THE PRODUCT OF GENERALIZED SUPERDERIVATIONS ON A PRIME SUPERALGEBRA

HE YUAN AND YU WANG

Abstract. In the paper, we extend the definition of generalized derivations to superalgebras and prove that a generalized superderivation $g$ on a prime superalgebra $A$ is represented as $g(x) = ax + d(x)$ for all $x \in A$, where $a$ is an element of $Q_{mr}$ (the maximal right ring of quotients of $A$) and $d$ is a superderivation on $A$. Using the result we study two generalized superderivations when their product is also a generalized superderivation on a prime superalgebra $A$.

Key words and phrases. Generalized derivation, Prime ring, Extended centroid, Generalized superderivation, Superalgebra.

1. Introduction

Let $R$ be a prime ring. According to Hvala [9] an additive mapping $g : R \to R$ is said to be a generalized derivation of $R$ if there exists a derivation $\delta$ of $R$ such that $g(xy) = g(x)y + x\delta(y)$ for all $x, y \in R$. In [14] Lee proved that every generalized derivation of $A$ can be uniquely extended to $Q_{mr}$ and there exists an element $a \in Q_{mr}$ such that $g(x) = ax + \delta(x)$ for all $x \in R$.

The study of the product of derivations in prime rings was initiated by Posner [18]. He proved that the product of two nonzero derivations can not be a derivation on a prime ring of characteristic not 2. Later a number of authors studied the problem in several ways (see [2], [4], [5], [9], [10], [12], [13], and [15]). Hvala [9] studied two generalized derivations $f_1, f_2$ when the product is also a generalized derivation on a prime ring $R$ of characteristic not 2 in 1998. In 2001 Lee [13] gave a description of Hvala’s Theorem without the assumption of char$R \neq 2$. In 2004 Fošner [5] extended Posner’s Theorem to prime superalgebras. Superalgebras first appeared in physics, in the Theory of Supersymmetry, to create an algebraic structure representing the behavior of the subatomic particles known as bosons and fermions ([11]). Recently there has been a considerable authors who are interested in superalgebras. They extended many results of rings to superalgebras (see [3], [5], [6], [7], [8], [11], [16], [17] and [19]).

In Section 3, we will extend the definition of generalized derivations to superalgebras and prove that every generalized superderivation of a prime superalgebra $A$ can be extended to $Q_{mr}$ (the maximal right ring of quotients of $A$). Further, we will prove that a generalized superderivation of a prime superalgebra is a sum of a left multiplication mapping and a superderivation. Using the result we will
study two generalized superderivations when their product is also a generalized superderivation on a prime superalgebra. As a result, Fošner’s theorem [5, Theorem 4.1] is the special case of the main theorem of the paper.

2. preliminaries

Let \( \Phi \) be a commutative ring with \( \frac{1}{2} \in \Phi \). An associative algebra \( A \) over \( \Phi \) is said to be an associative superalgebra if there exist two \( \Phi \)-submodules \( A_0 \) and \( A_1 \) of \( A \) such that \( A = A_0 \oplus A_1 \) and \( A_i A_j \subseteq A_{i+j} \), \( i, j \in \mathbb{Z}_2 \). A superalgebra is called trivial if \( A_1 = 0 \). The elements of \( A_i \) are homogeneous of degree \( i \) and we write \( |a_i| = i \) for all \( a_i \in A_i \). We define \([a,b]_s = ab - (-1)^{|a||b|}ba\) for all \( a, b \in A_0 \cup A_1 \). Thus, \([a,b]_s = [a_0,b_0]_s + [a_1,b_0]_s + [a_0,b_1]_s + [a_1,b_1]_s\), where \( a = a_0 + a_1, b = b_0 + b_1 \) and \( a_i, b_i \in A_i \) for \( i = 0, 1 \). It follows that \([a,b]_s = [a,b]\) if one of the elements \( a \) and \( b \) is homogeneous of degree 0. Let \( k \in \{0, 1\} \). A superderivation of degree \( k \) is actually a \( \Phi \)-linear mapping \( d_k : A \rightarrow A \) which satisfies \( d_k(A_i) \subseteq A_{k+i} \) for \( i \in \mathbb{Z}_2 \) and \( d_k(ab) = d_k(a)b + (-1)^{|a|}a d_k(b) \) for all \( a, b \in A_0 \cup A_1 \). If \( d = d_0 + d_1 \), then \( d \) is a superderivation on \( A \). For example, for \( a = a_0 + a_1 \in A \) the mapping \( ad_s(a)(x) = [a,x]_s = [a_0,x]_s + [a_1,x]_s \) is a superderivation, which is called the inner superderivation induced by \( a \). For a superalgebra \( A \), we define \( \sigma : A \rightarrow A \) by \((a_0 + a_1)^\sigma = a_0 - a_1 \), then \( \sigma \) is an automorphism of \( A \) such that \( \sigma^2 = 1 \). On the other hand, for an algebra \( A \), if there exists an automorphism \( \sigma \) of \( A \) such that \( \sigma^2 = 1 \), then \( A \) becomes a superalgebra \( A = A_0 \oplus A_1 \), where \( A_i = \{x \in A|x^\sigma = (-1)^ix\}, i = 0, 1 \). Clearly a superderivation \( d \) of degree 1 is a \( \sigma \)-derivation, i.e., it satisfies \( d(ab) = d(a)b + a^\sigma d(b) \) for all \( a, b \in A \). A superalgebra \( A \) is called a prime superalgebra if and only if \( aAb = 0 \) implies \( a = 0 \) or \( b = 0 \), where at least one of the elements \( a \) and \( b \) is homogeneous. The knowledge of superalgebras refers to [3], [5], [6], [7], [8], [16], [17] and [19].

In [17] Montaner obtained that a prime superalgebra \( A \) is not necessarily a prime algebra but a semiprime algebra. Hence one can define the maximal right ring of quotients \( Q_{mr} \) of \( A \), and the useful properties of \( Q_{mr} \) can be found in [1]. By [1, proposition 2.5.3] \( \sigma \) can be uniquely extended to \( Q_{mr} \) such that \( \sigma^2 = 1 \). Therefore \( Q_{mr} \) is also a superalgebra. Further, we can get that \( Q_{mr} \) is a prime superalgebra.

3. the product of generalized superderivations

Firstly, we extend the definition of generalized derivations to superalgebras.

Definition. Let \( A \) be a superalgebra. For \( i \in \{0, 1\} \), a \( \Phi \)-linear mapping \( g_i : A \rightarrow A \) is called a generalized superderivation of degree \( i \) if \( g_i(A_j) \subseteq A_{i+j} \), \( j \in \mathbb{Z}_2 \), and \( g_i(xy) = g_i(x)y + (-1)^{|x|}xd_i(y) \) for all \( x, y \in A_0 \cup A_1 \), where \( d_i \) is a superderivation of degree \( i \) on \( A \). If \( g = g_0 + g_1 \), then \( g \) is called a generalized superderivation on \( A \).

Let \( A \) be a prime superalgebra and \( Q = Q_{mr} \) be the maximal right ring of quotients of \( A \). Next, we prove that a generalized superderivation of a prime superalgebra is a sum of a left multiplication mapping and a superderivation. By [20, proposition 2] we have every \( \sigma \)-derivation \( d \) of a semiprime ring \( A \) can be uniquely extended to a \( \sigma \)-derivation of \( Q \).

Theorem 3.1. Let \( A \) be a prime superalgebra and \( g : A \rightarrow A \) be a generalized superderivation. Then \( g \) can be extended to \( Q \) and there exist an element \( a \in Q \)
and a superderivation $d$ of $A$ such that $g(x) = ax + d(x)$ for all $x \in A$, where both $a$ and $d$ are determined by $g$ uniquely.

**Proof.** To prove that the generalized superderivation $g$ on a prime superalgebra $A$ can be extended to $Q$, it suffices to prove that $g_0$ and $g_1$ can be extended to $Q$, respectively. The generalized superderivation of degree 1 $g_1$ is represented as $g_1(xy) = g_1(x)y + x^*d_1(y)$ for all $x,y \in A$, where $d_1$ is a superderivation of degree 1 on $A$. Note that $d_1(xy) = d_1(x)y + x^*d_1(y)$. So combining the two equations we have $(g_1 - d_1)(xy) = (g_1 - d_1)(x)y$. Let $g_1 - d_1 = f$. Clearly $f$ is a right $A$-module mapping. Then there exists $a_1 \in Q$ such that $f(x) = a_1x$. So $g_1(x) = a_1x + d_1(x)$ for all $x \in A$. Since $d_1$ can be extended to $Q$, then it follows that $g_1$ can be extended to $Q$. It is easy to prove that $g_0(x) = a_0x + d_0(x)$ and $g_0$ can be extended to $Q$ similarly, where $a_0$ is an element of $Q$ and $d_0$ is a superderivation of degree 0 on $A$. So $g$ can be extended to $Q$. Clearly $a_i \in Q_1$, $i \in \{0, 1\}$. Let $a = a_0 + a_1$ and $d = d_0 + d_1$. Then $g(x) = g_0(x) + g_1(x) = a_0x + d_0(x) + a_1x + d_1(x) = ax + d(x)$ for all $x \in A$, where $a$ is an element of $Q$ and $d$ is a superderivation of $A$.

Now we claim both $a$ and $d$ are determined by $g$ uniquely. It suffices to prove that $a = 0$ and $d = 0$ when $g = 0$. Since $g = 0$, we have $g_0 = g_1 = 0$. By $g_1 = 0$, we obtain $0 = g_1(ryr) = a_1yry + d_1(ry) = a_1y + r + y^*d_1(r) = g_1(yr) + y^*d_1(r) = g^*d_1(r)$ for all $y,r \in A$. Then $A^*d_1(A) = 0$. So $Ad_1(A) = 0$. Clearly $d_1(A) = 0$. Since $g_1(A) = 0$, it follows that $a_1A = 0$. Hence $a_1 = 0$. Similarly we can prove the case when $g_0 = 0$. So $a = 0$ and $d = 0$.

Next, we give two results which are used in the proof of the main result.

**Lemma 3.2.** Let $A$ be a prime superalgebra. If $A$ satisfies

1. $(a_0, x) + d_0(z) = 0$ for all $x, y, z \in A$,

where $a_0$, $b_0 \in Q_0$ and both $d_0$ and $k_0$ are superderivations of degree 0 on $A$. Then one of the following cases is true:

   (i) There exists $\mu \neq 0 \in C_0$ such that $\mu k_0(x) + d_0(x) = 0$;

   (ii) $[a_0, x] + d_0(x) = 0$;

   (iii) $[b_0, x] + k_0(x) = 0$

for all $x \in A$.

**Proof.** Let $d_0 = k_0 = 0$. Clearly there exists $0 \neq \mu \in C_0$ such that $\mu k_0(x) + d_0(x) = 0$. Hence (i) is true.

Next we assume either $d_0 \neq 0$ or $k_0 \neq 0$. By [5, Theorem 3.3] there exist $\lambda_1$ and $\lambda_2$ not all zero such that $\lambda_1([a_0, x] + d_0(x)) + \lambda_2([b_0, x] + k_0(x)) = 0$. Let $\lambda_1 = \lambda_{10} + \lambda_{11}$ and $\lambda_2 = \lambda_{20} + \lambda_{21}$. Then $\lambda_{10}([a_0, x] + d_0(x)) + \lambda_{11}([a_0, x] + d_0(x)) + \lambda_{20}([b_0, x] + k_0(x)) + \lambda_{21}([b_0, x] + k_0(x)) = 0$ for all $x \in A$, where $\lambda_{10}, \lambda_{20} \in C_0$, $\lambda_{11}, \lambda_{21} \in C_1$. By $A_0 \cap A_1 = 0$, we have

2. $\lambda_{11}([a_0, x] + d_0(x)) + \lambda_{21}([b_0, x] + k_0(x)) = 0$ for all $x \in A_0$,

3. $\lambda_{11}([a_0, x] + d_0(x)) + \lambda_{21}([b_0, x] + k_0(x)) = 0$ for all $x \in A_1$.

Using (2) and (3) we obtain

4. $\lambda_{11}([a_0, x] + d_0(x)) + \lambda_{21}([b_0, x] + k_0(x)) = 0$ for all $x \in A$.

We proceed by dividing three cases. Only one of $\lambda_{11}$ and $\lambda_{21}$ is nonzero. If $\lambda_{21} \neq 0$, then $[b_0, x] + k_0(x) = 0$. If $\lambda_{11} \neq 0$, then $[a_0, x] + d_0(x) = 0$. Hence either (ii) or (iii) is true.
Both $\lambda_{11} \neq 0$ and $\lambda_{21} \neq 0$. By (4) and [5, Lemma 3.1] we arrive at $[a_0, x] + d_0(x) = \lambda([b_0, x] + k_0(x))$, where $\lambda = -\lambda_{11}^{-1}\lambda_{21} \neq 0$. Using (1) we get $\lambda([b_0, x] + k_0(x))y [k_0(z) + ([b_0, x] + k_0(x))y d_0(z) = 0$. That is, $([b_0, x] + k_0(x))y (k_0(z) + d_0(z)) = 0$. If there exists $z \in A$ such that $\lambda k_0(z) + d_0(z) \neq 0$, then $[b_0, x] + k_0(x) = 0$ for all $x \in A_0 \cup A_1$. It follows that $[b_0, x] + k_0(x) = 0$ for all $x \in A$. Hence either (i) or (iii) is true. Similarly, when $\rho([a_0, x] + d_0(x)) = [b_0, x] + k_0(x)$, where $\rho = -\lambda_{11}^{-1}\lambda_{11} \neq 0$, we have either (i) or (ii) is true by using (1) again.

When $\lambda_{11} = \lambda_{21} = 0$, i.e., $\lambda_1, \lambda_2 \in C_0$. If one of $\lambda_1$ and $\lambda_2$ is zero, then either (ii) or (iii) is true. If both $\lambda_1$ and $\lambda_2$ are nonzero, the proof is similar to the above paragraph.

Similar to the proof of Lemma 3.2, we can get the following result.

**Lemma 3.3.** Let $A$ be a prime superalgebra. If $A$ satisfies

$$([a_1, x] + d_1(x))y k_1(z) - ([b_1, x] + k_1(x))y d_1(z) = 0$$

for all $x, y, z \in A$,

where $a_1, b_1 \in Q_1$ and both $d_1$ and $k_1$ are superderivations of degree 1 on $A$. Then one of the following cases is true:

(i) There exists $0 \neq \nu \in C_0$ such that $\nu k_1(x) + d_1(x) = 0$;

(ii) $[a_1, x] + d_1(x) = 0$;

(iii) $[b_1, x] + k_1(x) = 0$

for all $x \in A$.

Now, we are in a position to give the main result of this paper.

**Theorem 3.4.** Let $A$ be a prime superalgebra and let $f = a + d$ and $g = b + k$ be two nonzero generalized superderivations on $A$, where $a, b \in Q$ and both $d$ and $k$ are superderivations on $A$. If $fg$ is also a generalized superderivation on $A$. Then one of the following cases is true:

(i) There exists $0 \neq \omega \in C_0$ such that $\omega k_j(x) + d_j(x) = 0$;

(ii) $[a_i, x] + d_i(x) = 0$;

(iii) $[b_i, x] + k_i(x) = 0$

for all $x \in A$, where $i, j \in \{0, 1\}$, $a_i, b_i \in Q_i$ and both $d_i$ and $k_i$ are superderivations of degree $i$ on $A$, as well as $d_j$ and $k_j$.

**Proof.** According to Theorem 3.1 we assume $h(x) = fg(x) = cx + l(x)$ for all $x \in A$, where $c \in Q$ and $l$ is a superderivation on $A$, then

$$fg(x) = a(bx + k(x)) + d(bx + k(x))$$

$$= abx + ak(x) + d_0(b)x + bd_0(x) + d_1(b)x + b^2 d_1(x) + d_0k(x) + d_1k(x).$$

Hence

$$c = ab + d_0(b) + d_1(b) = ab + d(b),$$

$$l(x) = ak(x) + bd_0(x) + b^2 d_1(x) + d_0k(x) + d_1k(x),$$

$$l_0(x) = a_0k_0(x) + a_1k_1(x) + b_0d_0(x) - b_1d_1(x) + d_0k_0(x) + d_1k_1(x),$$

$$l_1(x) = a_1k_0(x) + a_0k_1(x) + b_1d_0(x) + b_0d_1(x) + d_0k_1(x) + d_1k_0(x).$$
On the one hand we get

\[ l_0(xy) = a_0 k_0(xy) + a_1 k_1(xy) + b_0 d_0(xy) - b_1 d_1(xy) + d_0 k_0(xy) + d_1 k_1(xy) \]
\[ = a_0 k_0(x) y + a_0 x d_0(y) + a_1 x^\sigma k_1(y) \]
\[ + b_0 d_0(x) y + b_0 x d_0(y) - b_1 x^\sigma d_1(y) \]
\[ + d_0 k_0(x) y + k_0(x) d_0(y) + d_0(x) k_0(y) + x d_0 k_0(y) \]
\[ + d_1 k_1(x) y + k_1(x)^\sigma d_1(y) + d_1(x^\sigma) k_1(y) + x d_1 k_1(y) \]

and on the other hand we get

\[ l_0(xy) = a_0 k_0(x) y + a_1 k_1(x) y + b_0 d_0(x) y - b_1 d_1(x) y + d_0 k_0(x) y + d_1 k_1(x) y \]
\[ + x [a_0 k_0(y) + a_1 k_1(y) + b_0 d_0(y) - b_1 d_1(y) + d_0 k_0(y) + d_1 k_1(y)]. \]

Combining the two equations we have

\[ 0 = [a_0, x] k_0(y) + a_1 x^\sigma k_1(y) - x a_1 k_1(y) + [b_0, x] d_0(y) - b_1 x^\sigma d_1(y) \]
\[ + x b_1 d_1(y) + k_0(x) d_0(y) + d_0(x) k_0(y) + k_1(x)^\sigma d_1(y) - d_1(x)^\sigma k_1(y). \]  

(5)

In particular, replacing \( y \) by \( y z \) in (5) we get

\[ 0 = [a_0, x] k_0(y z) + a_1 x^\sigma k_1(y z) - x a_1 k_1(y z) + [b_0, x] d_0(y z) - b_1 x^\sigma d_1(y z) \]
\[ + x b_1 d_1(y z) + k_0(x) d_0(y z) + d_0(x) k_0(y z) + k_1(x)^\sigma d_1(y z) - d_1(x)^\sigma k_1(y z). \]

Extending the identity above we arrive at

\[ 0 = [a_0, x] k_0(y z) + a_1 x^\sigma k_1(y z) - x a_1 y^\sigma k_1(z) + a_1 x^\sigma y^\sigma k_1(z) \]
\[ - x a_1 k_1(y) z - x a_1 y^\sigma k_1(z) + [b_0, x] d_0(y z) - b_1 x^\sigma d_1(z) \]
\[ - b_1 x^\sigma d_1(z) - b_1 x^\sigma y^\sigma d_1(z) + x b_1 d_1(z) + x b_1 y^\sigma d_1(z) \]
\[ + k_0(x) d_0(y z) + k_0(x) y d_0(z) + d_0(x) k_0(y z) + d_0(x) y k_0(z) \]
\[ + k_1(x)^\sigma d_1(z) + k_1(x)^\sigma y^\sigma d_1(z) - d_1(x)^\sigma k_1(z) - d_1(x)^\sigma y^\sigma k_1(z). \]

Using (5) we have

\[ 0 = [a_0, x] y k_0(z) + a_1 x^\sigma y^\sigma k_1(z) - x a_1 y^\sigma k_1(z) + [b_0, x] y d_0(z) - b_1 x^\sigma y^\sigma d_1(z) \]
\[ + x b_1 y^\sigma d_1(z) + k_0(x) y d_0(z) + d_0(x) y k_0(z) + k_1(x)^\sigma y^\sigma d_1(z) - d_1(x)^\sigma y^\sigma k_1(z). \]

[5, Corollary 3.6] gives

\[ p_i = [a_0, x_1] y k_0(z_j) + [b_0, x_1] y d_0(z_j) + k_0(x_1) y d_0(z_j) + d_0(x_1) y k_0(z_j) = 0, \]
\[ q_i = a_1 x_1^\sigma y k_1(z_j) - x a_1 y k_1(z_j) - b_1 x_1^\sigma y d_1(z_j) + x b_1 y d_1(z_j), \]
\[ + k_1(x_1)^\sigma y d_1(z_j) - d_1(x_1)^\sigma y k_1(z_j) = 0. \]

for all \( x_i \in A_i, y \in A, z_j \in A_j, i, j \in \{0, 1\} \). Therefore

\[ p_00 + p_{01} + p_{10} + p_{11} = [a_0, x] y k_0(z_0) + [b_0, x] y d_0(z_0) + k_0(x) y d_0(z_0) \]
\[ + d_0(x) y k_0(z_0) = 0, \]
\[ q_00 + q_{01} + q_{10} + q_{11} = a_1 x^\sigma y k_1(z_0) - x a_1 y k_1(z_0) - b_1 x^\sigma y d_1(z_0) + x b_1 y d_1(z_0) \]
\[ + k_1(x)^\sigma y d_1(z_0) - d_1(x)^\sigma y k_1(z_0) = 0. \]

According to (8) and Lemma 3.2 we see that either (i) or (ii) or (iii) is true.

By (9) we get

\[ [a_1, x_0] y k_1(z) - [b_1, x_0] y d_1(z) - k_1(x_0) y d_1(z) \]
\[ + d_1(x_0) y k_1(z) = 0 \]

for all \( x_0 \in A_0, y, z \in A, \)
Combining the identities above we give

\[-[a_1, x_1]_s y k_1(z) + [b_1, x_1]_s y d_1(z) + k_1(x_1) y d_1(z) - d_1(x_1) y k_1(z) = 0 \text{ for all } x_1 \in A_1, y, z \in A.\]

By Lemma 3.3 we have that either (i) or (ii) or (iii) is true. Similarly, using the same way to \( l_1(xy) \) we have

\[[a_0, x] y k_1(z) + [b_0, x] y d_1(z) + k_0(x) y d_1(z) + d_0(x) y k_1(z) = 0,\]

\[-x^\sigma a_1 y k_0(z) - x^\lambda b_1 y d_0(z) + k_1(x) y d_0(z) + d_1(x) y k_0(z) = 0\]

and either (i) or (ii) or (iii) is true. \( \square \)

In particular, taking \( a = b = 0 \) in Theorem 3.4 we obtain

**Corollary 3.5.** ([5, Theorem 4.1]) Let \( A \) be a prime associative superalgebra and let \( d = d_0 + d_1 \) and \( k = k_0 + k_1 \) be nonzero superderivations on \( A \). Then \( dk \) is a superderivation if and only if \( d_0 = k_0 = 0 \) and \( k_1 = \lambda_0 d_1 \) for some nonzero \( \lambda_0 \in C_0 \).

**Proof.** We assume that both \( d_0 \) and \( k_0 \) are nonzero. Since \( d \) and \( k \) are nonzero superderivations and \( dk \) is also a superderivation of \( A \), then there exists \( 0 \neq \mu \in C_0 \) such that \( k_0(x) = \mu d_0(x) \) by Theorem 3.4. We have \( 2 \mu d_0(x) y d_0(z) = 0 \) by taking \( z = x \) in (8), that is, \( d_0(x) A d_0(x) = 0 \). Since \( A \) is a semiprime algebra, then \( d_0(x) = 0 \). But it contradicts \( d_0 \neq 0 \). We set \( d_0 = 0 \). Then \( d_1 \neq 0 \). When \( k_1 \neq 0 \), there exists \( 0 \neq \lambda_0 \in C_0 \) such that \( k_1(x) = \lambda_0 d_1(x) \) and \( k_0(x) = d_0(x) = 0 \) by Theorem 3.4. When \( k_1 = 0 \) and \( k_0 \neq 0 \), we have \( d_1(x) = 0 \) by (10). It contradicts that \( d \) is a nonzero superderivation. So \( d_0 = k_0 = 0 \) and \( k_1 = \lambda_0 d_1 \) for some nonzero \( \lambda_0 \in C_0 \) when \( dk \) is a superderivation. It is easy to prove that \( dk \) is a superderivation when \( d_0 = k_0 = 0 \) and \( k_1 = \lambda_0 d_1 \) for some nonzero \( \lambda_0 \in C_0 \). \( \square \)

**References**


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Department of Mathematics, Jilin University, Changchun 130012, P. R. of China and Department of Mathematics, Jilin Normal University, Siping 136000, P. R. of China
E-mail address: yuanhe1983@126.com

Department of Mathematics, Shanghai Normal University, Shanghai 200234, P. R. of China
E-mail address: ywang2004@126.com