Generalized Class of Estimators for Population Median Using Auxiliary Information

Prayas Sharma∗ and Rajesh Singh†‡

Abstract
This article suggests a generalized class of estimators for population median of the study variable in simple random sampling using information on an auxiliary variable. Asymptotic expressions of bias and mean square error of the proposed class of estimators have been obtained. Asymptotic optimum estimator has been investigated along with its approximate mean square error. It has been shown that proposed generalized class of estimators are more efficient than estimators considered by [26], [5],[6], [22], [1], [19] and other estimators. In addition theoretical findings are supported by an empirical study based on two populations to show the superiority of the constructed estimators over others.

Keywords: Auxiliary variable, Simple random sampling, Bias, Mean Square Error.

2000 AMS Classification: 62D05

1. Introduction
In the sampling literature, Statisticians are often interested in dealing with variables that have highly skewed distributions such as consumptions and incomes. In such situations median is considered the more appropriate measure of location than mean. It has been well recognised that use of auxiliary information results in efficient estimators of population parameters. Initially, estimation of median without auxiliary variable analyzed, after that some authors including [6], [9], [24] and [7] used the auxiliary information in median estimation. [6], proposed the problem of estimating the population median \( M_y \) of study variable \( Y \) using the auxiliary variable \( X \) for the unites in the sample and its median \( M_x \) for the whole population. Some other important references in this context are [3], [11], [8], [15], [2], [4], [21, 20],[25] and [19].

Let \( Y_i \) and \( X_i \) (i = 1,2,..., N) be the values of the population unites for the study variable \( Y \) and auxiliary variable \( X \) respectively .Further suppose that \( y_i \) and \( x_i \) (i=1,2,...,n) be the values of the unites including in the sample say, \( s_n \) of size \( n \) drawn by simple random sampling without replacement scheme. [6] suggested a

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ratio estimator for population median \( M_y \) of the study variable \( Y \), assuming population median of auxiliary variable \( X \), \( M_x \) is known, given as

\[
\hat{M}_r = \frac{\hat{M}_y}{\hat{M}_x} \tag{1.1}
\]

where \( \hat{M}_y \) (due to [5]) and are the sample estimators of \( M_y \) and \( M_x \) respectively. Suppose that \( Y_1, Y_2, \ldots, Y_n \) are the \( y \) values of sample unites in ascending order. Further, suppose \( t \) be an integer satisfying and \( p = t/n \) be the proportion of \( y \) values in the sample that are less than or equal to the median value \( M_y \), an unknown population parameter. If \( Q_y(t) \) denote the \( t \)-quantile of \( Y \) then \( \hat{M}_y = Q_y(0.5) \).

[6] defined a matrix of proportion \( (p_{ij}) \) is

<table>
<thead>
<tr>
<th>( X \leq M_x )</th>
<th>( Y \leq M_y )</th>
<th>( Y &gt; M_y )</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>( X \leq M_x )</td>
<td>( p_{11} )</td>
<td>( p_{21} )</td>
<td>( p_{1.} )</td>
</tr>
<tr>
<td>( X &gt; M_x )</td>
<td>( p_{12} )</td>
<td>( p_{21} )</td>
<td>( p_{2.} )</td>
</tr>
<tr>
<td>Total</td>
<td>( p_1 )</td>
<td>( p_2 )</td>
<td>1</td>
</tr>
</tbody>
</table>

Following [14] and [10], the product estimator for population median \( M_y \) is defined as

\[
\hat{M}_p = \hat{M}_y \frac{\hat{M}_x}{M_x} \tag{1.2}
\]

The usual difference estimator for population median \( M_y \) is given by

\[
\hat{M}_d = \hat{M}_y + d(M_x - \hat{M}_x) \tag{1.3}
\]

Where \( d \) is a constant to be determined such that the mean square error of \( \hat{M}_d \) is minimum.

[21] proposed the following modified product and ratio estimator for population median \( M_y \), respectively, as

\[
\hat{M}_1 = \hat{M}_y \frac{a - \hat{M}_x}{a + \hat{M}_x} \tag{1.4}
\]

and

\[
\hat{M}_2 = \hat{M}_y \frac{a + \hat{M}_x}{a - \hat{M}_x} \tag{1.5}
\]

Where \( a \) is suitably chosen scalar.

[26] type estimator for median estimation is

\[
\hat{M}_3 = \hat{M}_y \frac{M_x}{\hat{M}_x} \tag{1.6}
\]

[12, 13] and [30]-type estimator is given by

\[
\hat{M}_4 = \hat{M}_y \left[ \frac{M_x}{M_x + \beta(M_x - \hat{M}_x)} \right] \tag{1.7}
\]
A sample article to present the class of estimators is given by

\[ \hat{M}_5 = \hat{M}_y \left[ 2 - \left( \frac{M_x}{\hat{M}_x} \right)^2 \right] \]

[29]-type estimator is given by

\[
\begin{align*}
\hat{M}_6 &= w \hat{M}_y + (1 - w) \hat{M}_y \frac{\hat{M}_x}{\hat{M}_y} \\
\hat{M}_7 &= w \hat{M}_y + (1 - w) \hat{M}_y \frac{\hat{M}_x}{\hat{M}_y}
\end{align*}
\]

Where \( w \) is suitably chosen scalar.

All the estimators considered from (1.1) to (1.9) and conventional estimator \( \hat{M}_y \) are members of the \([27, 28]\)-type class of estimators

\[ G = \left\{ \hat{M}_y^{(G)} : \hat{M}_y^{(G)} = G \left( \hat{M}_y, \hat{M}_x \right) \right\} \]

Where the function \( G \) assumes a value in a bounded closed convex subset \( Q \subset \mathbb{R}^2 \), which contains the point \((M_y, 1)\) and is such that \( G(M_y, 1) = 1 \).

Using first order Taylor’s series expansion about the point \((\hat{M}_y, 1)\), we have

\[ \hat{M}_y^{(G)} = G(M_y, 1) + (\hat{M}_y - M_y)G_{10}(M_y, 1) + O(n^{-1}) \]

Where \( U = \frac{\hat{M}_x}{\hat{M}_y} \)

and \( G_{01}(M_y, 1) = \frac{\partial G(\cdot)}{\partial U}(M_y, 1) \)

Using condition we have

\[ \hat{M}_y^{(G)} = M_y + (\hat{M}_y - M_y) + (U - 1)G_{01}(M_y, 1) + O(n^{-1}) \]

or

\[ \hat{M}_y^{(G)} = M_y + (\hat{M}_y - M_y) + (U - 1)G_{01}(M_y, 1) + O(n^{-1}) \]

Squaring and taking expectations both sides of (1.12), we get the MSE of \( \hat{M}_y^{(G)} \) to the first order of approximation as

\[ MSE(\hat{M}_y^{(G)}) = \left[ V(\hat{M}_y) + V(\hat{M}_y) \frac{\hat{M}_y^2}{M_x^2} G_{01}^2(M_y, 1) + 2 \frac{\text{Cov}(\hat{M}_y, \hat{M}_x)}{M_x} G_{01}^2(M_y, 1) \right] \]

Here as \( N \to \infty, n \to \infty \) then \( n/N \to f \) and we assumed that as \( N \to \infty \) the distribution of \((X, Y)\) approaches a continuous distribution with marginal densities \( f_x(x) \) and \( f_y(y) \) of \( X \) and \( Y \) respectively. Super population model framework is necessary for treating the values of \( X \) and \( Y \) in a realization of \( N \) independent observation from a continuous distribution. It is also assumed that \( f_x(M_x) \) and \( f_y(M_y) \) are positive. Under these conditions, sample median \( \hat{M}_y \) is consistent and
asymptotically normal (due to [5]) with mean $M_y$ and variance

\begin{align}
(1.14) \quad V(\hat{M}_y) &= \gamma M_y^2 C_y^2 \\
(1.15) \quad V(\hat{M}_x) &= \gamma M_x^2 C_x^2 \\
(1.16) \quad \text{Cov}(\hat{M}_y, \hat{M}_x) &= \gamma \rho_c M_y C_y C_x
\end{align}

Where $\gamma = (1 - f)/4n$, $f = n/N$, $C_y = (M_y f_y(M_y))^{-1}$, $C_x = (M_x f_x(M_x))^{-1}$ and $\rho_c = (4p_{11} - 1)$ with $p_{11} = P(M_x, M_y)$ goes from $-1$ to $+1$ as $p_{11}$ increases from 0 to 0.5.

Substituting these values we get the MSE of $\hat{M}_y^{(G)}$ to the first degree of approximation as

\begin{align}
(1.17) \quad \text{MSE}(\hat{M}_y^{(G)}) &= \gamma [M_y^2 C_y^2 + C_x^2 G_01(M_y, 1)]^2 + 2 \rho_c C_x C_y M_y G_01(M_y, 1) \\
&= \gamma C_y^2 M_y^2 (1 - \rho_c^2) = \text{MSE}_{\text{min}}(\hat{M}_d)
\end{align}

The MSE is minimum when

\begin{align}
(1.18) \quad G_01 &= (M_y, 1) = -k_c M_y
\end{align}

Thus the minimum MSE of $\hat{M}_y^{(G)}$ is given by

\begin{align}
(1.19) \quad \text{MSE}_{\text{min}}(\hat{M}_y^{(G)}) &= \gamma C_y^2 M_y^2 (1 - \rho_c^2) = \text{MSE}_{\text{min}}(\hat{M}_d)
\end{align}

Which equal to the minimum MSE of the estimator $\hat{M}_d$ defined at (1.3).

It is to be mentioned that minimum MSEs of the estimators $\hat{M}_r$, $\hat{M}_p$, and $\hat{M}_i (i = 1, 2, ..., 7)$ are equal to MSE expression given in equation (1.19). It is obvious from (1.19) that the estimators of the form $\hat{M}_y^{(G)}$ are asymptotically no more efficient than the difference estimator at its optimum value or the regression type estimator given as

\begin{align}
(1.20) \quad \hat{M}_d &= \hat{M}_y + \hat{d}(M_x - \hat{M}_x)
\end{align}

where $\hat{d} = \hat{f}_x M_x (4\hat{p}_{11} - 1)$

[19] Suggested following Classes of estimator

\begin{align}
(1.21) \quad \hat{M}_d^{(1)} &= d_1 \hat{M}_y + (1 - d_1)(M_x - \hat{M}_x) \\
(1.22) \quad \hat{M}_d^{(2)} &= d_1 \hat{M}_y + d_2 (M_x - \hat{M}_x) \\
(1.23) \quad \hat{M}_d^{(3)} &= d_1 \hat{M}_y + d_2 \hat{M}_x + (1 - d_1 - d_2) M_x \\
(1.24) \quad \hat{M}_d^{(4)} &= \left[d_1 \hat{M}_y + d_2 (M_x - \hat{M}_x) \right] \left(\frac{\phi M_x + \delta}{\phi M_x + \delta}\right)^{\beta}
\end{align}

where $d_1$ and $d_2$ are suitable constants to be determined such that MSEs of the estimators considered in (1.21) to (1.24) are minimum, $\phi$ and $\delta$ are either real numbers or the functions of the known parameters of auxiliary variable X.

Biases and minimum MSEs of the estimators considered in (1.21) to (1.24) are given as

\begin{align}
(1.25) \quad B(\hat{M}_d^{(1)}) &= (d_1 - 1) M_y
\end{align}
(1.26) \( B(\hat{M}_d^{(2)}) = (d_1 - 1)M_y \)

(1.27) \( B(\hat{M}_d^{(3)}) = (d_1 - 1)(1 - R)M_y \)

(1.28) \( B(\hat{M}_d^{(4)}) = M_y \left[ d_1^2 \{1 + \gamma \delta C_x^2 (\delta - k_c)\} + d_2 R \gamma \delta C_x^2 - 1 \right] \)

(1.29) \( \text{MSE}_{\text{min}}(\hat{M}_d^{(1)}) = M_y^2 \left[ 1 + R^2 \gamma C_x^2 \frac{\{1 + R \gamma C_x^2 (R + k_c)\}^2}{\{1 + \gamma (C_y^2 + R C_y^2 (R + 2k_c))\}} \right] \)

(1.30) \( \text{MSE}_{\text{min}}(\hat{M}_d^{(2)}) = \frac{M_y^2 \gamma C_y^2 (1 - \rho_c^2)}{\{1 + \gamma C_y^2 (1 - \rho_c^2)\}} \)

(1.31) \( \text{MSE}_{\text{min}}(\hat{M}_d^{(3)}) = \frac{M_y^2 \gamma C_y^2 (1 - \rho_c^2) (1 - R)^2}{\{1 - R\}^2 + \gamma C_y^2 (1 - \rho_c^2)} \)

(1.32) \( \text{MSE}_{\text{min}}(\hat{M}_d^{(4)}) = \frac{(1 - \delta^2 \gamma C_x^2) M_y^2 \gamma C_y^2 (1 - \rho_c^2)}{\{1 - \delta^2 \gamma C_x^2\} + \gamma C_y^2 (1 - \rho_c^2)} \)

2. The Suggested Generalised Class of Estimators

We propose a generalized family of estimators for population median of the study variable \( Y \), as

\[
(2.1) \quad t_m = \left\{ w_1 \hat{M}_y \left( \frac{M_x}{\hat{M}_x} \right)^\alpha \exp \left( \frac{\eta (M_x - \hat{M}_x)}{\eta (M_x + \hat{M}_x) + 2 \lambda} \right) \right\} + w_2 \hat{M}_x + (1 - w_1 - w_2)M_x
\]

where \( w_1 \) and \( w_2 \) are suitable constants to be determined such that MSE of \( t_m \) is minimum, \( \eta \) and \( \lambda \) are either real numbers or the functions of the known parameters of auxiliary variables such as coefficient of variation \( C_x \), skewness \( \beta_{1(x)} \), kurtosis \( \beta_{2(x)} \) and correlation coefficient \( \rho_c \) (see [17]).

It is to be mentioned that

(i) For \( (w_1, w_2) = (1, 0) \), the class of estimator \( t_m \) reduces to the class of estimator as

\[
(2.2) \quad t_{mp} = \left\{ \hat{M}_y \left( \frac{M_x}{\hat{M}_x} \right)^\alpha \exp \left( \frac{\eta (M_x - \hat{M}_x)}{\eta (M_x + \hat{M}_x) + 2 \lambda} \right) \right\}
\]

(ii) For \( (w_1, w_2) = (w_1, 0) \), the class of estimator \( t_m \) reduces to the class of estimator as

\[
(2.3) \quad t_{mq} = \left\{ w_1 \hat{M}_y \left( \frac{M_x}{\hat{M}_x} \right)^\alpha \exp \left( \frac{\eta (M_x - \hat{M}_x)}{\eta (M_x + \hat{M}_x) + 2 \lambda} \right) \right\}
\]

A set of new estimators generated from (2.1) using suitable values of \( w_1, w_2, \alpha, \eta \) and \( \lambda \) are listed in Table 2.1.
Table 2.1: Set of estimators generated from the class of estimators \( t_m \)

<table>
<thead>
<tr>
<th>Subset of proposed estimator</th>
<th>( w_1 )</th>
<th>( w_2 )</th>
<th>( \alpha )</th>
<th>( \eta )</th>
<th>( \lambda )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t_{m1} = M_y )</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>( t_{m2} = M_y \left( \frac{M_x}{M_s} \right) )</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>( t_{m3} = M_y \left( \frac{M_x}{M_s} \right)^\alpha )</td>
<td>1</td>
<td>0</td>
<td>( \alpha )</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>( t_{m4} = M_y \left( \frac{M_x}{M_s} \right) = M_p )</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>( t_{m5} = w_1 M_y \left( \frac{M_x}{M_s} \right) )</td>
<td>( w_1 )</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>( t_{m6} = w_1 M_y \left( \frac{M_x}{M_s} \right) )</td>
<td>( w_1 )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>( t_{m7} = w_1 M_y \left[ 1 \right] )</td>
<td>( w_1 )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

*Estimator proposed by [19] given in equation (1.23). Another set of estimators generated from class of estimator \( t_{mq} \) given in (2.3) using suitable values of \( \eta \) and \( \lambda \) are summarized in table 2.2

Table 2.2: Set of estimators generated from the estimator \( t_{mq} \)

<table>
<thead>
<tr>
<th>Subset of proposed estimator</th>
<th>( \alpha )</th>
<th>( \eta )</th>
<th>( \lambda )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t_{mq}^{(1)} = \left{ \begin{array}{l} w_1 M_y \frac{M_x}{M_s} \exp \left{ \frac{(M_x - M_s)}{M_s + M_x} \right} \end{array} \right} )</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( t_{mq}^{(2)} = \left{ \begin{array}{l} w_1 M_y \frac{M_x}{M_s} \exp \left{ \frac{(M_x - M_s)}{M_s + M_x + 2} \right} \end{array} \right} )</td>
<td>1</td>
<td>1</td>
<td>( \rho_c )</td>
</tr>
<tr>
<td>( t_{mq}^{(3)} = \left{ \begin{array}{l} w_1 M_y \frac{M_x}{M_s} \exp \left{ \frac{(M_x - M_s)}{M_s + M_x + 2 M_x} \right} \end{array} \right} )</td>
<td>1</td>
<td>1</td>
<td>( M_x )</td>
</tr>
<tr>
<td>( t_{mq}^{(4)} = \left{ \begin{array}{l} w_1 M_y \frac{M_x}{M_s} \exp \left{ \frac{(M_x - \hat{M}_s)}{M_s + M_x} \right} \end{array} \right} )</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( t_{mq}^{(5)} = \left{ \begin{array}{l} w_1 M_y \frac{\hat{M}_x}{M_s} \exp \left{ \frac{(M_x - \hat{M}_s)}{M_s + M_x} \right} \end{array} \right} )</td>
<td>(-1)</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( t_{mq}^{(6)} = \left{ \begin{array}{l} w_1 M_y \frac{M_x}{M_s} \exp \left{ \frac{(M_x - \hat{M}_s)}{M_s + M_x + 2 M_x} \right} \end{array} \right} )</td>
<td>1</td>
<td>( M_x )</td>
<td>( \rho_c )</td>
</tr>
<tr>
<td>( t_{mq}^{(7)} = \left{ \begin{array}{l} w_1 M_y \exp \left{ \frac{M_s (M_x - \hat{M}_s)}{M_s (M_x + M_x) + 2 \rho_c} \right} \end{array} \right} )</td>
<td>0</td>
<td>( M_x )</td>
<td>( \rho_c )</td>
</tr>
<tr>
<td>( t_{mq}^{(8)} = \left{ \begin{array}{l} w_1 M_y \frac{M_x}{M_s} \exp \left{ \frac{\rho_c (M_x - \hat{M}_s)}{\rho_c (M_x + M_x) + 2 M_x} \right} \end{array} \right} )</td>
<td>1</td>
<td>( \rho_c )</td>
<td>( M_x )</td>
</tr>
<tr>
<td>( t_{mq}^{(9)} = \left{ \begin{array}{l} w_1 M_y \frac{M_x}{M_s} \exp \left{ \frac{\rho_c (M_x - \hat{M}_s)}{\rho_c (M_x + M_x) + 2 M_x} \right} \end{array} \right} )</td>
<td>(-1)</td>
<td>( \rho_c )</td>
<td>( M_x )</td>
</tr>
</tbody>
</table>
Expressing (2.1) in terms of e’s, we have

\[
(2.4) \quad t_m = w_1 M_y (1+e_0)(1+e_1)^{-\alpha} \exp\{-ke_1(1+ke_1)^{-1}\} + w_2 M_x (1+e_1)+(1-w_1-w_2) M_x
\]

where \( k = \frac{\eta M_x}{2(\eta M_x + \lambda)} \)

Up to the first order of approximation we have,

\[
(2.5) \quad (t_m - M_y) = [(w_1 - 1)b + w_2 M_y (e_0 - ae_1 + de_1^2 - ae_0 e_1)] + w_2 M_x e_1
\]

where \( a = (\alpha + k), \ b = (M_y - M_x) \) and \( d = \left\{ \frac{3}{2} k^2 + \alpha k + \frac{\alpha(\alpha + 1)}{2} \right\} \)

Squaring both sides of equation (2.5) and neglecting terms of e’s having power greater than two, we have

\[
(2.6) \quad (t_m - \bar{Y})^2 = [(1 - 2w_1)b^2 + w_1^2 (b^2 + M_y^2 (e_0^2 + a^2 e_1^2 - 2ae_0 e_1)) + w_2^2 M_x^2 e_1^2 + 2w_1 w_2 M_y M_x (e_0 e_1 - a e_1^2)]
\]

Taking expectations both sides, we get the MSE of the estimator \( t_m \) to the first order of approximation as

\[
(2.7) \quad MSE(t_m) = [(1 - 2w_1)b^2 + w_1^2 A + w_2^2 B + 2w_1 w_2 C]
\]

where,

\[
A = b^2 + M_y^2 C_y + a^2 C_x^2 - 2a \rho_c C_y C_x
\]

\[
B = M_x^2 C_x^2
\]

\[
C = M_y M_x \gamma (\rho_c C_y - a C_x) C_x
\]

The optimum values of \( w_1 \) and \( w_2 \) are obtained by minimizing (2.7) and is given by

\[
(2.8) \quad w_1^* = \frac{b^2 B}{(AB - C^2)} \quad \text{and} \quad w_2^* = \frac{-b^2 C}{(AB - C^2)}
\]

Substituting the optimal values of \( w_1 \) and \( w_2 \) in equation (2.7) we obtain the minimum MSE of the estimator \( t_m \) as

\[
(2.9) \quad MSE_{min}(t_m) = b^2 \left[ 1 - \frac{b^2 B}{(AB - C^2)} \right]
\]

Putting the values of A, B, C and b and simplifying, we get the minimum MSE of estimator \( t_m \) as

\[
(2.10) \quad MSE_{min}(t_m) = \left[ \frac{M_y^2 (1-R)^2 \gamma C_y^2 (1-\rho_c^2)}{(1-R)^2 \gamma C_y^2 (1-\rho_c^2)} \right]
\]

where \( R = \frac{M_x}{M_y} \)

MSE expression given in (2.10) is same as the minimum MSE of Estimator \( \hat{M}_d \)
Therefore, the minimum MSE of the class of estimators $t_{mq}$ is given by

$$\text{(2.11)} \quad \text{MSE}_{\text{min}}(t_{mq}) = M_y^2 \left[ \frac{(\gamma C_y^2 + a^2 \gamma C_x^2 - 2a \gamma \rho_c C_y C_x)}{(1 + \gamma C_y^2 + a^2 \gamma C_x^2 - 2a \gamma \rho_c C_y C_x)} \right]$$

3. Efficiency Comparisons

From equations (1.19) and (2.10) we have

$$\text{(3.1)} \quad \left\{ \text{MSE}_{\text{min}}(\hat{M}_y^{(G)}) = \text{MSE}_{\text{min}}(\hat{M}_d) \right\} - \text{MSE}_{\text{min}}(\hat{t}_m) = \frac{(1 - R)^2 \text{MSE}_{\text{min}}(\hat{M}_d)}{(1 - R)^2 + \text{MSE}_{\text{min}}(\hat{M}_d)} > 0$$

From equations (1.19) and (2.11) we have

$$\text{MSE}_{\text{min}}(\hat{M}_y^{(G)}) = \text{MSE}_{\text{min}}(\hat{M}_d) - \text{MSE}_{\text{min}}(t_{mq}) > 0$$

$$\gamma C_y^2 M_y^2 (1 - \rho_y^2) - M_y^2 \left[ \frac{(\gamma C_y^2 + a^2 \gamma C_x^2 - 2a \gamma \rho_c C_y C_x)}{(1 + \gamma C_y^2 + a^2 \gamma C_x^2 - 2a \gamma \rho_c C_y C_x)} \right] > 0$$

$$\text{(3.2)} \quad \gamma C_y^2 M_y^2 (1 - \rho_y^2) (1 + 1 + \gamma C_y^2 + a^2 \gamma C_x^2 - 2a \gamma \rho_c C_y C_x) > \gamma C_y^2 + a^2 \gamma C_x^2 - 2a \gamma \rho_c C_y C_x$$

From equations (1.30) and (2.10)

$$\text{MSE}_{\text{min}}(t_{mq}) - \text{MSE}_{\text{min}}(\hat{M}_d)$$

$$\text{(3.3)} \quad \frac{M_y^2 R(R - 2) \text{MSE}_{\text{min}}(\hat{M}_d)}{M_y^2 + \text{MSE}_{\text{min}}(\hat{M}_d) \left\{ M_y^2 (1 - R)^2 + \text{MSE}_{\text{min}}(\hat{M}_d) \right\}} < 0, \quad \text{When } 0 < R < 2$$

Since, $\text{MSE}_{\text{min}}(\hat{M}_d^{(2)}) - \text{MSE}_{\text{min}}(\hat{M}_d^{(0)}) > 0$

$$\text{(3.4)} \quad \frac{\delta^2 \gamma C_x^2 M_y^2 \left\{ \text{MSE}_{\text{min}}(\hat{M}_d) \right\}^2}{M_y^2 + \text{MSE}_{\text{min}}(\hat{M}_d) \left\{ M_y^2 (1 - \delta^2 \gamma C_x^2) + \text{MSE}_{\text{min}}(\hat{M}_d) \right\}} > 0$$

and from (3.3) we have, $\text{MSE}_{\text{min}}(t_{mq}) - \text{MSE}_{\text{min}}(\hat{M}_d^{(2)}) < 0$

$$\text{(3.5)} \quad \text{Therefore, MSE}_{\text{min}}(t_{mq}) - \text{MSE}_{\text{min}}(\hat{M}_d^{(0)}) < 0, \text{ When } 0 < R < 2$$

It follows from (3.1), (3.2), (3.3), (3.4) and (3.5) that the proposed class of estimators $t_{mq}$ is better than the Conventional difference estimator $\hat{M}_d$, the class of estimators $M_y^{(G)}$ and estimator belonging to the class of estimators $M_y^{(G)}$ i.e. usual unbiased estimator $\hat{M}_y$ , due to [5], usual ratio-type estimator $\hat{M}_y$ due to [6], product estimator $\hat{M}_p$ and $\hat{M}_i$ (i=3,4,...7) at their optimum conditions. Further it is shown that the proposed class of estimators $t_{mq}$ is better than the estimators
$M_d^{(2)}, M_d^{(4)}$ and $M_d^{(1)}$ considered by [19].

**Remark 3.1: Estimator Based on optimum values**

Putting the optimum values of $w_1^*$ and $w_2^*$ in the equation (2.1) we get the optimum estimator as:

\[
(3.6) \quad t_m' = \left\{ w_1^* M_y \left( \frac{M_x}{M_x} \right)^{\alpha} \exp \left( \frac{\eta(M_x - \hat{M}_x)}{\eta(M_x + M_x) + 2\lambda} \right) \right\} + w_2^* \hat{M}_x(1 - w_1^* - w_2^*) \hat{M}_x
\]

If the experimenter is not able to specify the value precisely, then it may be desirable to estimate the optimum values from the samples, therefore the values of $w_1^*$ and $w_2^*$ are given as: $w_1^* = \frac{\hat{b}^2 \hat{B}}{(\hat{A} \hat{B} - \hat{C}^2)}$ and $w_2^* = \frac{\hat{b}^2 \hat{C}}{(\hat{A} \hat{B} - \hat{C}^2)}$

where, $A = \hat{b}^2 + \hat{M}_y \gamma (\hat{C}_y^2 + \hat{a}^2 \hat{C}_x^2 - 2a \hat{\rho_c} \hat{C}_y \hat{C}_x)$

$B = \hat{M}_x \gamma \hat{C}_x^2, \hat{\rho_c} = 4(4p_{11} - 1)$

$C = \hat{M}_y \hat{M}_x \gamma (\hat{\rho_c} \hat{C}_y - a \hat{C}_x) \hat{C}_x, \hat{C}_y = \left\{ \hat{M}_x \hat{f}_x \left( \hat{M}_x \right) \right\}^{-1}$

$\hat{\gamma} = \left\{ \hat{M}_y \hat{f}_y \left( \hat{M}_y \right) \right\}^{-1}$

$\hat{a} = (\alpha + \hat{k}), \hat{b} = (\hat{M}_y - \hat{M}_x)$ and $\hat{k} = \frac{\eta \hat{M}_x}{2(\eta \hat{M}_x + \lambda)}$

Here, we have assumed that the population median of auxiliary variable $x$ is known, therefore $M_x$ can also be remain as $M_x$.

Expressing (3.6) in terms of $e$'s, we have

\[
t_m' = w_1^* M_y (1 + e_0)(1 + e_1)^{-\alpha} \exp\{-\hat{k}e_1 (1 + \hat{k}e_1)^{-1}\} + w_2^* M_x (1 + e_1) + (1 - w_1^* - w_2^*) \hat{M}_x
\]

Proceeding as above, we get the minimum MSE of the estimator $t_m'$ given as:

\[
(3.7) \quad MSE_{min}(t_m') = \left[ \frac{\hat{M}_y^2 (1 - \hat{R})^2 \hat{C}_y^2 (1 - \hat{\rho_c}^2)}{(1 - \hat{R})^2 \hat{C}_y^2 (1 - \hat{\rho_c}^2)} \right]
\]

**Remark 3.2** It may be noted here that the minimum MSEs of the estimators considered in (2.10) and (2.11) are usable only if we know the exact values of $C_x, C_y, \hat{R}, k_c$ and $\hat{\rho_c}$ . If these values are unknown then we can estimate them from samples as $\hat{C}_y = \left\{ \hat{M}_y \hat{f}_y \left( \hat{M}_y \right) \right\}^{-1}, \hat{C}_x = \left\{ \hat{M}_x \hat{f}_x \left( \hat{M}_x \right) \right\}^{-1}$, $\hat{R} = \hat{M}_x / \hat{M}_y$, $k_c = \hat{\rho_c} \left( \hat{C}_y / \hat{C}_x \right)$ and $\hat{\rho_c} = 4(4p_{11} - 1)$ with $p_{11}$ being the sample values analogues of $p_{11}$ ([18]; [24]).

4. **Empirical study**

Data Statistics: To illustrate the efficiency of proposed generalized class of estimators in the application, we consider the following two population data sets.

**Population I.** (Source [23])

$y$: The number of fish caught by marine recreational fisherman in 1995.

$x$: The number of fish caught by marine recreational fisherman in 1964.

The values of the required parameters are:

$N$ = 69, $n$ = 17, $M_y$ = 2068, $M_x$ = 2011, $f_y(M_y) = 0.00014, f_x(M_x) = 0.00014$, $\rho_c = 0.1505, R = 0.97244$. 

**Population II.** (Source [23])

\( y \) : The number of fish caught by marine recreational fisherman in 1995.

\( x \) : The number of fish caught by marine recreational fisherman in 1993.

The values of the required parameters are:

\[\begin{align*}
N &= 69, \quad n = 17, \quad M_y = 2068, \quad M_x = 2307, \quad f_y(M_y) = 0.00014, \quad f_x(M_x) = 0.00013, \\
\rho_c &= 0.3166, \quad R = 1.11557
\end{align*}\]

<table>
<thead>
<tr>
<th>Table 3.1: Variances / MSEs/minimum MSEs of different Estimators</th>
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<tbody>
<tr>
<td>Estimators</td>
</tr>
<tr>
<td>( V(\hat{M}_y) )</td>
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<tr>
<td>( MSE(\hat{M}_r) )</td>
</tr>
<tr>
<td>( MSE_{min}(\hat{M}_d) )</td>
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<tr>
<td>( MSE_{min}(\hat{M}_d^{(G)}) )</td>
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<tr>
<td>( MSE_{min}(\hat{M}_i) )</td>
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<tr>
<td>( MSE_{min}(\hat{M}_d^1) )</td>
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<tr>
<td>( MSE_{min}(\hat{M}_d^2) )</td>
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<tr>
<td>( MSE_{min}(\hat{M}_d^3) )</td>
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<tr>
<td>( MSE_{min}(\hat{M}_d^4) )</td>
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<tr>
<td>( MSE_{min}(\hat{M}_d^5) )</td>
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<tr>
<td>( MSE_{min}(\hat{M}_d^6) )</td>
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<tr>
<td>( MSE_{min}(\hat{M}_d^7) )</td>
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<tr>
<td>( MSE_{min}(\hat{M}_d^8) )</td>
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<tr>
<td>( MSE_{min}(\hat{M}_d^9) )</td>
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<tr>
<td>( MSE_{min}(\hat{M}_d^{10}) )</td>
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<tr>
<td>( MSE_{min}(\hat{M}_d^{11}) )</td>
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<tr>
<td>( MSE_{min}(\hat{M}_d^{12}) )</td>
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<tr>
<td>( MSE_{min}(\hat{M}_d^{13}) )</td>
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<tr>
<td>( MSE_{min}(\hat{M}_d^{14}) )</td>
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</tbody>
</table>

(for \( i=1,2,\ldots,7 \))

Analysing table 3.1 we conclude that the estimators based on auxiliary information are more efficient than the one which does not use the auxiliary information as \( \hat{M}_y \). The members of the class of estimators \( t_{mq} \),obtained from generalized class of estimators \( t_m \), are almost equally efficient but more than the usual unbiased estimator \( \hat{M}_y \) (due to [5]), usual ratio estimator \( \hat{M}_r \) (due to [6]), difference type estimator \( \hat{M}_d \), the class of estimators \( \hat{M}_y^{(G)} \), the estimators \( \hat{M}_i \) (\( i=1,2,\ldots,7 \)) and the estimators \( \hat{M}_d^{(1)}, \hat{M}_d^{(2)} \) and \( \hat{M}_d^{(3)} \) (due to [19]). Among the proposed estimators \( t_m \) and \( t_{mq} \) (\( j=1,2,\ldots,9 \)) the performance of the estimator \( t_m \),which is equal efficient to the estimator \( \hat{M}_d^{(3)} \) (due to [19]) , is best in the sense of having the least MSE followed by the estimator \( \hat{M}_d^{(7)} \) which utilize the information on population median \( M_x \) and \( \rho_c \).
5. Conclusion

In this article we have suggested a generalized class of estimators for the population median of study variable $y$ when information is available on an auxiliary variable in simple random sampling without replacement (SRSWOR). In addition, some known estimators of population median such as usual unbiased estimator for population median $\hat{M}_y$ due to [5], estimators due to [6], [26], [10], [1] and [19] are found to be members of the proposed generalized class of estimators. Some new members are also generated from the proposed generalized class of estimators. We have determined the biases and mean square errors of the proposed class of estimators up to the first order of approximation. The proposed generalized class of estimators are advantageous in the sense that the properties of the estimators, which are members of the proposed class of estimators, can be easily obtained from the properties of the proposed generalized class. Thus the study unifies properties of several estimators for population median. In theoretical and empirical efficiency comparisons, it has been shown that the proposed generalized class of estimators are more efficient than the estimators considered here and equally efficient to the estimator $\hat{M}_d^{(3)}$

References

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