The Borel property for 4-dimensional matrices

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Abstract

In 1909 Borel has proved that “Almost all of the sequences of 0’s and 1’s are Cesàro summable to \( \frac{1}{2} \)”. Then Hill [6] has generalized Borel’s result to two dimensional matrices. In this paper we investigate the Borel property for 4-dimensional matrices.

Keywords: Double sequences, Pringsheim convergence, the Borel Property, double sequences of 0’s and 1’s.

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1. Introduction

The summability of sequences of 0’s and 1’s has been studied by various authors ([1], [3], [6], [7], [8], [10]). In 1909 Borel proved that “Almost all of the sequences of 0’s and 1’s are Cesàro summable to \( \frac{1}{2} \)”. Then Hill [6] has generalized Borel’s result to general matrices. We say that the matrix has the Borel property, if a matrix sums almost all of the sequences of 0’s and 1’s to \( \frac{1}{2} \). Establishing a one-to-one correspondence between the interval (0,1] and the collection of all sequences of 0’s and 1’s, Hill has given some necessary conditions and also some sufficient conditions for matrices to have the Borel property in [6], [7]. This property has also been examined in [5], [8].

In the present paper we investigate the Borel property for 4-dimensional matrices. In particular we exhibit some necessary and some sufficient conditions for 4-dimensional matrices to have the Borel property.

We first recall some basic notations and results related to double sequences.

A double sequence \( s = (s_{ij}) \) is said to be Pringsheim convergent (i.e., it is convergent in Pringsheim’s sense) to \( L \) if for every \( \varepsilon > 0 \) there exists an \( N \in \mathbb{N} \) such that \( |s_{ij} - L| < \varepsilon \) whenever \( i,j \geq N \) ([2], [11]). In this case \( L \) is called the Pringsheim limit of \( s \).

Throughout the paper when there is no confusion, convergence means the Pringsheim convergence.

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Let $X$ denote the set of all double sequences of 0’s and 1’s, that is

$$X = \{ x = (x_{jk}) : x_{jk} \in \{0, 1\} \text{ for each } j, k \in \mathbb{N} \}.$$

Let $\mathcal{R}$ be the smallest $\sigma$-algebra of subsets of the set $X$ which contains all sets of the form

$$\{ x = (x_{jk}) \in X : x_{j_1 k_1} = a_1, \ldots, x_{j_n k_n} = a_n \}$$

where each $a_i \in \{0, 1\}$ and the pairs $\{(j_i k_i)\}_{i=1}^n$ are pairwise distinct.

There exists a unique probability measure $P$ on the set $\mathcal{R}$, such that

$$P \left( \{ x = (x_{jk}) \in X : x_{j_1 k_1} = a_1, \ldots, x_{j_n k_n} = a_n \} \right) = \frac{1}{2^n}$$

for all choices of $n$ and all pairwise disjoint pairs $\{(j_i k_i)\}_{i=1}^n$, and all choices of $a_1, \ldots, a_n$.

Recall that the functions $r_{jk}(x) = 2x_{jk} - 1$, for $x \in X$, are the Rademacher functions (see [4]).

Four dimensional Cesàro matrix $(C, 1, 1) = (c_{jk}^{nm})$ is defined by

$$c_{jk}^{nm} = \begin{cases} \frac{1}{nm}, & 1 \leq j \leq n \text{ and } 1 \leq k \leq m \\ 0, & \text{otherwise.} \end{cases}$$

It is known that the $(C, 1, 1)$ matrix is an RH regular, i.e., it sums every bounded convergent sequence to the same limit.

An element $x$ of $X$ is said to be normal ([4]) if for each $\varepsilon > 0$ there is a natural number $N_{\varepsilon}$ such that for $n, m \geq N_{\varepsilon}$ we have $\left| \frac{1}{nm} \sum_{k \leq m} x_{jk} - \frac{1}{2} \right| < \varepsilon$. Let $\eta$ denote the set of all elements $x$ in $X$ that are normal. This means that normal elements are $(C, 1, 1)$-summable to $\frac{1}{2}$. It is also proved in [4] that $P(\eta) = 1$. So $(C, 1, 1)$ method has the Borel property.

It would be appropriate to recall the definition of bounded regularity.

1.1. Definition. Let $A = (a_{jk}^{nm})$ be a 4-dimensional matrix. If the limit

$$\lim_{n,m \to \infty} \sum_{j,k=1}^{\infty} a_{jk}^{nm} s_{jk} = L$$

exists, the double sequence $(s_{jk})$ is called $A$-summable to $L$ and denoted by $s_{jk} \rightarrow L$ ($A$). A matrix $A = (a_{jk}^{nm})$ is bounded regular if every bounded and convergent sequence $s = (s_{jk})$ is $A$-summable to the same limit and $A$-means are also bounded [9]. The next corollary characterizes bounded regular matrices.

1.2. Proposition. $A = (a_{jk}^{nm})$ is bounded regular if and only if

(i) $\lim_{n,m \to \infty} a_{jk}^{nm} = 0$, $(j, k = 1, 2, \ldots)$

(ii) $\lim_{n,m \to \infty} \sum_{j,k=1}^{\infty} a_{jk}^{nm} = 1$, 

(iii) $\lim_{n,m \to \infty} \sum_{k=1}^{\infty} |a_{jk}^{nm}| = 0$, $(j = 1, 2, \ldots)$

(iv) $\lim_{n,m \to \infty} \sum_{j=1}^{\infty} |a_{jk}^{nm}| = 0$, $(k = 1, 2, \ldots)$

(v) $\sum_{j,k=1}^{\infty} |a_{jk}^{nm}| \leq C < \infty$, $(m, n = 1, 2, \ldots)$. 


These conditions were first established by Robison [12].

2. The Borel Property

This section is devoted to the Borel property for 4-dimensional matrices.

2.1. Theorem. If \( A = (a_{jk}^{nm}) \) has the Borel property, then the \( \sum_{j,k=1}^{\infty,\infty} a_{jk}^{nm} \) series converges for each \( n,m \) and tends to 1 as \( n,m \to \infty \).

Proof. Since \( A \) has the Borel property, for almost all \( x \in X \), we obtain

\[
\lim_{n,m \to \infty} \sum_{j,k=1}^{\infty,\infty} a_{jk}^{nm} x_{jk} = \frac{1}{2}
\]

Indeed \( P(E) = 1 \) where \( E = \{ x = (x_{jk}) \in X: (Ax)_{nm} \to \frac{1}{2} \} \).

Let us define \( x = (\bar{x}_{jk}) \) by

\[
\bar{x}_{jk} = \begin{cases} 
0 & \text{if } x_{jk} = 1 \\
1 & \text{if } x_{jk} = 0 
\end{cases}
\]

Let \( Y = E \cap \eta \) and \( \mathcal{Y} = \{(\tau_{jk}) : x_{jk} \in Y\} \). We get \( \mathcal{Y} = E \cap \eta \). Since the mapping \( (x_{jk}) \to (\tau_{jk}) \) preserves \( P \) measure, we obtain \( P(\mathcal{Y}) = 1 \). If \( x = (x_{jk}) \in Y \cap \mathcal{Y} \), then \( x \in E, x \in \eta \) and \( \tau \in E, \tau \in \eta \). Since \( x, \tau \in E \), it follows that

\[
\sum_{j,k=1}^{\infty,\infty} a_{jk}^{nm} x_{jk} + \sum_{j,k=1}^{\infty,\infty} a_{jk}^{nm} \tau_{jk} = \sum_{j,k=1}^{\infty,\infty} a_{jk}^{nm} \to 1 \quad (n,m \to \infty)
\]

This completes the proof. \( \square \)

2.2. Theorem. If \( A = (a_{jk}^{nm}) \) has the Borel property, then we have

\[
\sum_{j,k=1}^{\infty,\infty} (a_{jk}^{nm})^2 < \infty
\]

for each \( n,m \in \mathbb{N} \).

Proof. Let \( r_{jk}(x) = 2x_{jk} - 1 \) be the Rademacher functions for double sequences. We have

\[
\sum_{j,k=1}^{\infty,\infty} a_{jk}^{nm} x_{jk} = \frac{1}{2} \sum_{j,k=1}^{\infty,\infty} a_{jk}^{nm} + \frac{1}{2} \sum_{j,k=1}^{\infty,\infty} a_{jk}^{nm} r_{jk}(x)
\]

Since \( A \) has the Borel property and it follows from Theorem 2.1 that the series \( \sum_{j,k=1}^{\infty,\infty} a_{jk}^{nm} r_{jk}(x) \) converges for each \( n,m \in \mathbb{N} \) and almost all \( x \in X \). Furthermore we obtain \( \lim_{n,m} \sum_{j,k=1}^{\infty,\infty} a_{jk}^{nm} r_{jk}(x) = 0 \) for almost all \( x \in X \). So \( \left( \sum_{j,k=1}^{\infty,\infty} a_{jk}^{nm} r_{jk}(x) \right) \) is convergent uniformly on a set \( D \) with
positive measure for each \( n, m \in \mathbb{N} \) with respect to \( x \). Hence for each \( n, m \in \mathbb{N} \) and for every \( \varepsilon > 0 \), there exists \( N_1, N_2 \in \mathbb{N} \) such that for \( p, \mu > N_1 \) and \( q, \nu > N_2 \)

\[
\left| \sum_{j,k=1,1}^{p,q} a_{jk}^m r_{jk}(x) - \sum_{j,k=1,1}^{\mu,\nu} a_{jk}^m r_{jk}(x) \right| < \varepsilon.
\]

From the last inequality we immediately get

\[
(2.1) \quad \varepsilon^2 P(D) > \int_D \left( \sum_{E[\mu,\nu,v,q]} a_{jk}^m r_{jk}(x) \right)^2 dP(x)
\]

\[
= P(D) \sum_{E[\mu,\nu,v,q]} (a_{jk}^m)^2 + R
\]

where

\[
E[\mu, p; \nu, q] = \{(j, k) : \mu < j \leq p \text{ or } \nu < k \leq q\},
\]

\[R = 2 \sum_{I[\mu, p; \nu, q]} a_{jk}^m a_{j’k’}^m \int_D r_{j_1 k_1}(x) r_{j_2 k_2}(x) dP(x)\]

and \( I[\mu, p; \nu, q] = E[\mu, p; \nu, q] \cap \{(j, k) : j_1 \neq j_2 \text{ or } k_1 \neq k_2\} \). On the other hand using the Hölder inequality, we obtain

\[
|R| \leq 2 \left\{ \sum_{I[\mu, p; \nu, q]} (a_{jk}^m a_{j’k’}^m)^2 \right\}^{\frac{1}{2}} \left\{ \sum_{I[\mu, p; \nu, q]} \left( \int_D r_{j_1 k_1}(x) r_{j_2 k_2}(x) dP(x) \right)^2 \right\}^{\frac{1}{2}}.
\]

Let \( v_{j_1 k_1 j_2 k_2}^2 = \left( \int_D r_{j_1 k_1}(x) r_{j_2 k_2}(x) dP(x) \right)^2 \). From the Bessel inequality, we get

\[
\sum_{1 \leq j_1 < j_2 < \infty} v_{j_1 k_1 j_2 k_2}^2 \leq \int_X (\chi_D(x))^2 dP(x) = P(D).
\]

For sufficiently large \( p, q, \mu \) and \( \nu \), we have

\[
\left\{ \sum_{I[\mu, p; \nu, q]} v_{j_1 k_1 j_2 k_2}^2 \right\}^{\frac{1}{2}} \leq \frac{P(D)}{4}.
\]

Hence we obtain

\[
|R| \leq \frac{P(D)}{2} \left\{ \sum_{I[\mu, p; \nu, q]} (a_{jk}^m a_{j’k’}^m)^2 \right\}^{\frac{1}{2}}
\]

\[
\leq \frac{P(D)}{2} \left\{ \sum_{E[\mu, p; \nu, q]} (a_{jk}^m a_{j’k’}^m)^2 \right\}^{\frac{1}{2}}
\]

\[
\leq \frac{P(D)}{2} \sum_{E[\mu, p; \nu, q]} (a_{jk}^m)^2.
\]
From (2.1) and last inequality, it follows that
\[
\varepsilon^2 P(D) > P(D) \sum_{E[\mu, \nu, \nu, q]} (a_{jk}^m)^2 - \frac{P(D)}{2} \sum_{E[\mu, \nu, \nu, q]} (a_{jk}^m)^2
\]
\[
= \frac{P(D)}{2} \sum_{E[\mu, \nu, \nu, q]} (a_{jk}^m)^2.
\]
Also since \( P(D) > 0 \), we obtain \( \sum_{E[\mu, \nu, \nu, q]} (a_{jk}^m)^2 < 2\varepsilon^2 \). So for each \( n, m \in \mathbb{N} \), the series
\[
\left\{ \sum_{j, k=1, 1} (a_{jk}^m)^2 \right\}
\]
is convergent. Hence we obtain the result. \( \square \)

2.3. Theorem. If \( A = (a_{jk}^m) \) has the Borel property and satisfies \((v)\), we have
\begin{equation}
(2.2) \quad \sum_{j, k=1, 1} (a_{jk}^m)^2 = o(1), \quad (n, m \to \infty).
\end{equation}

Proof. Let \( \sigma_{nm} (x) = \sum_{j, k=1, 1} a_{jk}^m r_{jk} (x) \). Using the equality
\[
\sigma_{nm}^2 (x) = \left( \sum_{j, k=1, 1} a_{jk}^m r_{jk} (x) \right) \left( \sum_{j, k=1, 1} a_{jk}^m r_{jk} (x) \right)
\]
and \((v)\), we can easily obtain
\[
|\sigma_{nm}^2 (x)| \leq \sum_{j, k=1, 1} |a_{jk}^m| \sum_{j, k=1, 1} |a_{jk}^m| < \infty
\]
and hence
\[
\sigma_{nm}^2 (x) = \sum_{1 \leq j_1, j_2 \leq \infty, 1 \leq k_1, k_2 \leq \infty} a_{j_1 k_1}^m a_{j_2 k_2}^m r_{j_1 k_1} (x) r_{j_2 k_2} (x)
\]
is convergent uniformly almost everywhere. So we have
\begin{equation}
(2.3) \quad \int_X \sigma_{nm}^2 (x) dP (x) = \sum_{1 \leq j_1, j_2 \leq \infty, 1 \leq k_1, k_2 \leq \infty} a_{j_1 k_1}^m a_{j_2 k_2}^m \int_X r_{j_1 k_1} (x) r_{j_2 k_2} (x) dP (x)
\end{equation}
\[
= \sum_{j, k=1, 1} (a_{jk}^m)^2.
\]
Since \( A \) has the Borel property, the uniformly bounded sequence \( \{ \sigma_{nm} (x) \} \) converges to 0 for almost all \( x \). From (2.3) and the Lebesgue convergence theorem, it follows that
\[
\lim_{n, m \to \infty} \sum_{j, k=1, 1} (a_{jk}^m)^2 = 0.
\]
This completes the proof. \( \square \)
Now let us give sufficient conditions for the Borel property. First we consider the following sets

\[
D_0 (A) = \{ x \in X : (Ax)_{n,m} \text{ diverges} \}, \\
D_1 (A) = \{ x \in X : (Ax)_{n,m} \text{ converges} \}, \\
D_2 (A) = \left\{ x \in X : (Ax)_{n,m} \to \frac{1}{2} \ (n,m \to \infty) \right\}.
\]

We examine the relationship between these sets in the sense of $P$-measure.

2.4. Theorem. Let $A = (a_{jk}^{nm})$ be a 4-dimensional bounded regular matrix. The sets $D_1 (A)$ and $D_2 (A)$ have the same measure and the value is either 0 or 1.

Proof. Choose an arbitrary $x \in D_1 (A)$ (or $D_2 (A)$). Let $\hat{x}$ be a sequence obtained by altering a finite term of $x$. We have the following equality

\[
\sum_{j,k=1}^{\infty,\infty} a_{jk}^{nm} x_{jk} = \sum_{j,k=1}^{j_0,k_0} a_{jk}^{nm} x_{jk} + \sum_{j>j_0 \text{ veyka } k>k_0} a_{jk}^{nm} x_{jk} \\
\quad = \sum_{j,k=1}^{j_0,k_0} a_{jk}^{nm} x_{jk} + \sum_{j>j_0 \text{ veyka } k>k_0} a_{jk}^{nm} x_{jk}.
\]

From Proposition 1.2 (i), it follows $\hat{x} \in D_1 (A)$ (or $D_2 (A)$). Hence the sets $D_1 (A)$ and $D_2 (A)$ are homogeneous [14]. Since homogeneous sets have measure 0 or 1 and $D_2 (A) \subset D_1 (A)$, the proof will be completed if $P (D_1 (A)) = 1$ implies $P (D_2 (A)) = 1$.

On the other hand we have

\[
\lim_{n,m} \sum_{j,k=1}^{\infty,\infty} a_{jk}^{nm} x_{jk} = \lim_{n,m} \frac{1}{2} \sum_{j,k=1}^{\infty,\infty} a_{jk}^{nm} + \lim_{n,m} \frac{1}{2} \sum_{j,k=1}^{\infty,\infty} a_{jk}^{nm} r_{jk} (x)
\]

where $r_{jk} (x) = 2x_{jk} - 1$. If we choose $x \in D_1 (A)$, we get $\lim_{n,m} \sum_{j,k=1}^{\infty,\infty} a_{jk}^{nm} r_{jk} (x) = h (x)$ for almost all $x \in X$. From (v), interchanging integral and sum we have

\[
\int_X h (x) \, dx = \int_X \left( \lim_{n,m} \sum_{j,k=1}^{\infty,\infty} a_{jk}^{nm} r_{jk} (x) \right) \, dP (x) \\
\quad = \lim_{n,m} \int_X \left( \sum_{j,k=1}^{\infty,\infty} a_{jk}^{nm} r_{jk} (x) \right) \, dP (x) \\
\quad = \lim_{n,m} \sum_{j,k=1}^{\infty,\infty} a_{jk}^{nm} \left( \int_X r_{jk} (x) \, dP (x) \right) = 0.
\]

Hence we have $h (x) = 0$ for almost all $x \in X$. Also since first part of the right hand side of (2.4) is $\frac{1}{2}$ we get $x \in D_2 (A)$. This completes the proof. \hfill \Box

2.5. Corollary. Let $A = (a_{jk}^{nm})$ be a 4-dimensional bounded regular matrix. The set $D_0 (A)$ has measure 0 or 1.

2.6. Corollary. If $A = (a_{jk}^{nm})$ is a 4-dimensional bounded regular matrix sums almost all sequences of 0’s and 1’s, then the matrix has the Borel property.
2.7. Theorem. Let $A = (a_{jk}^{nm})$ be a 4-dimensional matrix. If $P(D_1(A)) = 1$, then we have

$$p_{nm} = \sum_{j,k=1}^{\infty} a_{jk}^{nm} \text{ converges for each } n, m \text{ and } \lim_{n,m} p_{nm} = p \text{ exists,}$$

$$\mathcal{A}_{nm} = \sum_{j,k=1}^{\infty} (a_{jk}^{nm})^2 < \infty \text{ for each } n, m.$$  

The proof of the theorem is similar to those of Theorems 2.1 and 2.2, and therefore is omitted.

2.8. Lemma. If $A$ satisfies condition (v), then we have

$$\int_X |\psi_{nm}(x)|^{2r} \, dP(x) \leq \frac{(2r)!}{2^r r!} (A_{nm})^r$$

where $r$ is a positive integer, $\psi_{nm}(x) = \sum_{j,k=1}^{\infty} a_{jk}^{nm} r_{jk}(x)$ and $A_{nm} = \sum_{j,k=1}^{\infty} (a_{jk}^{nm})^2$.

The proof can be proved using Lemma 1 of [13].

2.9. Theorem. If $A = (a_{jk}^{nm})$ satisfies (ii), (v) and the series

$$\sum_{n,m=1}^{\infty, \infty} \left( \sum_{j,k=1}^{\infty, \infty} (a_{jk}^{nm})^2 \right)^r$$

converges for some $r > 0$, then $A$ has the Borel property.

Proof. To complete the proof it is sufficient to show that

$$\sum_{j,k=1}^{\infty, \infty} a_{jk}^{nm} x_{jk} = \frac{1}{2} \sum_{j,k=1}^{\infty, \infty} a_{jk}^{nm} + \frac{1}{2} \sum_{j,k=1}^{\infty, \infty} a_{jk}^{nm} r_{jk}(x)$$

the limit of the right hand side of (2.7) equals $\frac{1}{2}$ for almost all $x \in X$. From Lemma 2.8, the inequality (2.5) holds for every positive integer $r$. On the other hand since the series in (2.6) converges for some $r > 0$, we easily get

$$\sum_{n,m=1}^{\infty, \infty} \int_X |\psi_{nm}(x)|^{2r} \, dP(x) < \infty.$$  

Using the Beppo-Levi theorem, we have $\sum_{n,m=1}^{\infty, \infty} |\psi_{nm}(x)|^{2r} < \infty$ for almost all $x \in X$. Hence we obtain for almost all $x \in X$ that

$$\lim_{n,m \to \infty} \psi_{nm}(x) = 0.$$  

This completes the proof. □

It is shown in [4] that the 4-dimensional Cesàro matrix method $(C, 1, 1)$ has the Borel property. We can also deduce this result from Theorem 2.9. We have already observed that (2.2) is a necessary condition for the Borel property. We raise the question whether the converse of Theorem 2.3 is true. The answer is no as the following example shows.

Since a 4-dimensional matrix can be considered as a matrix of infinite matrices, we can look at every entry as a matrix.
Consider the 4-dimensional Cesàro matrix, \((C, 1, 1) = (c^{nm}_{jk})\). Now we construct a 4-dimensional matrix \(A = (a^{nm}_{jk})\) as follows:

Shift the every column to the right in every possible order as the number of nonzero elements.

For example since there exist two possible order, we have

\[
\begin{align*}
(a^{11}_{jk}) &= \begin{bmatrix} 1 & 0 & 0 & \cdots \\
0 & 0 & 0 & \cdots \\
\vdots & & & \ddots 
\end{bmatrix}, & (a^{12}_{jk}) &= \begin{bmatrix} 0 & 1 & 0 & \cdots \\
& & & \ddots 
\end{bmatrix}, \\
(a^{21}_{jk}) &= \begin{bmatrix} \frac{1}{2} & 0 & 0 & \cdots \\
0 & \frac{1}{2} & 0 & \cdots \\
\vdots & & & \ddots 
\end{bmatrix}, & (a^{13}_{jk}) &= \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 & \cdots \\
& & & \ddots 
\end{bmatrix}, \\
(a^{14}_{jk}) &= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & \cdots \\
& & & \ddots 
\end{bmatrix}, & (a^{15}_{jk}) &= \begin{bmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & \cdots \\
& & & \ddots 
\end{bmatrix}. \\
(a^{16}_{jk}) &= \begin{bmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & \cdots \\
& & & \ddots 
\end{bmatrix}, & (a^{17}_{jk}) &= \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\
& & & \ddots 
\end{bmatrix}, \\
(a^{18}_{jk}) &= \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\
& & & \ddots 
\end{bmatrix}, \\
(a^{19}_{jk}) &= \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\
& & & \ddots 
\end{bmatrix}, & (a^{20}_{jk}) &= \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\
& & & \ddots 
\end{bmatrix}.
\end{align*}
\]

in the above we have six possible orders. Now let us obtain 

\[
\begin{align*}
(a^{21}_{jk}) &= \begin{bmatrix} \frac{1}{2} & 0 & \cdots \\
0 & \frac{1}{2} & 0 & \cdots \\
\vdots & & & \ddots 
\end{bmatrix}, & (a^{22}_{jk}) &= \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 & \cdots \\
& & & \ddots 
\end{bmatrix}, \\
(a^{23}_{jk}) &= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & \cdots \\
& & & \ddots 
\end{bmatrix}, & (a^{24}_{jk}) &= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & \cdots \\
& & & \ddots 
\end{bmatrix}, \\
(a^{25}_{jk}) &= \begin{bmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & \cdots \\
& & & \ddots 
\end{bmatrix}, & (a^{26}_{jk}) &= \begin{bmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & \cdots \\
& & & \ddots 
\end{bmatrix}. \\
(a^{27}_{jk}) &= \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\
& & & \ddots 
\end{bmatrix}, & (a^{28}_{jk}) &= \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\
& & & \ddots 
\end{bmatrix}, \\
(a^{29}_{jk}) &= \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\
& & & \ddots 
\end{bmatrix}, & (a^{30}_{jk}) &= \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\
& & & \ddots 
\end{bmatrix}.
\end{align*}
\]

Continuing this procedure we can construct the matrix \(A\).

Observe that the matrix \(A\) constructed above satisfies the condition (2.2).

Now let us consider the sequence \(\{x_{jk}\}\) having \((\eta \mu + p)\) times 1 ve \((\eta \mu - p)\) times 0 in the rectangle \((\eta, 2\mu)\).

In the case of \(p = 0\), an element of the matrix \(A\) which consists of \(0\)'s and \(\frac{1}{\eta \mu}\)'s sums \(\{x_{jk}\}\) to 0 and the another one sums to 1. Let these terms be \((n_0, m_0)\) and \((n_1, m_1)\) respectively.

If \((a^{n_0,m_0}_{jk})\) containing \(\frac{1}{\eta \mu}\)'s, such that all the \(0\)'s of the sequence in the rectangle \((\eta, 2\mu)\) correspond with \(\frac{1}{\eta \mu}\)'s, we have

\[
\sum_{j,k} x_{jk} = 0.
\]

Also if \((a^{n_1,m_1}_{jk})\) containing \(\frac{1}{\eta \mu}\)'s, such that all the \(1\)'s of the sequence in the rectangle \((\eta, 2\mu)\) correspond with \(\frac{1}{\eta \mu}\)'s, we have

\[
\sum_{j,k} x_{jk} = 1.
\]

In the case of \(p > 0\) there is an entry \((a^{n_0,m_0}_{jk})\) containing \(\frac{1}{\eta \mu}\)'s, such that all the \(1\)'s of the sequence in the rectangle \((\eta, 2\mu)\) correspond with \(\frac{1}{\eta \mu}\)'s, we have

\[
\sum_{j,k} x_{jk} = 1.
\]
Also there is another entry \( (a_{j,k}^{n_{1},m_{1}}) \) containing \( \frac{1}{\eta\mu} \)'s, such that all the 0's of the sequence in the rectangle \((\eta, 2\mu)\) correspond with \( \frac{1}{\eta\mu} \)'s, we have
\[
\sum_{j,k} a_{j,k}^{n_{1},m_{1}} x_{jk} = \frac{p}{\eta\mu}.
\]
In the case of \( p < 0 \) there is an entry \( (a_{j,k}^{n_{0},m_{0}}) \) containing \( \frac{1}{\eta\mu} \)'s, such that all the 0's of the sequence in the rectangle \((\eta, 2\mu)\) correspond with \( \frac{1}{\eta\mu} \)'s, we have
\[
\sum_{j,k} a_{j,k}^{n_{0},m_{0}} x_{jk} = 0.
\]
Also there is another entry \( (a_{j,k}^{n_{1},m_{1}}) \) containing \( \frac{1}{\eta\mu} \)'s, such that all the 1's of the sequence in the rectangle \((\eta, 2\mu)\) correspond with \( \frac{1}{\eta\mu} \)'s, we have
\[
\sum_{j,k} a_{j,k}^{n_{1},m_{1}} x_{jk} = 1 + \frac{p}{\eta\mu}.
\]
In any cases above, the oscillation of the sum \( \sum a_{j,k}^{n,m} x_{jk} \) in the inner matrix containing \( \frac{1}{\eta\mu} \)'s is at least \( 1 - |\frac{p}{\eta\mu}| \). In order that \( \{x_{jk}\} \) is \( A \)-summable we necessarily have \( \frac{|p|}{\eta\mu} \to 1 \), as \( \eta, \mu \to \infty \).

Since almost all double sequences of 0's and 1's is \((C, 1, 1)\)-summable to \( \frac{1}{2} \), the set of sequences which \( \frac{|p|}{\eta\mu} \) tends to 1 has \( P \)-measure 1. From this it follows that the set of sequences for which \( \frac{|p|}{\eta\mu} \) tends to 1 is of \( P \)-measure 0. Therefore, \( A \) does not have the Borel property. That is condition (2.2) can not be sufficient.

References