Sharp Wilker and Huygens type inequalities for trigonometric and hyperbolic functions

Yun Hua∗†

Abstract

In the article, some sharp Huygens and Wilker type inequalities involving trigonometric and hyperbolic functions are established.

Keywords: Huygens inequality, Wilker inequality, Trigonometric function, Hyperbolic function.

2000 AMS Classification: 26D05, 26D15, 33B10.

Received: 16.12.2013 Accepted: 14.05.2015 Doi: 10.15672/HJMS.20164513099

1. Introduction

The trigonometric and hyperbolic inequalities have been in recent years in the focus of many researchers. For many results and a long list of references we quote the papers [6, 10, 24], where many further references may be found. The following inequality

\[(1.1) \quad \left( \sin \frac{x}{x} \right)^2 + \tan \frac{x}{x} > 2, \quad 0 < x < \frac{\pi}{2}\]

is due to Wilker [13]. It has attracted attention of several researchers (see, e.g., [4], [7], [8], [9], [14], [15], [21]). A hyperbolic counterpart of Wilker’s inequality

\[(1.2) \quad \left( \sinh \frac{x}{x} \right)^2 + \tanh \frac{x}{x} > 2, \quad (x \neq 0)\]

has been established by L. Zhu [16].

In [12], it was proved that

\[(1.3) \quad 2 + \frac{8}{45} x^3 \tan x > \left( \sin \frac{x}{x} \right)^2 + \tan \frac{x}{x} > 2 + \left( \frac{2}{\pi} \right)^4 x^3 \tan x, \quad \text{for} \quad 0 < x < \frac{\pi}{2}. \quad \text{The constants} \quad \frac{8}{45} \quad \text{and} \quad \left( \frac{2}{\pi} \right)^4 \quad \text{in the inequality} \quad (1.3) \quad \text{are the best possible.}

∗ Department of Information Engineering, Weihai Vocational College, Weihai City 264210, ShanDong province, P. R. CHINA., Email: xxgcxhy@163.com
† Corresponding Author.
The famous Huygens inequality\[11\] for the sine and tangent functions states that for\(x \in \left(0, \frac{\pi}{2}\right)\)
\[
2 \sin x + \tan x > 3x. 
\]
(1.4)

The hyperbolic counterpart of (1.4) was established in [6] as follows: For \(x > 0\)
\[
2 \sinh x + \tanh x > 3x. 
\]
(1.5)

The inequalities (1.4) and (1.5) were respectively refined in [6, Theorem 2.6] as
\[
2 \frac{\sin x}{x} + \frac{\tan x}{x} > 2 \frac{x}{\sin x} + \frac{x}{\tan x} > 3, 
\]
(1.6)

and
\[
2 \frac{\sinh x}{x} + \frac{\tanh x}{x} > 2 \frac{x}{\sinh x} + \frac{x}{\tanh x} > 3, \quad x \neq 0. 
\]
(1.7)

In the most recent paper [5], the inequalities (1.6), (1.7) and (1.1) were respectively further refined as
\[
2 \frac{\sin x}{x} + \frac{\tan x}{x} > \frac{\sin x}{x} + 2 \frac{\tan(x/2)}{x/2} > 2 \frac{x}{\sin x} + \frac{x}{\tan x} > 3. 
\]
(1.8)

and
\[
2 \frac{\sinh x}{x} + \frac{\tanh x}{x} > \frac{\sinh x}{x} + 2 \frac{\tanh(x/2)}{x/2} > 2 \frac{x}{\sinh x} + \frac{x}{\tanh x} > 3. 
\]
(1.9)

and
\[
\left( \frac{\sin x}{x} \right)^2 + \frac{\tan x}{x} > \left( \frac{x}{\sin x} \right)^2 + \frac{x}{\tan x} > \frac{\sin x}{x} + \left( \frac{\tan(x/2)}{x/2} \right)^2 
\]
(1.10)

The hyperbolic counterparts of the last two inequalities in (1.10) were also given in [5] as follows:
\[
\frac{\sinh x}{x} + \left[ \frac{\tanh(x/2)}{x/2} \right]^2 > \frac{x}{\sinh x} \left[ \frac{x/2}{\tanh(x/2)} \right]^2 > 2. 
\]
(1.11)

Inspired by (1.3), Jiang et al. [19] first proved
\[
3 + \frac{1}{60} x^3 \sin x < 2 \frac{x}{\sin x} + \frac{x}{\tan x} < 3 + \frac{8\pi - 24}{\pi^3} x^3 \sin x. 
\]
(1.12)

and
\[
2 + \frac{17}{720} x^3 \tan x < \frac{x}{\sin x} + \left( \frac{\tan\frac{x}{2}}{2} \right)^2 < 2 + \frac{\pi^2 + 8\pi - 32}{2\pi^3} x^3 \tan x. 
\]
(1.13)

holds for \(0 < |x| < \frac{\pi}{2}\). Furthermore the constants \(\frac{1}{60}, \frac{8\pi - 24}{\pi^3}\) in (1.12) and the constants \(\frac{17}{720}, \frac{\pi^2 + 8\pi - 32}{2\pi^3}\) in (1.13) are the best possible.

Recently, Chen and Sándor [20] proved that
\[
3 + \frac{3}{20} x^3 \tan x < 2 \left( \frac{\sin x}{x} \right) + \frac{\tan x}{x} < 3 + \left( \frac{2}{\pi} \right)^4 x^3 \tan x. 
\]
for \(0 < |x| < \frac{\pi}{2}\). The constants \(\frac{3}{20}\) and \(\left( \frac{2}{\pi} \right)^4\) are the best possible.

This paper is a continuation of our work [25] and is organized as follows. In Section 2, we give some lemmas and preliminary results. In Section 3, we prove some new sharp Wilker- and Huygens-type inequalities for trigonometric and hyperbolic functions.
2. some Lemmas

In order to establish our main result we need several lemmas, which we present in this section.

2.1. Lemma. The Bernoulli numbers $B_{2n}$ for $n \in \mathbb{N}$ have the property

\[(2.1) \quad (-1)^{n-1}B_{2n} = |B_{2n}|,\]

where the Bernoulli numbers $B_i$ for $i \geq 0$ are defined by

\[(2.2) \quad x e^x - 1 = \sum_{i=0}^{\infty} B_i \frac{x^i}{i!} = 1 - \frac{x}{2} + \sum_{i=1}^{\infty} B_{2i} \frac{x^{2i}}{(2i)!}, \quad |x| < 2\pi.\]

Proof. In [2, p. 16 and p. 56], it is listed that for $q \geq 1$

\[(2.3) \quad \zeta(2q) = (-1)^{q-1} \frac{(2\pi)^{2q} B_{2q}}{(2q)!}.\]

where $\zeta$ is the Riemann zeta function defined by

\[\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.\]

In [22, p.18, theorem 3.4], the following formula was given

\[(2.4) \quad \sum_{n=1}^{\infty} \frac{1}{n^{2q}} = \frac{2^{2q-1} x^{2q} |B_{2q}|}{(2q)!}.\]

From (2.3) and (2.4), the formula (2.1) follows. \qed

2.2. Lemma. [17, 18] Let $B_{2n}$ be the even-indexed Bernoulli numbers. Then

\[(2.5) \quad \frac{2(2n)!}{(2\pi)^{2n}} \frac{1}{1 - 2^{-2n}} < |B_{2n}| < \frac{2(2n)!}{(2\pi)^{2n}} \frac{1}{1 - 2^{-2n}}, n = 1, 2, 3, \ldots.\]

2.3. Lemma. For $0 < |x| < \pi$, we have

\[(2.6) \quad \frac{x}{\sin x} = 1 + \sum_{n=1}^{\infty} \frac{2(2^{2n-1} - 1)|B_{2n}|}{(2n)!} x^{2n}.\]

Proof. This is an easy consequence of combining the equality

\[(2.7) \quad \csc x = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 2(2^{2n-1} - 1)B_{2n}}{(2n)!} x^{2n-1},\]

see [1, p. 75, 4.3.68], with Lemma 2.1. \qed

2.4. Lemma ([1, p. 75, 4.3.70]). For $0 < |x| < \pi$,

\[(2.8) \quad \cot x = \frac{1}{x} - \sum_{n=1}^{\infty} \frac{2^{2n} |B_{2n}|}{(2n)!} x^{2n-1}.\]

The following Lemma 2.5 and Lemma 2.6 can be found in [25].

2.5. Lemma. For $0 < |x| < \pi$,

\[(2.9) \quad \frac{1}{\sin^2 x} = \frac{1}{x^2} + \sum_{n=1}^{\infty} \frac{2^{2n} (2n-1) |B_{2n}|}{(2n)!} x^{2(n-1)}.\]
2.6. Lemma. For $0 < |x| < \pi$,
\[
\cos x = \frac{1}{x^2} - \frac{2(2n-1)(2^{2n-1}-1)|B_{2n}|}{(2n)!} x^{2(n-1)}.
\]

2.7. Lemma. For $0 < |x| < \pi$,
\[
\frac{1}{\sin^3 x} = \frac{1}{x^3} + \frac{1}{2} \sum_{n=2}^{\infty} \frac{2^{2n}-2}{(2n)!} |B_{2n}| (2n-1)(2n-2)x^{2n-3} \]
\[
+ \frac{1}{2x} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{2^{2n}-2}{(2n)!} |B_{2n}| x^{2n-1},
\]
and
\[
\frac{\cos x}{\sin^3 x} = \frac{1}{x^3} - \frac{1}{2} \sum_{n=2}^{\infty} \frac{(2n-1)(n-1)2^{2n} |B_{2n}|}{(2n)!} x^{2n-3}.
\]

Proof. Combining
\[
\frac{1}{\sin^3 x} = \frac{1}{2\sin x} - \frac{1}{2} \left( \frac{\cos x}{\sin^2 x} \right)'
\]
with Lemma 2.6, the identity (2.6), and Lemma 2.1 gives (2.10).

The equality (2.11) follows from combination of
\[
\frac{\cos x}{\sin^3 x} = -\frac{1}{2} \left( \frac{1}{\sin^2 x} \right)'
\]
with Lemma 2.5.

2.8. Lemma. [23, 3, 15] Let $a_n$ and $b_n (n = 0, 1, 2, \cdots)$ be real numbers, and let the power series $A(t) = \sum_{n=0}^{\infty} a_n t^n$ and $B(t) = \sum_{n=0}^{\infty} b_n t^n$ be convergent for $|t| < R$. If $b_n > 0$ for $n = 0, 1, 2, \cdots$, and if \( a_n \) is strictly increasing (or decreasing) for $n = 0, 1, 2, \cdots$, then the function $A(t)^{B(t)}$ is strictly increasing (or decreasing) on $(0, R)$.

3. Main results

Now we are in a position to state and prove our main results.

3.1. Theorem. For $0 < |x| < \frac{\pi}{2}$, we have
\[
2 + \frac{23}{720} x^3 \sin x < \frac{\sin x}{x} + \left( \frac{\tan \frac{x}{2}}{\frac{\pi}{2}} \right)^2 < 2 + \frac{128 - 16\pi^2 + 16\pi^2 x^3 \sin x}{\pi^3}.
\]

The constants $\frac{23}{720}$ and $\frac{128 - 16\pi^2 + 16\pi^2 x^3 \sin x}{\pi^3}$ in (3.1) are the best possible.

Proof. Let
\[
f(x) = \frac{\sin x}{x^3 \sin x} + \left( \frac{\tan \frac{x}{2}}{\frac{\pi}{2}} \right)^2 - 2
\]
\[
= \frac{x \sin^3 x - 8 \cos x - 4 \sin^2 x - 2x^2 \sin^2 x + 8}{x^3 \sin^3 x}
\]
\[
= \frac{1}{x^5} \left( x + \frac{8 \cos x}{\sin^3 x} - \frac{8 \cos x}{\sin^3 x} \right) - \frac{4}{\sin x} - \frac{2x^2}{\sin x}
\]
for $x \in \left( 0, \frac{\pi}{2} \right)$. By virtue of (2.10), (2.11), and (2.6), we have
3.3. Theorem. For all $x$ since

$$f(x) = \frac{1}{x^3} \left[ x + \frac{8}{x^3} + \sum_{n=2}^{\infty} \frac{4(2n-1)(2n-2)(2^{2n}-2)}{(2n)!} |B_{2n}| x^{2n-3} \right]$$

$$+ \frac{4}{x} + \sum_{n=1}^{\infty} \frac{4(2^2n-2)}{(2n)!} |B_{2n}| x^{2n-1}$$

$$- \frac{8}{x^3} + \sum_{n=2}^{\infty} \frac{8 \cdot 2^{2n}(2n-1)(n-1)}{(2n)!} |B_{2n}| x^{2n-3}$$

$$- \frac{4}{x} - \sum_{n=1}^{\infty} \frac{4(2^2n-2)}{(2n)!} |B_{2n}| x^{2n-1}$$

$$- 2x - \sum_{n=2}^{\infty} \frac{2(2^2n-2)}{(2n)!} |B_{2n}| x^{2n+1} \right]$$

$$= \frac{1}{x^3} \left[ -x + \sum_{n=2}^{\infty} \frac{16(2n-1)(n-1)(2^{2n}-1)}{(2n)!} |B_{2n}| x^{2n-3} - \sum_{n=1}^{\infty} \frac{2(2^2n-2)}{(2n)!} |B_{2n}| x^{2n+1} \right]$$

$$= \frac{1}{x^3} \left[ \sum_{n=3}^{\infty} \frac{16(2n-1)(n-1)(2^{2n}-1)}{(2n)!} |B_{2n}| x^{2n-3} - \sum_{n=1}^{\infty} \frac{2(2^2n-2)}{(2n)!} |B_{2n}| x^{2n+1} \right]$$

$$= \frac{1}{x^3} \left[ \sum_{n=2}^{\infty} \frac{16(2n+3)(n+1)(2^{2n+4}-1)}{(2n+4)!} |B_{2n+4}| x^{2n+1} - \sum_{n=1}^{\infty} \frac{2(2^2n-2)}{(2n)!} |B_{2n}| x^{2n+1} \right]$$

$$= \frac{1}{x^3} \sum_{n=2}^{\infty} \frac{16(2n+3)(n+1)(2^{2n+4}-1)}{(2n+4)!} |B_{2n+4}| x^{2n+1} - \sum_{n=1}^{\infty} \frac{2(2^2n-2)}{(2n)!} |B_{2n}| x^{2n+1}$$

Let $a_n = \frac{16(2n+3)(n+1)(2^{2n+4}-1)}{(2n+4)!} |B_{2n+4}| - \frac{2(2^2n-2)}{(2n)!} |B_{2n}|$ for $n \geq 2$.

By a simple computation, we have $a_2 = \frac{23}{720}$. Furthermore, when $n \geq 3$, from Lemma 2.2 one can get

$$a_n = \frac{16(2n+3)(n+1)(2^{2n+4}-1)}{(2n+4)!} |B_{2n+4}| - \frac{2(2^2n-2)}{(2n)!} |B_{2n}|$$

$$> 16(2n+3)(n+1)(2^{2n+4}-1) \cdot \frac{2(2n+4)!}{(2n+4)!} \cdot \frac{2(2n+4)!}{(2n)!} \cdot \frac{1}{(2^2n+4) - 2n - 4}$$

$$- \frac{2(2^2n-2)}{(2n)!} \cdot \frac{2(2n)!}{(2n+4)!} \cdot \frac{1}{(2^2n+4) - 2n - 4}$$

$$= \frac{4}{(2n)!} \left[ \frac{8(2n+3)(n+1)}{(2n+4)!} - 1 \right] > 0.$$ 

So the function $f(x)$ is strictly increasing on $(0, \frac{\pi}{2})$. Moreover, it is easy to obtain

$$\lim_{x \to 0^+} f(x) = a_2 = \frac{23}{720} \quad \text{and} \quad \lim_{x \to (\pi/2)^-} f(x) = \frac{128 - 16\pi^2 + 16\pi}{\pi^5}.$$ 

The proof of Theorem 3.1 is complete.

3.2. Remark. Since $f(x)$ is an even function we conclude that Theorem 3.1 holds for all $x$ which satisfy $0 < |x| < \frac{\pi}{2}$.

3.3. Theorem. For $x \neq 0$, we have

$$3 + \frac{1}{40} x^3 \tanh x < \frac{\sinh x}{x} + 2 \left(\frac{\tanh \frac{x}{2}}{\frac{x}{2}}\right) < 3 + \frac{1}{40} x^3 \sinh x.$$
The constant \( \frac{1}{40} \) is the best possible.

**Proof.** Without loss of generality, we assume that \( x > 0 \).

We firstly prove the first inequality of (3.2).

Consider the function \( F(x) \) defined by

\[
F(x) = \frac{\sinh x + 2 \tanh x}{x^3 \tanh x} - 3 = \frac{\cosh 3x - 17 \cosh x + 8 \cosh 2x - 6x \sinh 2x + 8}{2x^4 (\cosh 2x - 1)}.
\]

and let \( f(x) = \cosh 3x - 17 \cosh x + 8 \cosh 2x - 6x \sinh 2x + 8 \) and \( g(x) = 2x^4 (\cosh 2x - 1) \).

From the power series expansions

\[
\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \quad \text{and} \quad \cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!},
\]

it follows that

\[
f(x) = \cosh 3x - 17 \cosh x + 8 \cosh 2x - 6x \sinh 2x + 8
\]

\[
= \sum_{n=0}^{\infty} \frac{3^{2n} x^{2n}}{(2n)!} - \sum_{n=0}^{\infty} \frac{17 x^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{2^{2n+3} x^{2n}}{(2n)!} - \sum_{n=0}^{\infty} \frac{6 \cdot 2^{2n+1} x^{2n+2}}{(2n+1)!} + 8
\]

\[
= \sum_{n=0}^{\infty} \frac{(3^{2n} + 2^{2n+3} - 17) x^{2n}}{(2n)!} - \sum_{n=0}^{\infty} \frac{6 \cdot 2^{2n+1} x^{2n+2}}{(2n+1)!} + 8
\]

\[
= \sum_{n=0}^{\infty} \frac{(3^{2n} + 2^{2n+3} - 17) x^{2n}}{(2n)!} - \sum_{n=0}^{\infty} \frac{6n 2^{2n} x^{2n}}{(2n)!}
\]

\[
= \sum_{n=3}^{\infty} \frac{3^{2n} + 2^{2n+3} - 17 - 6n 2^{2n}}{(2n)!} = \sum_{n=3}^{\infty} a_n x^{2n}
\]

and

\[
g(x) = 2x^4 (\cosh 2x - 1)
\]

\[
= \sum_{n=1}^{\infty} \frac{2^{2n+4}}{(2n)!}
\]

\[
= \sum_{n=3}^{\infty} \frac{4n(n-1)(2n-3)(2n-1)2^{2n-3} x^{2n}}{(2n)!}
\]

\[
= \sum_{n=3}^{\infty} b_n x^{2n}.
\]

It is easy to see that the quotient

\[
c_n = \frac{a_n}{b_n} = \frac{3^{2n} + 2^{2n+3} - 17 - 6n 2^{2n}}{4n(n-1)(2n-3)(2n-1)2^{2n-3}}
\]

satisfies \( c_3 = \frac{1}{45}, c_4 = \frac{21}{1120}, c_5 = \frac{507}{507000} \) and

\[
c_{n+1} - c_n = \frac{f_1 + f_2 + f_3}{2n(2n+3)(4n^2-1)(n^2-1)}, (n \geq 6),
\]
where
\[ f_1 = \left( \frac{9}{4} \right)^n (10n^2 - 57n + 23) = \left( \frac{9}{4} \right)^n (10n(n-6) + 3(n-6) + 41) > 0, \]
\[ f_2 = \frac{1}{4^n} (102n^2 + 298n + 17) > 0, \]
\[ f_3 = 144n^2 - 184n - 8 = 144(n-6) + 680(n-6) + 4072 > 0. \]

for \( n \geq 6 \). This means that the sequence \( c_n \) is increasing. By Lemma 2.8, the function \( F(x) \) is increasing on \( (0, \infty) \). Moreover, it is not difficult to obtain \( \lim_{x \to 0^+} F(x) = c_3 = \frac{140}{3} \). Therefore, the first inequality in (3.2) holds.

Finally, we prove the second inequality of (3.2).

Define a function \( G(x) \) by
\[
G(x) = \frac{\sinh x}{x} + 2\frac{\tanh \frac{x}{2}}{x} - 3
\]
\[
= \frac{\cosh 2x + 8 \cosh x - 6x \sinh x - 9}{x^4(\cosh 2x - 1)}.
\]

and let
\[ f(x) = \cosh 2x + 8 \cosh x - 6x \sinh x - 9 \quad \text{and} \quad g(x) = x^4(\cosh 2x - 1). \]

By using (3.3), it follows that
\[
f(x) = \cosh 2x + 8 \cosh x - 6x \sinh x - 9
\]
\[
= \sum_{n=0}^{\infty} \frac{2^{2n} x^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{8x^{2n}}{(2n)!} - \sum_{n=0}^{\infty} \frac{6x^{2n+2}}{(2n+1)!} + 9
\]
\[
= \sum_{n=1}^{\infty} (2^{2n} + 8) x^{2n} - \sum_{n=0}^{\infty} 6x^{2n+2} - \sum_{n=1}^{\infty} (2n+1)!
\]
\[
= \sum_{n=1}^{\infty} (2^{2n} + 8) x^{2n} - \sum_{n=0}^{\infty} 12n x^{2n} - \sum_{n=1}^{\infty} (2n+1)!
\]
\[
= \sum_{n=3}^{\infty} (2^{2n} + 8 - 12n) x^{2n}
\]
\[ \triangleq \sum_{n=3}^{\infty} a_n x^{2n} \]

and
\[
g(x) = x^4(\cosh 2x - 1)
\]
\[
= \sum_{n=1}^{\infty} \frac{2^{2n} x^{2n+4}}{(2n)!}
\]
\[
= \sum_{n=3}^{\infty} \frac{4n(n-1)(2n-1)(2n-3)2^{2n-4} x^{2n}}{(2n)!}
\]
\[ \triangleq \sum_{n=3}^{\infty} b_n x^{2n}.
\]

Let
\[ c_n = \frac{a_n}{b_n} = \frac{2^{2n} - 12n + 8}{4n(n-1)(2n-1)(2n-3)2^{2n-4}} \]
satisfies $c_3 = \frac{1}{40}$. Furthermore, when $n \geq 3$, by a simple computation, we have

$$c_{n+1} - c_n = -4 \frac{(8n - 2)4^n - (18n^3 + 33n^2 - 16n - 11)}{n(2n - 3)(4n^2 - 1)(n^2 - 1)4^n},$$

for $n \geq 3$.

Since

$$(8n - 2)4^n - (18n^3 + 33n^2 - 16n - 11)$$

$$> (8n - 2)4^n - (18n^3 + 33n^2 - 16n - 11)$$

$$= 14n^3 - 41n^2 + 16n + 11$$

$$= 14n(n - 3)^2 + 43n(n - 3) + 19(n - 3) + 68 > 0.$$

This means that the sequence $c_n$ is decreasing. By Lemma 2.8, the function $G(x)$ is decreasing on $(0, \infty)$. Moreover, it is not difficult to obtain $\lim_{x \to 0^+} G(x) = c_3 = \frac{1}{40}$.

This completes the proof of Theorem 3.3.

□

3.4. Remark. Since $F(x)$ and $G(x)$ both are even functions, we conclude that Theorem 3.3 holds for all $x \neq 0$.

3.5. Theorem. For $x \neq 0$,

$$2 + \frac{23}{720} x^3 \tanh x < \frac{\sinh x}{x} + \left[ \frac{\tanh(x/2)}{x/2} \right]^2 < 2 + \frac{23}{720} x^3 \sinh x. \tag{3.4}$$

The both constants $\frac{23}{720}$ in (3.4) are the best possible.

Proof. The left-hand side of inequality in (3.4) has been proved in [19], so we only need to prove the right-hand side of the inequality in (3.4).

Without loss of generality, we assume that $x > 0$.

Consider the function $H(x)$ defined by

$$H(x) = \frac{\sinh x}{x} + \frac{[\tanh(x/2)]^2}{x^2} - 2$$

$$= \frac{x \sinh x \cosh x + x \sinh x + 4 \cosh x - 2x^2 \cosh x - 2x^2 - 4}{x^5 \sinh x (1 + \cosh x)}$$

and let

$$f(x) = x \sinh x \cosh x + x \sinh x + 4 \cosh x - 2x^2 \cosh x - 2x^2 - 4$$

and

$$g(x) = x^5 \sinh x (1 + \cosh x).$$

By the power series expansions in (3.3), we obtain
\[ f(x) = x \sinh x \cosh x + x \sinh x + 4 \cosh x - 2x^2 \cosh x - 2x^2 - 4 \]
\[ = \sum_{n=0}^{\infty} \frac{2^{2n}}{(2n+1)!} x^{2n+2} + \sum_{n=0}^{\infty} \frac{x^{2n+2}}{(2n)!} + \sum_{n=0}^{\infty} \frac{4x^{2n}}{(2n)!} - \sum_{n=0}^{\infty} \frac{2x^{2n+2}}{(2n)!} - 2x^2 - 4 \]
\[ = \sum_{n=0}^{\infty} \frac{2^{2n} + 1 - 2(2n+1)}{(2n+1)!} x^{2n+2} + \sum_{n=0}^{\infty} \frac{4}{(2n)!} x^{2n} \]
\[ = \sum_{n=1}^{\infty} \frac{2^{2n-2} + 1 - 2(2n-1)}{(2n-1)!} x^{2n} + \sum_{n=2}^{\infty} \frac{4}{(2n)!} x^{2n} \]
\[ = \sum_{n=3}^{\infty} \frac{2n(2^{2n-2} - 4n + 3) + 4}{(2n)!} x^{2n} \]
\[ \equiv \sum_{n=3}^{\infty} a_n x^{2n} \]

and
\[ g(x) = x^5 \left[ \frac{1}{2} \sinh(2x) + \sinh x \right] \]
\[ = \sum_{n=0}^{\infty} \frac{1 + 2^{2n}}{(2n+1)!} x^{2n+6} = \sum_{n=3}^{\infty} \frac{1 + 2^{2n-6}}{(2n-5)!} x^{2n} \]
\[ = \sum_{n=3}^{\infty} \frac{(1 + 2^{2n-6})(2n-4)(2n-3)(2n-2)(2n-1)}{(2n)!} 2n x^{2n} \]
\[ \equiv \sum_{n=3}^{\infty} b_n x^{2n} \]

Let
\[ c_n = \frac{a_n}{b_n} = \frac{2n(2^{2n-2} - 4n + 3) + 4}{(1 + 2^{2n-6})(2n-4)(2n-3)(2n-2)(2n-1)2n} \]
satisfies
\[ c_3 = \frac{23}{720} = 0.031 \ldots \]
\[ c_4 = \frac{17}{336} = 0.01226 \ldots \]

Furthermore, when \( n \geq 4 \), by a simple computation, we have
\[ c_{n+1} - c_n = -4 \frac{f_1(n) + f_2(n) + f_3(n)}{n(16 + 4^n)(64 + 4^n)(n-2)(2n-3)(4n^2-1)(n^2-1)}, \]
where
\[ f_1(n) = 16^n \left( 8n^2 + 2n - 6 \right) \]
\[ f_2(n) = 4^n \left( -24n^4 - 138n^3 + 391n^2 + 153n - 382 \right) \]
\[ f_3(n) = -1536n^3 - 256n^2 + 2944n - 256 \]

Since \( n \geq 4 \), one can easily check that \( 4^n \geq 16n^2 \), this implies that
\[ f_1(n) + f_2(n) > 4^n 16n^2(8n^2 + 2n - 6) + 4^n \left( -24n^4 - 138n^3 + 391n^2 + 153n - 382 \right) \]
\[ = 4^n (104n^4 - 106n^3 + 295n^2 + 153n - 382) \]
By a simple computation, one has
\[ 104n^4 - 106n^3 + 295n^2 + 153n - 382 = 104(n - 4)^3 + 1142(n - 4)^2 + 4439(n - 4) + 6293(n - 4) + 24790 > 0. \]

On the other hand, when \( n \geq 4 \), one has \( 4^n > 16 \), Hence
\[
f_1(n) + f_2(n) + f_3(n) > 4^n(104n^4 - 106n^3 + 295n^2 + 153n - 382) - 1536n^3 - 256n^2 + 2944n - 256 \]
\[ = 1664n^4 - 3323n^3 + 4464n^2 + 5392n - 6368 \]
\[ = 1664(n - 4)^3 + 16736(n - 4)^2 + 58480(n - 4) + 78032(n - 4) + 305760 > 0. \]

This means that the sequence \( c_n \) is decreasing. By Lemma 2.8, the function \( H(x) \) is decreasing on \( (0, \infty) \). Moreover, it is not difficult to obtain \( \lim_{x \to 0^+} H(x) = c_3 = \frac{23}{720} \). □

3.6. Remark. Since \( H(x) \) is an even function, we conclude that Theorem 3.5 holds for all \( x \neq 0 \).

ACKNOWLEDGEMENTS The author is deeply indebted to the editor and the referee for useful suggestions and valuable comments.

References


