}{2} \} \\
= \min \left\{ \left( \bigcap_{i \in \Lambda} \mu_i \right)(x), \left( \bigcap_{i \in \Lambda} \mu_i \right)(y), \frac{1-k}{2} \right\}.
\]
and
\[
\left( \bigcap_{i \in \Lambda} \mu_i \right)(xay) = \inf_{i \in \Lambda} \mu_i (xay) \\
\geq \inf_{i \in \Lambda} \min \{ \mu_i(x), \mu_i(y), \frac{1-k}{2} \} \\
= \min \{ \inf_{i \in \Lambda} \min \{ \mu_i(x), \mu_i(y) \}, \frac{1-k}{2} \} \\
= \min \left\{ \left( \bigcap_{i \in \Lambda} \mu_i \right)(x), \left( \bigcap_{i \in \Lambda} \mu_i \right)(y), \frac{1-k}{2} \right\}.
\]
Therefore \( \bigcap_{i \in \Lambda} \mu_i \) is an \((\epsilon, \in \vee q_k)\)-fuzzy bi-ideal of \( S \). \( \square \)

Taking \( k = 0 \) in Theorem 3.9 induces the following corollary.

3.10. Corollary. \( [11] \) If \( \{ \mu_i \mid i \in \Lambda \} \) is a family of \((\epsilon, \in \vee q)\)-fuzzy bi-ideals of \( S \), then
\[
\bigcap_{i \in \Lambda} \mu_i \text{ is an } (\epsilon, \in \vee q)\text{-fuzzy bi-ideal of } S, \text{ where } \left( \bigcap_{i \in \Lambda} \mu_i \right)(x) = \inf_{i \in \Lambda} \mu_i(x). \text{ qed}
\]

Let \( S \) be the semigroup given in Example 3.2. Consider the \((\epsilon, \in \vee q_{0.4})\)-fuzzy bi-ideal \( \mu \) of \( S \) in Example 3.2(1) and let \( \nu \) be a fuzzy set in \( S \) defined by
\[
\nu(x) := \begin{cases} 
  0.3 & \text{if } x \in \{a, c\}, \\
  0.1 & \text{if } x \in \{b, d\}.
\end{cases}
\]
Then \( \nu \) is an \((\epsilon, \in \vee q_{0.4})\)-fuzzy bi-ideal of \( S \). Note that
\[
(\mu \cup \nu)(bc) = (\mu \cup \nu)(d) = \max \{ \mu(d), \nu(d) \} = 0.1
\]
\[
< 0.3 = \min \{ (\mu \cup \nu)(b), (\mu \cup \nu)(c), \frac{1-0.4}{2} \}.
\]
This shows that a union of \((\epsilon, \in \vee q_k)\)-fuzzy bi-ideals may not be an \((\epsilon, \in \vee q_k)\)-fuzzy bi-ideal.

Using Theorem 3.4, we provide a condition for an \((\epsilon, \in \vee q_k)\)-fuzzy bi-ideal to be a fuzzy bi-ideal.

3.11. Theorem. Let \( \mu \) be an \((\epsilon, \in \vee q_k)\)-fuzzy bi-ideal of \( S \) such that \( \mu(x) < \frac{1-k}{2} \) for all \( x \in S \). Then \( \mu \) is a fuzzy ideal of \( S \).

Proof. Straightforward by using Theorem 3.4. \( \square \)

Taking \( k = 0 \) in Theorem 3.11, we have the following corollary.

3.12. Corollary. Let \( \mu \) be an \((\epsilon, \in \vee q)\)-fuzzy bi-ideal of \( S \) such that \( \mu(x) < 0.5 \) for all \( x \in S \). Then \( \mu \) is a fuzzy ideal of \( S \).

3.13. Theorem. For a fuzzy set \( \mu \) in \( S \), the following are equivalent:
(1) \( \mu \) is an \((\epsilon, \in \vee q_k)\)-fuzzy bi-ideal of \( S \).
(2) \( \forall t \in \left( 0, \frac{1-k}{2} \right) \), \( (\bigcup_{\mu} t) \neq \emptyset \implies U(\mu; t) \) is a bi-ideal of \( S \).
Proof. (1) $\implies$ (2) Assume that $\mu$ is an $(\in, \in \lor q_k)$-fuzzy bi-ideal of $S$. Let $t \in (0, \frac{1-k}{2})$ be such that $U(\mu; t) \neq \emptyset$. Let $x, y \in U(\mu; t)$. Then $\mu(x) \geq t$ and $\mu(y) \geq t$, i.e., $(x, t) \in \mu$ and $(y, t) \in \mu$. It follows from Definition 3.1(i) that $(xy, t) \in \lor q_k \mu$, i.e., $(xy, t) \in \mu$ or $(xy, t) q_k \mu$. If $(xy, t) \in \mu$, then $\mu(xy) \geq t$ and so $xy \in U(\mu; t)$. If $(xy, t) q_k \mu$, then $\mu(xy) + t > 1 - k$. Thus if $\mu(xy) < t$, then

$$1 - k < \mu(xy) + t < t + t \leq \frac{1-k}{2} + \frac{1-k}{2} = 1 - k.$$ 

This is a contradiction. Hence $\mu(xy) \geq t$, i.e., $xy \in U(\mu; t)$. Therefore $U(\mu; t)$ is a subsemigroup of $S$. Let $x, y \in U(\mu; t)$ and $a \in S$. Then $\mu(x) \geq t$ and $\mu(y) \geq t$, i.e., $(x, t) \in \mu$ and $(y, t) \in \mu$. It follows from Definition 3.1(ii) that $(xay, t) \in \lor q_k \mu$, that is, $(xay, t) \in \mu$ or $(xay, t) q_k \mu$. If $(xay, t) \in \mu$, then $\mu(xay) \geq t$ and thus $xay \in U(\mu; t)$. If $(xay, t) q_k \mu$, then $\mu(xay) + t > 1 - k$. Thus if $\mu(xay) < t$, then

$$1 - k < \mu(xay) + t < t + t \leq \frac{1-k}{2} + \frac{1-k}{2} = 1 - k.$$ 

This is impossible, and so $\mu(xay) \geq t$. Hence $xay \in U(\mu; t)$. Consequently, $U(\mu; t)$ is a bi-ideal of $S$.

(2) $\implies$ (1) Assume that $\mu$ does not satisfy Theorem 3.4(1). Then $\mu(xy) < s \leq \min \{\mu(x), \mu(y), \frac{1-k}{2}\}$ for some $x, y \in S$ and $s \in (0, 1]$. Clearly, $s \in (0, \frac{1-k}{2}]$, $(x, s) \in \mu$ and $(y, s) \in \mu$, but $(xy, s) \not\in \mu$. Moreover,

$$\mu(xy) + s < s + s \leq \frac{1-k}{2} + \frac{1-k}{2} = 1 - k,$$

and so $(xy, s) \not\in \lor q_k \mu$. Hence $(xy, s) \not\in \lor q_k \mu$, a contradiction. Thus Theorem 3.4(1) is valid.

Now, suppose that $\mu$ does not satisfy Theorem 3.4(2). Then $\mu(xay) < r \leq \min \{\mu(x), \mu(y), \frac{1-k}{2}\}$ for some $x, a, y \in S$ and $r \in (0, 1]$. It follows that $(x, r) \in \mu$, $(y, r) \in \mu$, and $r \in (0, \frac{1-k}{2}]$, but $(xay, r) \not\in \mu$. Also,

$$\mu(xay) + r < r + r \leq \frac{1-k}{2} + \frac{1-k}{2} = 1 - k,$$

which implies that $(xay, r) \not\in \lor q_k \mu$. Hence $(xay, r) \not\in \lor q_k \mu$, a contradiction. Therefore $\mu$ satisfies Theorem 3.4(2). Using Theorem 3.4, we know that $\mu$ is an $(\in, \in \lor q_k)$-fuzzy bi-ideal of $S$.

If we take $k = 0$ in Theorem 3.13, then we have the following corollary.

3.14. Corollary. [11] For a fuzzy set $\mu$ in $S$, the following are equivalent:

(1) $\mu$ is an $(\in, \in \lor q_k)$-fuzzy bi-ideal of $S$.

(2) $\forall t \in (0, 0.5)], (U(\mu; t) \not= \emptyset \implies U(\mu; t)$ is a bi-ideal of $S$).

3.15. Theorem. Let $S$ be a semigroup. If $0 \leq k < r < 1$, then every $(\in, \in \lor q_k)$-fuzzy bi-ideal of $S$ is an $(\in, \in \lor q_r)$-fuzzy bi-ideal of $S$.

Proof. Straightforward.

The following example shows that if $0 \leq k < r < 1$, then an $(\in, \in \lor q_r)$-fuzzy bi-ideal of $S$ may not be an $(\in, \in \lor q_k)$-fuzzy bi-ideal of $S$.

3.16. Example. Consider a semigroup $S = \{a, b, c, d, e\}$ with a multiplication table given by Table 3.
Define a fuzzy set \( \mu \) in \( S \) by
\[
\mu = \left( \begin{array}{ccccc}
0.7 & 0.6 & 0.3 & 0.4 & 0.5 \\
\end{array} \right).
\]
Then \( \mu \) is an \((\varepsilon, \in \lor q_{0.2})\)-fuzzy bi-ideal of \( S \). If \( t \in (0.4, \frac{1-0.14}{2}) = (0.4, 0.43) \), then \( U(\mu; t) = \{a, b, c\} \) is not a bi-ideal of \( S \). Hence \( \mu \) is not an \((\varepsilon, \in \lor q_{0.14})\)-fuzzy bi-ideal of \( S \) by Theorem 3.13.

3.17. Theorem. Let \( \mu \) be an \((\varepsilon, \in \lor q_{k})\)-fuzzy bi-ideal of \( S \). If \( S \) is completely regular, then \( \min \{\mu(a), \frac{1-k}{2}\} = \min \{\mu(a^2), \frac{1-k}{2}\} \) for all \( a \in S \).

Proof. Let \( a \in S \). Then there exists \( x \in S \) such that \( a = a^2 x a^2 \). Hence
\[
\min \{\mu(a), \frac{1-k}{2}\} = \min \{\mu(a^2 x a^2), \frac{1-k}{2}\} 
\leq \min \{\mu(a^2), \mu(x), \frac{1-k}{2}\} 
= \min \{\mu(a^2), \frac{1-k}{2}\} 
\leq \min \{\mu(a), \mu(a), \frac{1-k}{2}\} 
= \min \{\mu(a), \frac{1-k}{2}\},
\]
which implies that \( \min \{\mu(a), \frac{1-k}{2}\} = \min \{\mu(a^2), \frac{1-k}{2}\} \) for all \( a \in S \). \( \square \)

3.18. Corollary. Let \( \mu \) be an \((\varepsilon, \in \lor q_{k})\)-fuzzy bi-ideal of \( S \). If \( S \) is completely regular and \( \mu(a) < \frac{1-k}{2} \) for all \( a \in S \), then \( \mu(a) = \mu(a^2) \) for all \( a \in S \). \( \square \)

3.19. Corollary. Let \( \mu \) be an \((\varepsilon, \in \lor q_{k})\)-fuzzy bi-ideal of \( S \). If \( S \) is completely regular, then \( \min \{\mu(a), 0.5\} = \min \{\mu(a^2), 0.5\} \) for all \( a \in S \). \( \square \)

For any fuzzy set \( \mu \) in \( S \) and \( t \in (0, 1] \), we define
\[
Q(\mu; t) := \{x \in X \mid (x; t) q_{\mu}\}, \quad Q^k(\mu; t) := \{x \in X \mid (x; t) q_{\mu}\}.
\]

3.20. Theorem. If \( \mu \) is an \((\varepsilon, \in \lor q_{k})\)-fuzzy bi-ideal of \( S \), then
\[
(\forall t \in (\frac{1-k}{k}, 1]) \quad (Q^k(\mu; t) \neq \emptyset) \quad \Rightarrow \quad Q^k(\mu; t) \text{ is a bi-ideal of } S
\]
Proof. Let \( t \in (\frac{1-k}{k}, 1] \) be such that \( Q^k(\mu; t) \neq \emptyset \). Let \( x, a, y \in S \). If \( x, y \in Q^k(\mu; t) \), then \( (x, t) q_{\mu} \) and \((y, t) q_{\mu} \), i.e., \( \mu(x) + t + k > 1 \) and \( \mu(y) + t + k > 1 \). It follows from Theorem 3.4 that
\[
\mu(xy) \geq \min \{\mu(x), \mu(y), \frac{1-k}{2}\} \geq 1 - t - k, \\
\mu(xay) \geq \min \{\mu(x), \mu(y), \frac{1-k}{2}\} \geq 1 - t - k
\]
so that \((xy, t) q_{\mu} \) and \((xay, t) q_{\mu} \). Hence \( xy, xay \in Q^k(\mu; t) \), and therefore \( Q^k(\mu; t) \) is a bi-ideal of \( S \). \( \square \)

Table 3. Multiplication table

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
</tr>
</thead>
<tbody>
<tr>
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<td>a</td>
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<td>b</td>
<td>a</td>
<td>b</td>
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</tr>
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<td>c</td>
<td>a</td>
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<td>c</td>
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<td>a</td>
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<td>c</td>
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</tbody>
</table>
3.21. Corollary. Let $\mu$ be an $(\in, \in \lor q)$-fuzzy bi-ideal of $S$. If $k < r < 1$, then
\[
(\forall t \in \left(\frac{2r-k}{2}, 1\right]) \ (Q^r(\mu; t) \neq \emptyset \implies Q^r(\mu; t) \text{ is a bi-ideal of } S)
\]

Proof. Straightforward by Theorems 3.15 and 3.20. □

If we take $k = 0$ in Theorem 3.20, then we have the following corollary.

3.22. Corollary. If $\mu$ is an $(\in, \in \lor q)$-fuzzy bi-ideal of $S$, then
\[
(\forall t \in (0.5, 1]) \ (Q(\mu; t) \neq \emptyset \implies Q(\mu; t) \text{ is a bi-ideal of } S)
\]

References