Morita theory for group corings

Quanguo Chen*, Dingguo Wang † and M. Liu§

Abstract

Using the theory of group corings, we study (graded) Morita contexts associated to a comodule over a group coring, which generalize and unify some classical morita contexts. Some applications of our theory are also discussed.

Keywords: group coring, Morita context, cofree.

2000 AMS Classification: 16W30.

Received: 27.05.2014 Accepted: 29.09.2014 Doi: 10.15672/HJMS.2015449670

1. Introduction

An A-coring is a coalgebra in the monoidal category of A-bimodules over an arbitrary ring A. The concept was introduced by M. Sweedler [14]. In 2000, Takeuchi pointed out that to each entwining structure \((A, C, \psi)\) over a commutative ring k, which was introduced by T. Brzezinski and S. Majid [2], there corresponds an A-coring structure on \(C := A \otimes_k C\). This motivated the revival of the theory of corings and comodules and Brzezinski’s paper [3] was the engine behind the revival of the theory of corings and comodules over corings. Many examples of classical categories in noncommutative algebra are special cases of comodules over corings. Let us mention a few of them: the category of a descent datum of a ring extension, graded modules, Hopf modules, Long dimodules, Yetter-Drinfeld modules, Doi-Koppinen modules or entwined modules, and several other categories studied earlier by Hopf algebrists.

One of the important observations is that coring theory provides an elegant approach to descent theory and Galois theory. A systematic study of coring has been carried out in [1, 3, 6, 7, 15]. As the generalization of coring, Caenepeel, Janssen and Wang [5] introduced the group coring and developed Galois theory for group corings.

*School of Mathematics and Statistics, Yili Normal University Xinjiang, Yining 835000, China. Email: cqg2110163.com
†Corresponding Author.
‡School of Mathematical Sciences, Qufu Normal University Shandong, Qufu 273165, China. Email: dingguo95@126.com
§School of Mathematics and Statistics, Yili Normal University Xinjiang, Yining 835000, China.
It is well-known that the Morita context plays an important role in the theory of Hopf algebras. The first Morita context was constructed by Chase and Sweedler [9], which was generalized by Doi [12]. Morita contexts similar to the one of Doi were studied by Cohen, Fischman and Montgomery in [11]. As the generalization of both contexts, Caenepeel, Vercruysse and Wang associate different types of Morita contexts to a coring with a fixed grouplike element, which was generalized by Caenepeel, Janssen and Wang to group coring with a grouplike family [5]. Without the assumption of a coring with a fixed grouplike element, Caenepeel, De Groot and Vercruysse associated a Morita context to a comodule over a coring in [8]. Morita theory for group corings with fixed grouplike family is a remarkable tool to discuss Hopf-Galois extensions. In order to further discuss coalgebra-Galois extensions, we need to generalize the Morita context for group corings. Naturally, it occurs to us how to develop (graded) Morita context associated to a comodule over a group coring. This is the motivation of this paper.

The paper is organized as follows.

In Section 2, we recall some basic definitions such as group corings, comodules over a group coring and graded Morita contexts. In Section 3, we associate a Morita context to group coring with a grouplike family [5]. Without the assumption of a coring with a fixed grouplike element, which was generalized by Caenepeel, Janssen and Wang to group coring with a group coring, we need to generalize the Morita context for group corings. Morita theory for group corings with fixed grouplike family is an important tool to discuss Hopf-Galois extensions. In order to further discuss coalgebra-Galois extensions, we need to generalize the Morita context for group corings. Naturally, it occurs to us how to develop (graded) Morita context associated to a comodule over a group coring. This is the motivation of this paper.

2. Preliminaries

Throughout this paper, let \( G \) be a group with unit \( e \), and \( A \) a ring with unit \( 1_A \), and \( M \) an \( A \)-module. We will often need collections of \( A \)-modules isomorphic to \( M \) and indexed by \( G \). We will consider these modules as isomorphic, but distinct. Let \( M \times \{ \alpha \} \) be the module with index \( \alpha \). We then have isomorphisms

\[
\mu_\alpha : M \to M \times \{ \alpha \}, \quad \mu_\alpha(m) = (m, \alpha).
\]

We can then write \( M \times \{ \alpha \} = \mu_\alpha(M) \). \( \mu \) can be considered as a dummy variable, and we will also use the symbols \( \nu, \kappa, \cdots \). We will identify \( M \) and \( M \times \{ e \} \) using \( \mu_e \).

2.1. Group Corings. Let \( A \) be an algebra. Recall from [5] that a \( G \)-group \( A \)-coring (or shortly a \( G \)-coring) \( C \) is a family \( \{ C_\alpha \}_{\alpha \in G} \) of \( A \)-bimodules together with a family of \( A \)-bimodule maps

\[
\Delta_{\alpha, \beta} : C_{\alpha \beta} \to C_\alpha \otimes_A C_\beta, \quad \varepsilon : C_e \to A
\]

such that the following conditions hold:

\[
(\Delta_{\alpha, \beta} \otimes_A id) \circ \Delta_{\alpha, \beta, \gamma} = (id \otimes_A \Delta_{\beta, \gamma}) \circ \Delta_{\alpha, \beta, \gamma},
\]

\[
(id \otimes_A \varepsilon) \circ \Delta_{\alpha, e} = id = (\varepsilon \otimes_A id) \circ \Delta_{e, \alpha}
\]

for all \( \alpha, \beta, \gamma \in G \).

For a \( G \)-coring \( C \), we also use the following Sweedler-type notation for the comultiplication maps \( \Delta_{\alpha, \beta} \):

\[
\Delta_{\alpha, \beta}(c) = c_{(1, \alpha)} \otimes_A c_{(2, \beta)}
\]

for all \( c \in C_{\alpha \beta} \).

A morphism between two \( G \)-coring \( C \) and \( D \) consists of a family of \( A \)-bimodule maps \( f = \{ f_\alpha : C_\alpha \to D_\alpha \}_{\alpha \in G} \) such that

\[
(f_\alpha \otimes_A f_\beta) \circ \Delta_{\alpha, \beta} = \Delta_{\alpha, \beta} \circ f_{\alpha \beta}, \quad \varepsilon \circ f_e = \varepsilon.
\]

Over a \( G \)-coring \( C \), we can define two different types of comodules. A right \( C \)-comodule is a right \( A \)-module \( M \) together with a family of right \( A \)-linear maps \( \rho^M = \{ \rho^M_\alpha : M \to M \otimes_A C_\alpha \}_{\alpha \in G} \) such that

\[
(id \otimes_A \Delta_{\alpha, \beta}) \circ \rho^M_{\alpha \beta} = (\rho^M_{\alpha} \otimes_A id) \circ \rho_{\beta}, \quad (id \otimes_A \varepsilon) \circ \rho^M_e = id.
\]
We use the following Sweedler-type notation:

\[ p^M_{\alpha}(m) = m_{[0, \alpha]} \otimes_A m_{[1, \alpha]} \]

for all \( m \in M_{\alpha} \).

A morphism of right \( C \)-comodules is a right \( A \)-linear map \( f : M \to N \) satisfying the condition

\[ (f \otimes_A id) \circ \rho^M_{\alpha} = \rho^N_{\alpha} \circ f \]

for all \( \alpha \in G \). Let \( M_C \) denote the category of right \( C \)-comodules.

Similarly, we can define the left \( C \)-comodule and the category \( C_M \) of all left \( C \)-comodules. We use the following Sweedler-type notation for the left \( C \)-comodule structure maps \( M \rho_{\alpha} \):

\[ M \rho_{\alpha}(m) = m_{[-1, \alpha]} \otimes_A m_{[0, \alpha]} \]

for all \( m \in M_{\alpha} \).

A right \( G \)-\( C \)-comodule \( M \) is a family of right \( A \)-modules \( \{ M_{\alpha} \}_{\alpha \in G} \) (meaning that each \( M_{\alpha} \) is right \( A \)-module), together with a family of right \( A \)-linear maps \( \rho = \{ \rho_{\alpha, \beta} \}_{\alpha, \beta \in G} \), where \( \rho_{\alpha, \beta} : M_{\alpha} \otimes_A C_{\beta} \to M_{\alpha} \otimes_A C_{\beta} \), such that the following conditions hold:

\[ (id \otimes_A \Delta_{\beta, \gamma}) \circ \rho_{\alpha, \beta, \gamma} = (\rho_{\alpha, \beta} \otimes_A id) \circ \rho_{\alpha, \beta, \gamma}, \ (id \otimes_A \epsilon) \circ \rho_{\alpha, \beta} = id \]

for all \( \alpha, \beta, \gamma \in G \).

We use the following standard notation:

\[ \rho_{\alpha, \beta}(m) = m_{[0, \alpha]} \otimes_A m_{[1, \beta]} \]

for \( m \in M_{\alpha, \beta} \).

A morphism between two right \( G \)-\( C \)-comodules \( M = \{ M_{\alpha} \}_{\alpha \in G} \) and \( N = \{ N_{\alpha} \}_{\alpha \in G} \) is a family of right \( A \)-linear maps \( f = \{ f_{\alpha} : M_{\alpha} \to N_{\alpha} \}_{\alpha \in G} \) such that

\[ (f_{\alpha} \otimes_A id) \circ \rho_{\alpha, \beta} = \rho_{\alpha, \beta} \circ f_{\alpha, \beta} \]

The category of right \( G \)-\( C \)-comodules will be denoted by \( M^G_C \).

Let \( C \) be a \( G \)-\( A \)-coring. A family \( g = \{ g_{\alpha} \}_{\alpha \in G} \in \prod_{\alpha \in G} C_{\alpha} \) is called grouplike, if \( \Delta_{\alpha, \beta}(g_{\alpha, \beta}) = g_{\alpha} \otimes_A g_{\beta} \) and \( \epsilon(g_{\alpha}) = 1 \) for all \( \alpha, \beta \in G \).

Let \( C \) be a \( G \)-\( A \)-coring with a fixed grouplike family \( g = \{ g_{\alpha} \}_{\alpha \in G} \). Then \( A \) can be endowed with a structure of right \( C \)-comodule via the coaction maps

\[ \rho_{\alpha} : A \to A \otimes_A C_{\alpha}, \ \rho_{\alpha}(a) = 1_A \otimes_A g_{\alpha} \cdot a. \]

For \( M \in M^G_C \), we define

\[ M^{coG} = \{ m \in M | \rho_{\alpha}(m) = m \otimes_A g_{\alpha}, \forall \alpha \in G \} \]

In particular,

\[ A^{coG} = \{ a \in A | a \cdot g_{\alpha} = g_{\alpha} \cdot a, \forall \alpha \in G \} \]

Let \( A \otimes_B A \) be the canonical Sweedler coring associated to the ring morphism \( B \to A \) with its comultiplication and counit given by the formulas

\[ \Delta(a \otimes_B b) = (a \otimes_B 1_A) \otimes_A (1_A \otimes_B b), \ \epsilon(a \otimes_B b) = ab. \]

2.2. Graded Rings and Modules. Let \( A \) be a ring and \( R = \bigoplus_{\alpha \in G} R_{\alpha} \) a \( G \)-graded ring. Suppose that we have a ring morphism \( \iota : A \to R_e \). Then we call \( R \) a \( G \)-graded \( A \)-ring. Every \( R_{\alpha} \) is then an \( A \)-bimodule and the decomposition of \( R \) is a decomposition of \( A \)-bimodules. The category of \( G \)-graded right \( R \)-modules will be denoted by \( M^G_C \).

Let \( C \) be a \( G \)-\( A \)-coring. For every \( \alpha \in G \), \( R_{\alpha} = \iota HOM(C_{\alpha-1}, A) \) is an \( A \)-bimodule via

\[ (a \cdot f_{\alpha} \cdot b)(c) = f_{\alpha}(c \cdot a)b \]
for all $f_a \in \mathcal{R}_a$, $a, b \in A$ and $c \in C_{a-1}$. Take $f_a \in \mathcal{R}_a$, $g_b \in \mathcal{R}_b$ and define $f_a \ast g_b \in \mathcal{R}_{a\beta}$ by the following formula:

$$(f_a \ast g_b)(c) = g_b(c_{1, \beta-1} \cdot f_a(c_{2, a-1}))$$

for all $c \in C_{(a\beta)-1}$. This defines maps $m_{a\beta} : \mathcal{R}_a \otimes_A \mathcal{R}_b \to \mathcal{R}_{a\beta}$, which make $\mathcal{R} = \oplus_{a \in G} \mathcal{R}_a$ into a $G$-graded ring with the unit $\varepsilon$. Define $i : A \to \mathcal{R}_a$, $i(a)(c) = \varepsilon(c)a$ is a ring homomorphism, which make $\mathcal{R} = \bigoplus_{a \in G} \mathcal{R}_a$ be a $G$-graded $A$-ring, called the (left) dual (graded) ring of the group coring $\mathcal{C}$. We will also write $\mathcal{R} = \mathcal{C}^\ast$.

2.3. Graded Morita Contexts. Let $\mathcal{R}$ be a $G$-graded ring, and $M, N \in M^G_\mathcal{R}$. A right $\mathcal{R}$-linear map $f : M \to N$ is called homogeneous of degree $\sigma$, if $f(M_\alpha) \subset M_{\alpha\sigma}$ for all $\alpha \in G$. $\text{HOM}_\mathcal{R}(M, N)_\sigma$ denotes the additive group of all right $\mathcal{R}$-module maps of degree $\sigma$.

Let $S$ and $\mathcal{R}$ be $G$-graded rings. A $G$-graded Morita context connecting $S$ and $\mathcal{R}$ is a Morita context $(S, \mathcal{R}, P, Q, \varphi, \psi)$ with the following additional structure: $P$ and $Q$ are graded bimodules, and the maps

$$\varphi : P \otimes_{\mathcal{R}} Q \to S, \psi : Q \otimes_S P \to \mathcal{R}$$

are homogeneous of degree $e$.

Given two graded Morita contexts $(S, \mathcal{R}, P, Q, \varphi, \psi)$ and $(\hat{S}, \hat{\mathcal{R}}, \hat{P}, \hat{Q}, \hat{\varphi}, \hat{\psi})$, if there exist two graded ring morphism $\Phi : S \to \hat{S}$, $\Psi : \mathcal{R} \to \hat{\mathcal{R}}$ and two graded bimodule morphism $\Theta : Q \to \hat{Q}$, $\Xi : P \to \hat{P}$ such that the following two diagrams

$$\begin{array}{ccc}
P \otimes_{\mathcal{R}} Q & \xrightarrow{\varphi} & S \\
\downarrow{\varepsilon \otimes \Theta} & & \downarrow{\psi} \\
\hat{P} \otimes_{\hat{\mathcal{R}}} \hat{Q} & \xrightarrow{\varphi} & \hat{S} \\
\end{array}$$

are commutative, then we say a quadruple $\hat{\Gamma} = (\Phi, \Psi, \Theta, \Xi)$ a morphism from $(S, \mathcal{R}, P, Q, \varphi, \psi)$ to $(\hat{S}, \hat{\mathcal{R}}, \hat{P}, \hat{Q}, \hat{\varphi}, \hat{\psi})$.

Let $P$ be a $G$-graded right $\mathcal{R}$-module. Then $S = \text{END}_\mathcal{R}(P)$ is a $G$-graded ring, and $Q = \text{HOM}_\mathcal{R}(P, R) \in M^G_\mathcal{R}$ with structure

$$(r \cdot q \cdot s)(p) = rq(s(p))$$

for all $r \in \mathcal{R}$, $s \in S$, $q \in Q$ and $p \in P$. The connecting maps are the following

$$\varphi : P \otimes_{\mathcal{R}} Q \to S, \varphi(p \otimes_{\mathcal{R}} q)(p') = pq(p'),$$

$$\psi : Q \otimes_S P \to R, \psi(q \otimes_S p) = q(p).$$

Then $(S, \mathcal{R}, P, Q, \varphi, \psi)$ is a graded Morita context.

2.4. Cofree Group Corings. A $G$-$A$-coring $\mathcal{C} = \{C_\alpha\}_{\alpha \in G}$ is called cofree, if there exist $A$-bimodule isomorphisms $\gamma_\alpha : C_e \to C_\alpha$ such that

$$\Delta_{\alpha, \beta}(c_{\alpha, \beta}) = \gamma_\alpha(c_{1, \alpha}) \otimes_A \gamma_\beta(c_{2, \beta})$$

for all $c \in C_e$. If $\mathcal{C}$ is a cofree group coring, then, for every $\alpha \in G$, we have $A$-bimodule isomorphisms

$$\gamma_{\alpha^{-1}} : C_\alpha \to C_{\alpha^{-1}}, \alpha_{\alpha^{-1}} : \mathcal{R}_a \to \mathcal{R}_e,$$

and

$$\chi_\alpha = (\gamma_{\alpha^{-1}})^{-1} : \mathcal{R}_e \to \mathcal{R}_a.$$
2.1. Example. If $\mathcal{C} = C_\gamma(G)$ is a cofree $G$-coring and $g = (g_\alpha)_{\alpha \in G}$ a grouplike family of $\mathcal{C}$ such that $g_\alpha = \gamma_\alpha(g_\alpha)$. Then $A$ can be endowed with a structure of right $\mathcal{C}$-comodule via the coaction maps

$$\rho^A_\alpha : A \to A \otimes_A C_\alpha, \quad \rho^A_\alpha(a) = 1_A \otimes_A g_\alpha \cdot a.$$ 

For all $a \in A$, we have

$$(id \otimes_A \gamma_\alpha) \circ \rho^A_\alpha(a) = 1_A \otimes_A \gamma_\alpha(g_\alpha \cdot a) = 1_A \otimes_A g_\alpha \cdot a = \rho^A_\alpha(a),$$

this shows that $A$ is a cofree $\mathcal{C}$-module.

2.5. Group Entwining Structures. Let $\mathcal{C} = \{C_\alpha\}_{\alpha \in G}$ be a $G$-coalgebra and $A$ an algebra. We say that the $G$-coalgebra $\mathcal{C}$ and the algebra $A$ are $G$-entwined, if there is a family of linear maps $\psi = \{\psi_\alpha : C_\alpha \otimes A \to A \otimes C_\alpha\}_{\alpha \in G}$ such that

- $(ab)_\psi \otimes c^{\psi'} = a_{\psi_\alpha} b_{\psi_\beta} \otimes c^{\psi_\alpha \psi_\beta}$,
- $1_{A \psi_\alpha} \otimes c^{\psi_\alpha} = 1_A \otimes c$, for any $c \in C_\alpha$,
- $a_{\psi_\alpha \beta} \otimes c^{\psi_\alpha \beta(1, \alpha)} \otimes c^{\psi_\alpha \beta(2, \beta)} = a_{\psi_\beta} \psi_\alpha \otimes c_{(1, \alpha)}^{\psi_\alpha} \otimes c_{(2, \beta)}^{\psi_\beta}$,
- $a_{\psi_\alpha}(c \otimes a) = a_{\psi_\alpha} \otimes c^{\psi_\alpha} = a_{\psi_\alpha} \otimes c^{\psi_\alpha} = \cdots$, for $a \in A$ and $c \in C_\alpha$. The triple $(A, \mathcal{C}, \psi)$ is called a right and right $G$-entwining structure and is denoted by $(A, \mathcal{C})_{G, \psi}$.

Given a right-right $G$-entwining structure $(A, \mathcal{C})_{G, \psi}$, then $U^G_A(\psi)$ is the category of right $(A, \mathcal{C})$-modules. The object of $U^G_A(\psi)$ are right $\mathcal{C}$-comodules $(M, \rho^M_\alpha)$ which is also $A$-module such that

$$\rho^M_\alpha(m \cdot a) = m_{[0, a]} \cdot a_{\psi_\alpha} \otimes m_{[1, a]}^{\psi_\alpha}$$

for all $m \in M$ and $a \in A$. Morphisms in $U^G_A(\psi)$ are right $\mathcal{C}$-comodule and right $A$-module maps and let $U^G_A(\psi)$ be the category of right $(A, \mathcal{C})_{G, \psi}$ of which the objects are right $G$-$\mathcal{C}$-comodules $(M, \rho^M_{\alpha, \beta})$ which is also right $A$-module, i.e., each $M_\alpha$ is right $A$-module, such that

$$\rho^M_{\alpha, \beta}(m \cdot a) = m_{[0, a]} \cdot a_{\psi_\beta} \otimes m_{[1, \beta]}^{\psi_\beta}$$

for all $m \in M_{\alpha \beta}$ and $a \in A$. Morphisms in $U^G_A(\psi)$ are right $G$-$\mathcal{C}$-comodule and right $A$-module maps.

2.6. Group Coalgebra Galois Extensions. Let $\mathcal{C}$ be a $G$-coalgebra and $A$ an algebra. Let $A$ be a right $\mathcal{C}$-comodule. Let

$$B = A^{co\mathcal{C}} = \{a \in A | \rho^A_\alpha(ab) = a \rho^A_\beta(b), \forall b \in A, \alpha, \beta \in G\}.$$ 

We say that $A$ is a right $G$-$\mathcal{C}$-Galois extension of $B$, if the canonical left $A$-module right $G$-$\mathcal{C}$-comodule map $can = \{can_\alpha : A \otimes B A \to A \otimes C_\alpha\}$, by $a \otimes b \mapsto ab_{[0, a]} \otimes b_{[1, a]}$ for all $a, b \in A$ is bijective, i.e., every map $can_\alpha$ is bijective for all $\alpha \in G$.

3. Morita Context associated to a Comodule over a Group Coring

Let $\mathcal{C}$ be a $G$-coring, and $M \in \text{Comod}_\mathcal{C}$. We can associate a Morita context to $M$. The context will connect $T = \text{END}(M)^{op}$ and $^*\mathcal{C} = \mathfrak{R}$.

For every $\alpha \in G$, $Q_\alpha = \text{A-HOM}(C_{\alpha-1}, M) \in \text{M}_{TF}$ is a left $A$-module with

$$(a \cdot f_\alpha)(c) = f_\alpha(c \cdot a)$$

for all $f_\alpha \in Q_\alpha$, $a \in A$ and $c \in C_{\alpha-1}$. Let

$$Q = \bigoplus_{\alpha \in G} Q_\alpha, |q_{\alpha \beta}(c)_{[1, \beta-1]} \otimes_A q_{\alpha \beta}(c)_{[0, \beta-1]}|$$

$$= c_{(1, \beta-1)} \otimes_A q_{\alpha}(c_{(2, \alpha-1)}), \forall c \in C_{\beta-1, \alpha-1}.$$
3.1. Lemma. With the notation as above, $\ast M = \ast \text{HOM}(M, A) \in T \mathcal{M}_G$ and $Q \in \mathcal{M}_G$.

Proof. Let $\zeta \in \ast M$, $f_\alpha \in \mathcal{R}_\alpha$, $t \in T$, $q_\beta \in Q_\beta$ and $m \in M$. We define the bimodule structure on $\ast M$ as follows:

$$((\zeta \cdot f_\alpha) \cdot g_\beta)(m) = f_\alpha(m) \cdot \zeta((m)_{[0,0]}(1)) \quad \text{and} \quad t \cdot \zeta = \zeta \circ t.$$  

For all $g_\beta \in \mathcal{R}_\beta$, we have

$$((\zeta \cdot f_\alpha) \cdot g_\beta)(m) = f_\alpha(m_{[-1,\alpha^{-1}]} \cdot \zeta((m)_{[0,\alpha^{-1}]}(1)))$$

This shows that $\ast M$ is a $G$-graded right $\mathcal{R}$-module. Let us show that the two actions commute. Indeed, we compute

$$(t \cdot (\zeta \cdot f_\alpha))(m) = (\zeta \cdot f_\alpha)(t(m)) = f_\alpha(t)(m_{[-1,\alpha^{-1}]} \cdot \zeta((m)_{[0,\alpha^{-1}]}(1))) = f_\alpha(m_{[-1,\alpha^{-1}]} \cdot \zeta((m)_{[0,\alpha^{-1}]}(1))) = (t \cdot \zeta) \cdot f_\alpha.$$  

The bimodule structure on $Q$ is defined by

$$(f_\alpha \cdot q_\beta)(c) = q_\beta(c_{[1,\beta^{-1}]} \cdot f_\alpha(c_{[2,\alpha^{-1}]}))$$

for all $c \in C_{[1,\alpha^{-1}]}$ and $q_\beta \cdot t = t \circ q_\beta$. \hfill $\square$

3.2. Lemma. With the notation as above, we have well-defined bimodule maps

$$\mu : Q \otimes_T \ast M \to \mathcal{R}, \mu(q \otimes_T \zeta) = \sum_{\alpha \in G} \zeta \circ q_\alpha,$$

$$\tau : \ast M \otimes Q \to T, \tau(\zeta \otimes q)(m) = \sum_{\alpha \in G} q_\alpha(m_{[-1,\alpha^{-1}]} \cdot \zeta((m)_{[0,\alpha^{-1}]}(1))).$$

3.3. Theorem. With the notation as above, we have a Morita context $(T, \mathcal{R}, \ast M, Q, \tau, \mu)$.

Proof. Here we only check that, for $\zeta, \zeta' \in \ast M$, $q, q' \in Q$ and $m \in M$,

$$(q' \cdot \tau(\zeta \otimes q))(c) = \mu(q' \otimes_T \zeta) \cdot q \cdot \zeta \cdot \mu(q \otimes_T \zeta') = \tau(\zeta \otimes q) \cdot \zeta'$$

hold. Indeed, for all $c \in C_{[1,\gamma^{-1}]}$, we compute

$$(q'_{\alpha \gamma} \cdot \tau(\zeta \otimes q))(c) = \tau(\zeta \otimes q)(q'_{\alpha \gamma}(c)) = \sum_{\beta \in G} q_\beta(q'_{\alpha \gamma}(c)_{[-1,\beta^{-1}] \cdot \zeta((m)_{[0,\beta^{-1}]}(1))) = \sum_{\beta \in G} q_\beta(c_{1,\beta^{-1}} \cdot \zeta((q'_{\alpha \gamma \beta^{-1}}(c_{[2,\beta^{-1} \cdot \alpha^{-1}]}) = \sum_{\beta \in G} ((\zeta \circ q'_{\alpha \gamma \beta^{-1}} \cdot q_\beta)(c) = (\mu(q'_{\alpha \gamma} \otimes_T \zeta) \cdot q)(c).$$

Thus we show that the first identity in (3.1) holds. The other identity can be checked similarly. \hfill $\square$
Next, we want to make an application of Theorem 3.3 in order to get a new Morita context.

Let \( M \) be a right \( C \)-comodule. Assume that \( M \) is finitely generated and projective with the finite dual basis \( \{ e_i, e_i^* \} \) or \( \{ e_i', e_i'^* \} \). \( M^* \) = Hom\(_A(M, A)\) can be viewed as a left \( A \)-module via \((a \cdot f)(m) = af(m)\). Then \( M^* \) is a left \( C \)-comodule with the coaction maps
\[
M^* \rho_\alpha : M^* \to C_\alpha \otimes_A M^*, \quad M^* \rho_\alpha(f) = \sum_i f(e_i[0, \alpha]) \cdot e_i[1, \alpha] \otimes_A e_i^*
\]
Indeed, we compute
\[
(id \otimes M^* \rho_\beta) \circ M^* \rho_\alpha(f) = (id \otimes M^* \rho_\beta) (\sum_i f(e_i[0, \alpha]) \cdot e_i[1, \alpha] \otimes_A e_i^*)
\]
\[
= \sum_{i,j} f(e_i[0, \alpha]) \cdot e_i[1, \alpha] \otimes_A e_j^* (e_j[0, \beta]) \cdot e_j[1, \beta] \otimes_A e_j'^*
\]
\[
= \sum_{i,j} f(e_i[0, \alpha]) \cdot e_i[1, \alpha] \cdot e_j^* (e_j[0, \beta]) \otimes_A e_j[1, \beta] \otimes_A e_j'^*
\]
\[
= \sum_i f(e_i[0, \beta][0, \alpha]) \cdot e_i[0, \beta][1, \alpha] \otimes_A e_i[1, \beta] \otimes_A e_i^*
\]
\[
= \sum_i f(e_i[0, \beta]) \cdot e_i[1, \alpha][1, \beta] \otimes_A e_i[1, \alpha][2, \beta] \otimes_A e_i^*. 
\]
This shows that \( M^* \rho = \{ M^* \rho_\alpha \}_{\alpha \in G} \) is \( C \)-colinear.

**3.4. Lemma.** Let \( M \) be a right \( C \)-comodule. Assume that \( M \) is finitely generated and projective. Then
\[
\widetilde{\text{END}}(M^*)^\text{op} \cong \text{END}C(M).
\]

*Proof.* Let \( \{ e_i, e_i^* \} \) be the dual basis of \( M \). We construct the desired maps as follows:
\[
\Phi : \widetilde{\text{END}}(M^*)^\text{op} \to \text{END}C(M), \quad \Phi(f)(m) = \sum_i e_i \cdot f(e_i^*)(m)
\]
and
\[
\Psi : \text{END}C(M) \to \widetilde{\text{END}}(M^*)^\text{op}, \quad \Psi(f)(g)(m) = g(f(m)).
\]
The other verifications are straightforward. \( \square \)

From Lemma 3.4 and Theorem 3.3, we have the following result.

**3.5. Corollary.** Let \( M \) be a right \( C \)-comodule. Assume that \( M \) is finitely generated and projective. We obtain a Morita context
\[
(\text{END}C(M), R, M, Q = \text{HOM}(C, M^*), \tau, \mu)
\]
with \( M \in \mathcal{M}_R \) by
\[
m \cdot f_\alpha = m_{[0, \alpha-1]} \cdot f_\alpha(m_{[1, \alpha-1]}) \quad \text{and} \quad t \cdot m = t(m)
\]
for all \( m \in M, f \in \mathcal{R}_\alpha, t \in T, \) and \( Q \in \mathcal{M}_T \) by
\[
(f_\alpha \cdot q_\beta)(c) = q_\beta(\alpha_{(1, \beta-1)} \cdot f_\alpha(\epsilon_{(2, \alpha-1)}))
\]
for all \( c \in C_{\beta-1, \alpha-1} \) and \( q_\beta \in Q_\beta \) and \( q_\beta \cdot t(c') = q_\beta(c') \circ t \) for all \( c' \in C_{\beta-1} \), and
\[
\mu : Q \otimes_T M \to R, \quad \mu(q \otimes_T m_\alpha)(c) = q_\alpha(c)(m), \forall c \in C_{\alpha-1}
\]
\[
\tau : M \otimes_R Q \to T, \quad \tau(m \otimes_R q)(m') = \sum_{\alpha \in G} m_{[0, \alpha-1]} \cdot (q_\alpha(\alpha_{[0, \alpha-1]})(m'))
\]
3.6. Example. Let \( C \) be a \( G \)-\( A \)-coring with a grouplike family \( g = (g_a)_{a \in G} \). Then \( A \) is a right \( C \)-comodule via

\[
\rho^A_a : A \to A \otimes_A C_a, \quad \rho^A_a(a) = 1_A \otimes_A g_a \cdot a.
\]

By [10], \( T = End^G_C(A) \) is nothing but the \( A^{co}C \). Since \( A^* \cong A \), we have

\[
Q = \{ q \in \mathbb{R}[g_{\alpha \beta}(c)g_{\beta^{-1}} = c_{(1, \beta^{-1})} \cdot g_{\alpha}(c_{(2, \alpha^{-1})}), \forall c \in C_{\beta^{-1}, \alpha^{-1}} \}.
\]

Applying Corollary 3.5, we have a Morita context as in [5].

4. Graded Morita Context associated to a Comodule over a Group Coring

In this section, we assume that \( M \in M^G_C \) is finitely generated and projective with the dual basis \( \{e_i,e_i^*\} \). We say that a \( G \)-\( A \)-coring \( C \) is left homogeneously finite, if each \( C_a \) is finitely generated and projective as a left \( A \)-module. For \( M \in M^G_C \), it follows that \( \{\mu_{\alpha}(M)\}_{\alpha \in G} \in M^G_C \) with the coaction maps

\[
\rho_{\alpha,\beta} : \mu_{\alpha,\beta}(M) \to \mu_{\alpha}(M) \otimes_A C_{\beta}, \quad \rho_{\alpha,\beta}(\mu_{\alpha,\beta}(m)) = \mu_{\alpha}(m_{[0,\beta]} \otimes_A m_{[1,\beta]}).
\]

From Proposition 4.1 in [5], we then obtain that

\[
M\{G\} = \bigoplus_{\alpha \in G} \mu_{\alpha}(M) \in M^G_R.
\]

The right \( R \)-action is defined by the following formula,

\[
\mu_{\alpha}(m) \cdot f_\beta = \mu_{\alpha,\beta}(m_{[0,\beta^{-1}]} \cdot f_\beta(m_{[1,\beta^{-1}]})
\]

for all \( f_\beta \in R_\beta \) and \( m \in M \).

Next, we will compute the graded Morita context associated to the graded right \( R \)-module \( M\{G\} \). Consider the ring

\[
S = \{ f = (f_a)_{a \in G} \in \prod_{a \in G} End_A(M)[f_a(m)_{[0,\beta^{-1}]} \otimes_A f_a(m)_{[1,\beta^{-1}]} = f_{\alpha,\beta}(m_{[0,\beta^{-1}]} \otimes_A m_{[1,\beta^{-1}]} \}.
\]

Observe that we have a ring monomorphism

\[
i : T \to S, \quad i(f) = f = (f_a).
\]

On \( S \), we have the following right \( G \)-action:

\[
f^g = f \cdot \sigma = (f_{\sigma}a)_{a \in G}.
\]

Indeed, if \( f \in S \), we have \( f \cdot \sigma \in S \), since

\[
f_{\sigma a}(m)_{[0,\beta^{-1}]} \otimes_A f_{\sigma a}(m)_{[1,\beta^{-1}]} = f_{\sigma a,\beta}(m_{[0,\beta^{-1}]} \otimes_A m_{[1,\beta^{-1}]}).
\]

Now, we consider the twisted group ring \( G * S = \bigoplus_{a \in G} \mu_a S \) with multiplication

\[
\mu_a f_{\sigma}a g = \mu_{\alpha,\beta}(f_\beta g).
\]

4.1. Proposition. If \( G \)-\( A \)-coring \( C \) is left homogeneously finite, We then have a graded ring isomorphism

\[
\Omega : End_A(M\{G\}) \to G * S.
\]
Proof. For each $\sigma \in G$, we construct a map by
$$\Omega_\sigma : \text{END}_R(M\{G\})_\alpha \to \mu_\sigma S, \ \Omega_\sigma(h) = \mu_\sigma f$$
with $f_\sigma(m) = \mu_\sigma^\omega(h(\mu_\sigma(m)))$. Since $h$ is right $R$-linear, we have, for all $m \in M$ and $g \in R_\beta$ that
$$h(\mu_\sigma(m) \cdot g_\beta) = h(\mu_\sigma(\mu_\sigma(m)[0,\beta^{-1}] \cdot g_\beta(m[1,\beta^{-1}]))) = \mu_\sigma \alpha \beta(f_{\alpha \beta}(\mu_\sigma(m)[0,\beta^{-1}])) \cdot g_\beta(m[1,\beta^{-1}]).$$
Since
$$h(\mu_\sigma(m) \cdot g_\beta) = h(\mu_\sigma(m)) \cdot g_\beta = \mu_\sigma \alpha \beta(f_{\alpha \beta}(m[0,\beta^{-1}])), \quad g_\beta = \mu_\sigma \alpha \beta(\mu_\sigma(f_{\alpha \beta}(m[0,\beta^{-1}])),$$
it follows that
$$(f_{\alpha \beta}(m[0,\beta^{-1}]) \cdot g_\beta(f_{\alpha \beta}(m[1,\beta^{-1}])) = f_{\alpha \beta}(\mu_\sigma(m[0,\beta^{-1}]) \cdot g_\beta(m[1,\beta^{-1}]).$$
Since $C$ is left homogeneously finite (also see Lemma 4.2 in [5]), we have
$$(f_{\alpha \beta}(m[0,\beta^{-1}]) \otimes_A f_{\alpha \beta}(m[1,\beta^{-1}]) = f_{\alpha \beta}(m[0,\beta^{-1}]) \otimes_A m[1,\beta^{-1}].$$
This means $f \in S$. Next, we define a map
$$\Upsilon_\sigma : \mu_\sigma S \to \text{END}_R(M\{G\})_\alpha, \ \Upsilon_\sigma(f) = h$$
where $h$ satisfies $h(\mu_\sigma(m)) = \mu_\sigma \alpha \beta(f_{\alpha \beta}(m))$. It is straightforward to check that $\Upsilon_\sigma$ and $\Omega_\sigma$ are mutually inverses. It is routine to check that
$$\Omega = \bigoplus_{\alpha \in G} \Omega_\alpha : \text{END}_R(M\{G\}) \to G * S$$
preserves the multiplication and the unit.

Our next aim is to describe $\text{HOM}_R(M\{G\}, R)$. Consider
$$Q = \{ q = (q_\alpha)_{\alpha \in G} \in \prod_{\alpha \in G} A\text{HOM}(C_{\alpha^{-1}}, M^\ast) | \ c_{(1,\beta^{-1})} \otimes_A q_\alpha (c_{(2,\alpha^{-1})}) = q_{\alpha \beta}(c_{(1,\beta^{-1})}) \cdot e_{\alpha \gamma} \otimes_A e_i \}, \ c \in C_{(\alpha \beta)^{-1}} \}.$$

4.2. Lemma. If $f_\gamma, q_\gamma \in Q, q \in Q$, then
$$f_\gamma \cdot q = f_\gamma \cdot q_{\gamma^{-1} \alpha} \in Q.$$  

Proof. For all $c \in C_{(\alpha \beta)^{-1}}$ and $m \in M$, we have
$$c_{(1,\beta^{-1})} \otimes_A (f_\gamma \cdot q_{\gamma^{-1} \alpha}) (c_{(2,\alpha^{-1})}) = c_{(1,\beta^{-1})} \otimes_A q_{\gamma^{-1} \alpha}(c_{(2,\alpha^{-1})} \cdot f_\gamma(c_{(3,\gamma^{-1})})) = (c_{(1,\beta^{-1} - 1,\gamma^{-1})} \cdot f_\gamma(c_{(2,\gamma^{-1})}) \cdot q_{\gamma^{-1} \alpha}(c_{(1,\beta^{-1} - 1,\gamma^{-1})} \cdot f_\gamma(c_{(2,\gamma^{-1})}) \cdot e_{\alpha \gamma} \cdot e_i \otimes_A e_i) = (f_\gamma \cdot q_{\gamma^{-1} \alpha})(c)(c_{(1,\beta^{-1} - 1,\gamma^{-1})} \cdot f_\gamma(c_{(2,\gamma^{-1})}) \cdot e_{\alpha \gamma} \otimes_A e_i)$$
for all $c \in C_{\alpha^{-1}}$.
4.4. Lemma.

\[ QG = \bigoplus_{\alpha \in G} \omega_\alpha(Q) \in \mathcal{M}_G^G \]

with bimodule structures defined as follows: for all \( f \in \mathcal{R}_\beta \), \( q \in Q \) and \( b \in S \),

\[ f_\beta \cdot \omega_\alpha(q) = \omega_{\beta \alpha}(f_\beta \cdot q), \quad \omega_\alpha(q) \cdot \mu_b = \omega_{\alpha \tau}(q \cdot (b \cdot (\alpha \tau)^{-1})). \]

4.5. Proposition. If \( G \)-A coring \( C \) is left homogeneously finite, and \( M \in \mathcal{M}_G^G \) is finitely generated and projective as a right \( A \)-module. We then have an isomorphism of graded bimodules

\[ \Psi : \text{HOM}_K(M\{G\}, \mathcal{R}) \to QG \]

Proof. For each \( \sigma \in G \), we construct a map by

\[ \Psi_\sigma : \text{HOM}_K(M\{G\}, \mathcal{R})_\sigma \to \omega_\sigma(Q), \quad \Psi_\sigma(g) = \omega_\sigma(g) \]

with \( q_\alpha(c)(m) = g(\mu_{\alpha^{-1}\alpha}(m))(c) \) for all \( c \in C_{\alpha^{-1}} \) and \( m \in M \). Take \( \beta \in G \) and \( f_\beta \in \mathcal{R}_\beta \). Since \( g \) is right \( \mathcal{R} \)-linear, we have, for all \( m \in M \) that

\[ g(\mu_{\alpha^{-1}\alpha}(m) \cdot f_\beta) = g(\mu_{\alpha^{-1}\alpha}(m) \cdot f_\beta(m_{[1,\beta^{-1}]})) = g(\mu_{\alpha^{-1}\alpha}(m) \cdot f_\beta(m_{[1,\beta^{-1}]})). \]

Notice that

\[ g(\mu_{\alpha^{-1}\alpha}(m) \cdot f_\beta) = g(\mu_{\alpha^{-1}\alpha}(m)) \cdot f_\beta. \]

Thus, for all \( c \in C_{(\alpha,\beta^{-1})} \), we have

\[ (g(\mu_{\alpha^{-1}\alpha}(m) \cdot f_\beta)(c)) = f_\beta(c_{[1,\beta^{-1}]} \cdot (q_\alpha(c_{(2,\alpha^{-1})}))(m)). \]

it follows that

\[ f(q_\alpha(c)(m_{[1,\beta^{-1}]} \cdot m_{[1,\beta^{-1}]})) = f(c_{[1,\beta^{-1}]} \cdot g(\mu_{\alpha^{-1}\alpha}(m))(c_{(2,\alpha^{-1})})). \]

Since \( C \) is left homogeneously finite, we have

\[ q_\alpha(c)(m_{[0,\beta^{-1}]} \cdot m_{[1,\beta^{-1}]} \cdot c_{[1,\beta^{-1}]} \cdot (q_\alpha(c_{(2,\alpha^{-1})}))(m)). \]

Using the above equation and by \( M \) being finitely generated and projective (also see Lemma 4.2 in [5]), we have

\[ c_{[1,\beta^{-1}] \cdot A, q_\alpha(c_{(2,\alpha^{-1})}) = q_\alpha(c_{(0,\beta^{-1})}) \cdot c_{[1,\beta^{-1}] \cdot A, e_i^*}. \]

This means \( q \in Q \). Next, we define a map

\[ \Phi_\sigma : \omega_\sigma(Q) \to \text{HOM}_K(M\{G\}, \mathcal{R})_\sigma, \quad \Phi_\sigma(\omega_\sigma(q)) = g \]

where \( g \) satisfies \( g(\mu_{\alpha^{-1}\alpha}(m))(c) = q_\alpha(c)(m) \) for all \( c \in C_{\alpha^{-1}} \) with \( \alpha \in G \). It is straightforward to check that \( \Psi_\sigma \) and \( \Phi_\sigma \) are mutually inverse. It is routine to check that the bijection

\[ \Psi = \bigoplus_{\alpha \in G} \Psi_\alpha : \text{HOM}_K(M\{G\}, \mathcal{R}) \to QG \]

preserves the bimodule structure. \( \square \)

Now, we will achieve the main goal in this section.
4.6. Theorem. If \(G\)-\(A\) coring \(C\) is left homogeneously finite, and \(M \in \mathcal{M}\) is finitely generated and projective as a right \(A\)-module. Consider the graded Morita context \((\text{END}_{\mathcal{R}}(M\{G\}), \mathcal{R}, M\{G\}, \text{HOM}_{\mathcal{R}}(M\{G\}, \mathcal{R}), \mathcal{R}, \varphi, \psi)\) associated to the graded \(\mathcal{R}\)-module \(M\{G\}\). Using the isomorphism \(\Omega\) and \(\Psi\) from Proposition 4.1 and 4.5, we find an isomorphic graded Morita context \(\otimes_{\mathcal{M}} = (G * S, \mathcal{R}, M\{G\}, QG, \omega', \nu')\) with connecting map \(\omega'\) and \(\nu'\) given by the formulas
\[
\omega'(\mu_\alpha(m) \otimes_{\mathcal{R}} \omega_\sigma(q)) = \mu_{\alpha\sigma}(\{f_{\beta}\}_{\beta \in C}),
\]
\[
f_{\beta}(m') = m_{[0,(\sigma\beta)-1]} \cdot q_\sigma\beta(m_{[1,(\sigma\beta)-1]})(m'),
\]
\[
\nu'(\omega_\sigma(q) \otimes_{G*S} \mu_\alpha(m)(c)) = q_\sigma\alpha(c)(m), \quad \forall c \in C_{(\sigma\alpha)-1}.
\]

Proof. It is routine to check that the following two diagrams are commutative
\[
\begin{array}{cccc}
M\{G\} \otimes_{\mathcal{R}} \text{HOM}_{\mathcal{R}}(M\{G\}, \mathcal{R}) & \xrightarrow{id \otimes \Psi} & \text{END}_{\mathcal{R}}(M\{G\}) & \\
\downarrow & & \downarrow & \\
M\{G\} \otimes_{\mathcal{R}} QG & \xrightarrow{\omega'} & G * S & \\
\end{array}
\]
\[
\begin{array}{cccc}
\text{HOM}_{\mathcal{R}}(M\{G\}, \mathcal{R}) \otimes_{\text{END}_{\mathcal{R}}(M\{G\})} M\{G\} & \xrightarrow{\Psi \otimes id} & M\{G\} & \\
\downarrow & & \downarrow & \\
QG \otimes_{G*S} M\{G\} & \xrightarrow{\nu'} & \mathcal{R} & \\
\end{array}
\]
This ends the proof. \(\square\)

4.7. Remark. Let \((\overline{C}, \varepsilon)\) be a \(G\)-\(A\)-coring with a fixed grouplike family \(\varepsilon = (x_\alpha)_{\alpha \in G}\). The Morita context in Theorem 4.6 is just the Morita context studied in [5].

Let \(C_e\) be an \(A\)-coring and \(M\) a \(C_e\)-comodule such that \(M\) is finitely generated and projective as right \(A\)-module. Recall from [8] that we have a Morita context \(M_e = (T = \text{END}^{C_e}(M), \mathcal{R}_e, M, Q_e = \text{C}^*\text{HOM}(C_e, M^*), \tau_e, \mu_e)\) with \(M \in T\mathcal{M}_{\mathcal{R}_e}\) by
\[
m \cdot f_e = m_{[0,e]} \cdot f_e(m_{[1,e]}) \text{ and } t \cdot m = t(m)
\]
for all \(m \in M, f_e \in \mathcal{R}_e, t \in T,\) and \(Q_e \in \mathcal{M}_T\) by
\[
(f_e \cdot q_e)(c) = q_e(c_{(1,e)} \cdot f_e(c_{(2,e)}))
\]
for all \(c \in C_e\) and \(q_e \in Q_e\) and \((q_e \cdot t)(c') = q_e(c') \circ t\) for all \(c' \in C_e\), and
\[
\mu_e : Q_e \otimes_T M \to \mathcal{R}_e, \quad \mu_e(q_e \otimes_T m)(c) = q_e(c)(m), \forall c \in C_e
\]
\[
\tau_e : M \otimes_{\mathcal{R}_e} Q_e \to T, \quad \tau_e(m \otimes_{\mathcal{R}_e} q_e)(m') = m_{[0,e]} \cdot (q_e(m_{[1,e]})(m')).
\]

4.8. Proposition. Let \(M_e\) be the Morita context defined above. Consider the group rings \(T[G]\) and \(\mathcal{R}_e[G]\). Then \(M[G] = \bigoplus_{\sigma \in G} M_{\mu_\sigma} \in T[G] \mathcal{M}_{\mathcal{R}_e[G]}\) and \(Q_e[G] = \bigoplus_{\sigma \in G} Q_e \mu_\sigma \in \mathcal{R}_e[G]\) with
\[
f_{\mu_\sigma} \cdot m_{\mu_\alpha} \cdot r_{\mu_\beta} = (f \cdot m \cdot r)(\mu_\sigma \mu_\alpha \cdot r_{\mu_\beta} \cdot q_e \mu_\alpha \cdot f \mu_\sigma = (r \cdot q_e \cdot f)(\mu_\sigma \mu_\alpha)
\]
for all \(\sigma, \alpha, \beta \in G, f \in T, r_e \in \mathcal{R}_e, m \in M \) and \(q_e \in Q_e\). We have well-defined maps
\[
\mu : Q_e[G] \otimes_T G[M[G] \to \mathcal{R}_e[G], \quad \mu_{q_e \mu_\sigma} = \mu_{q_e \otimes_T m}(m_{\mu_\alpha}) = \mu_{(q_e \otimes_T m)}(m_{\mu_\alpha}).
\]
\[
\tau : M[G] \otimes_{\mathcal{R}_e[G]} Q_e[G] \to T[G], \quad \tau(m \otimes_{\mathcal{R}_e} q_e)(m \cdot q_e) = \tau_e(m \otimes_{\mathcal{R}_e} q_e)(m \cdot q_e)(m \cdot q_e)\mu_\alpha.
Then \( \mathcal{M}_c[G] = (T[G], R_c[G], M[G], Q_c[G], \tau, \mu) \) is a graded Morita context.

**4.9. Lemma.** Let \( C \) be a cofree group coring and \( M \) a cofree \( \underline{C} \)-comodule such that \( M \) is finitely generated and projective. Then \( i : T \rightarrow S \) is isomorphism, and \( \text{END}_R(M\{G\}) \cong G * S \) is isomorphic as a graded ring to the group ring \( T[G] \).

**Proof.** It suffices to show that \( i \) is surjective. For \( f \in S \), we have that

\[
\begin{align*}
f_{[0, e]}(m) & \otimes_A \gamma_{\beta^{-1}}(f_{[1, e]}(m)) = f_{[0, e]}(m) \otimes_A f_{[1, e]}(m) \\
& = f_{\alpha \beta}(m) \otimes_A m_{[1, \beta^{-1}]} \\
& = f_{\alpha \beta}(m) \otimes_A \gamma_{\beta^{-1}}(m). 
\end{align*}
\]

Applying \( i \otimes_A \varepsilon \circ \gamma_{\beta^{-1}} \), we have \( f_{[0, e]}(m) = f_{[1, e]}(m) \), hence \( f_{e} = f_{\beta} \) for all \( \beta \in G \), and \( f = i(f) \). \( \square \)

**4.10. Proposition.** Let \( C \) be a cofree group coring and \( M \) a cofree \( \underline{C} \)-comodule. Then we have an isomorphism of \( G \)-graded \( (G * S, \mathcal{R}) \)-bimodules

\[
\vartheta : M\{G\} \rightarrow M[G], \quad \vartheta(\mu_{\alpha}(m)) = m\mu_{\alpha}.
\]

**Proof.** Straightforward. \( \square \)

**4.11. Lemma.** Let \( C \) be a cofree group coring and \( M \) a cofree \( \underline{C} \)-comodule such that \( M \) is finitely generated and projective. Then \( Q \cong Q_c \). Consequently \( \text{HOM}_R(M\{G\}, \mathcal{R}) \cong Q_c[G] \).

**Proof.** Let us take \( Q = \{ q_{\alpha} \}_{\alpha \in \mathcal{C}} \in Q \). Then for all \( \alpha, \beta \in G \) and \( c \in C_c \), we have

\[
\begin{align*}
\gamma_{\beta^{-1}}(c_{[1, e]}) & \otimes_A q_{\alpha}(\gamma_{\alpha^{-1}}(c_{[2, e]})) \\
& = \gamma_{(\alpha \beta)^{-1}}(c_{[1, \beta^{-1}]} \otimes_A q_{\alpha}(\gamma_{(\alpha \beta)^{-1}}(c_{[2, \alpha^{-1}]}) \\
& = q_{\alpha \beta}(\gamma_{(\alpha \beta)^{-1}}(c_{[1, \beta^{-1}]} \otimes_A \gamma_{\alpha^{-1}}(e_{[0, e]} \cdot \gamma_{\alpha^{-1}}(e_{[1, e]}).
\end{align*}
\]

Taking \( \alpha = \beta = e \), we find that \( q_{e} \in Q_c \). For all \( m \in M \), it follows that

\[
\gamma_{\beta^{-1}}(c_{[1, e]}) \cdot q_{\alpha}(\gamma_{\alpha^{-1}}(c_{[2, e]}))(m) = q_{\alpha \beta}(\gamma_{(\alpha \beta)^{-1}}(c_{[1, \beta^{-1}]}) \cdot \gamma_{\alpha^{-1}}(m_{[1, e]}).
\]

Applying \( \gamma_{\beta^{-1}} \) to both sides of the equation above, we have

\[
\begin{align*}
q_{\alpha}(\gamma_{\alpha^{-1}}(c_{[2, e]}))(m) & = q_{\alpha \beta}(\gamma_{(\alpha \beta)^{-1}}(c_{[2, e]})) \cdot m_{[1, e]}.
\end{align*}
\]

Applying \( \varepsilon \) to both sides, we find that

\[
q_{\alpha}(\gamma_{\alpha^{-1}}(c))(m) = q_{\alpha \beta}(\gamma_{(\alpha \beta)^{-1}}(c))(m),
\]

and

\[
\varepsilon(c) = q_{\beta}(\gamma_{\beta^{-1}}(e)).
\]

Hence, we have \( q_{\beta} = q_{e} \circ \gamma_{\beta^{-1}} \). These arguments show that the map

\[
j : Q_{e} \rightarrow Q, \quad j(q) = (\sigma_{\alpha}(q))_{\alpha \in \pi}
\]

is a well-defined isomorphism. \( \square \)

**4.12. Theorem.** Let \( C \) be a cofree group coring and \( M \) a cofree \( \underline{C} \)-comodule such that \( M \) is finitely generated and projective. Then the graded Morita contexts \( GM \) and \( \mathcal{M}_c[G] \) are isomorphic.
Proof. Let \( \Xi : G \ast S \to T[G] \) be the isomorphism in Lemma 4.9. We will show that the diagram

\[
\begin{array}{ccc}
M\{G\} \otimes_{\mathbb{R}} QG & \xrightarrow{\omega'} & G \ast S \\
\phi \otimes j^{-1}G & & \Xi \\
M[G] \otimes_{R[G]} Q[G] & \xrightarrow{\varphi} & T[G]
\end{array}
\]
commutes. Indeed, for \( \alpha, \sigma \in G, \ a \in A \) and \( q \in Q \), we have

\[
(\Xi \circ \omega')(\mu_\alpha(m) \otimes \omega_\sigma(q)) = \Xi(\mu_{\alpha \sigma}((f_{\sigma \beta})_{\beta \in G}))
\]
where \( f_{\sigma \beta}(m') = m_{\{0, (\sigma \beta) \} - 1} \cdot q_{\sigma \beta}(m_{\{1, (\sigma \beta) \} - 1})(m') \), and

\[
(\varphi \circ (\vartheta \otimes j^{-1}G))(\mu_\alpha(m) \otimes \omega_\sigma(q)) = \varphi(m_{\mu_\alpha} \otimes q_{\mu_\alpha}) = \varphi_c(m \otimes q_{\sigma})_{\mu_{\alpha \sigma}},
\]
for all \( m' \in M \), since

\[
f_{\sigma}(m') = m_{\{0, \sigma \} - 1} \cdot q_{\sigma}(m_{\{1, \sigma \} - 1})(m') = m_{\{0, \sigma \} - 1} \cdot q_{\sigma}(m_{\{1, \sigma \}})(m') = m_{\{0, \sigma \} - 1} \cdot q_{\sigma}(m_{\{1, \sigma \}})(m') = \varphi_c(m \otimes q_{\sigma})(m'),
\]

it follows that \( \Xi \circ \omega = \varphi \circ (\vartheta \otimes j^{-1}G) \). Let

\[
\Gamma : R[G] \to \mathbb{R}, \ \Gamma(f_{\mu_\alpha}) = f \circ \gamma_{\alpha}^{-1}
\]
be the isomorphism from Proposition 4.6 in [5]. We will show that the diagram

\[
\begin{array}{ccc}
QG \otimes_{G\ast S} M\{G\} & \xrightarrow{\nu'} & \mathbb{R} \\
j^{-1}G \otimes \vartheta & & \Gamma^{-1} \\
Q[G] \otimes_{T[G]} M[G] & \xrightarrow{\varphi} & R[G]
\end{array}
\]
commutes. Take \( \sigma, \alpha \in G, \ q \in Q \) and \( a \in A \),

\[
(\Gamma \circ \psi \circ (j^{-1}G \otimes \vartheta))(\omega_\sigma(q) \otimes \mu_\alpha(m)) = (\Gamma \circ \psi)(q_{\sigma \mu_\alpha} \otimes m_{\mu_\alpha}) = \Gamma(q_{\sigma})(m_{\mu_\alpha}) = q_{\sigma}(-)(m) \circ \gamma_{(\sigma \alpha)}^{-1}
\]
and

\[
\nu'(\omega_\sigma(q) \otimes G_{\ast S} \mu_\sigma(m)(-)) = q_{\sigma \alpha}(-)(m).
\]

For \( \gamma_{(\sigma \alpha)}^{-1}(c) \in C_{(\sigma \alpha)}^{-1} \), we compute that

\[
(q_{\sigma}(-)(m) \circ \gamma_{(\sigma \alpha)}^{-1})(\gamma_{(\sigma \alpha)}^{-1}(c))(m) = q_{\sigma \alpha}(\gamma_{(\sigma \alpha)}^{-1}(c))(m) = \nu'(\omega_\sigma(q) \otimes G_{\ast S} \mu_\sigma(m)(\gamma_{(\sigma \alpha)}^{-1}(c))).
\]

Thus we have \( \psi \circ (j^{-1}G \otimes \vartheta) = \Gamma^{-1} \circ \nu' \). \( \square \)
5. Applications

In this section, we will give the application of our theory to \( G \)-entwining structure.

Given a \( G \)-entwining structure \( (A, \underline{C}, \underline{G}, \psi) \), then we have a \( G \)-A-coring \( \{A \otimes C_{\alpha}\}_{\alpha \in G} \) arising from \( (A, \underline{C}, \underline{G}, \psi) \). First observe that

\[
\mathcal{R}_\alpha = A \operatorname{HOM}(A \otimes C_{\alpha}, A) \cong \operatorname{HOM}(C_{\alpha^{-1}}, A)
\]
as spaces. This graded ring structure on \( \mathcal{R} \) induces a graded ring structure on \( \bigoplus_{\alpha \in G} \operatorname{HOM}(C_{\alpha^{-1}}, A) \), and this graded ring is denoted by \( \underline{G}(\underline{C}, A) \). The product is given by the formula

\[
(f_\alpha \otimes g_\beta)(c) = f_\alpha(c_{(2, \alpha^{-1})}) \psi_{\beta^{-1}} g_\beta(c_{(1, \beta^{-1})}) \psi_{\beta^{-1}}
\]
for all \( f_\alpha \in \operatorname{HOM}(C_{\alpha^{-1}}, A) \), \( g_\beta \in \operatorname{HOM}(C_{\beta^{-1}}, A) \) and \( c \in C_{(\alpha\beta)^{-1}} \).

Fix a group-like family \( x = \{x_\alpha\}_{\alpha \in G} \) of \( \underline{C} \), then we have that \( A \in \underline{G}(\underline{C}, \psi) \) with right \( \underline{C} \)-coaction \( \rho_\alpha^A(a) = a_{\psi_{\alpha^{-1}}} \times x_\alpha \).

Generally, let \( A \) be a right \( \underline{C} \)-comodule. Suppose that \( A \) is an object of \( \underline{G}(\underline{C}, \psi) \) with the structure maps \( m_A \) and \( \rho^A = \{\rho_\alpha^A\} \). Then we have \( \rho_\alpha^A(ab) = a_{[0, \alpha]}^b b_{\psi_{\alpha^{-1}}} \times a_{[1, \alpha]}^b \).

Specially, the coaction can be written as \( \rho_\alpha^A(b) = 1_{A[0, \alpha]}^b b_{\psi_{\alpha^{-1}}} \times 1_{A[1, \alpha]}^b \). The ring of coinvariants is

\[
B = \{a \in A | 1_{A[0, \alpha]}^a a_{\psi_{\alpha^{-1}}} \times 1_{A[1, \alpha]}^a = a 1_{A[0, \alpha]} \times 1_{A[1, \alpha]}, \forall \alpha \in G\}.
\]

Then we have the twisted group ring \( G \ast S = \bigoplus_{\alpha \in G} \mu_\alpha S \) with the multiplication given by \( \mu_\alpha \cdot \mu_\beta = \mu_\beta(\underline{b}_\alpha) \), where \( \underline{b}_\beta = (b_{\beta \alpha})_{\alpha \in G} \).

From Theorem 4.6, we have the following result.

5.1. Theorem. With the notation as above, we have a graded Morita context \( \mathcal{GM} = (G \ast S, \underline{G}(\underline{C}, A), A[G], QG, \omega', \nu') \) with connecting map \( \omega' \) and \( \nu' \) given by the formulas

\[
\omega' : A[G] \otimes \underline{G}(\underline{C}, A) QG \to G \ast S,
\]

\[
\nu' : \underline{G}(\underline{C}, A) QG \otimes G S A[G] \to \underline{G}(\underline{C}, A),
\]

\[
\nu'(\omega_\alpha(g) \otimes \mu_\alpha(a))(c) = q_{\omega_\alpha}(c) a, \forall c \in C_{(\alpha \beta)^{-1}}.
\]

As was stated above, if we fix a group-like family \( x = \{x_\alpha\}_{\alpha \in G} \) of \( \underline{C} \), then we have that \( A \in \underline{G}(\underline{C}, \psi) \) with right \( G \)-\( C \)-coaction \( \rho_\alpha^A(a) = a_{\psi_{\alpha^{-1}}} \times x_\alpha \). In particular, it follows that \( \rho_\alpha^A(1_A) = 1_A \times x_\alpha \). Then \( B \), \( S \) and \( Q \) have the following forms:

\[
B = \{a \in A | a_{\psi_{\alpha^{-1}}} \times x_\alpha = a \times x_\alpha, \forall \alpha \in G\},
\]

\[
Q = \{q = (q_\alpha)_{\alpha \in G} \in \prod_{\alpha \in G} \underline{G}(\underline{C}_{\alpha^{-1}}, A)_{\alpha} \}
\]

\[
q_\alpha(c_{(2, \alpha^{-1})}) \psi_{\beta^{-1}} \times c_{(1, \beta^{-1})} \psi_{\beta^{-1}} = q_{\alpha \beta}(c) \times x_{\beta^{-1}}, c \in C_{(\alpha \beta)^{-1}}
\]
and \[ S = \{ b = (b_\alpha)_{\alpha \in G} \in \prod_{\alpha \in G} A | b_\alpha \psi_{\beta^{-1}} \otimes x_{\beta^{-1} \psi_\beta}^{-1} = b_\alpha \otimes x_{\beta^{-1}} \}. \]

From Theorem 5.1, we have a graded Morita context \( \mathbb{G}M = (G \ast S, \sharp(C, A), A\{G\}, Q\{G\} \ast, \omega', \nu') \) with connecting map \( \omega' \) and \( \nu' \) given by the formulas

\[
\omega'(\mu_\alpha(a) \otimes \sharp(C, A) \omega_\alpha(q)) = \mu_\alpha(a \psi_{(\beta^{-1} \psi_\beta)}^{-1} q \sigma_\beta(x_{(\sigma^{-1})^{-1}})),
\]

\[
\nu' : QG \otimes_{G \ast S} A\{G\} \rightarrow \sharp(C, A),
\]

\[
\nu'(\omega_\alpha(q) \otimes \sharp G \ast S \mu_\alpha(a))(c) = q_{\sigma_\alpha(c)} a, \forall c \in C_{(\sigma_\alpha)}^{-1}.
\]

Furthermore, if \( G \) is a trivial group, then \( B = S \) and the graded Morita context \( \mathbb{G}M = (G \ast S, \sharp(C, A), A\{G\}, QG, \omega', \nu') \) recovers to the Morita context in the sense of \([7, \text{Section 4}]\).

In order to proceed the further discussion, we need the following result \([10]\).

**5.2. Proposition.** Let \( A \) and \( E \) be rings, and \( C \) a \( G \)-\( A \)-coring, and \( M \) both a \( C \)-comodule and a \( (E, A) \)-bimodule such that the comodule maps \( \rho_\alpha \) are left \( E \)-linear. Then we have a pair of adjoint functors \((F, U)\):

\[ F : M_E \rightarrow M^{G, C}_E, \quad F(N) = \{ \mu_\alpha(N \otimes E M) \}_{\alpha \in G}. \]

The coaction maps are

\[
\rho_{\alpha, \beta} : \mu_{\alpha \beta}(N \otimes E M) \rightarrow \mu_\alpha(N \otimes E M) \otimes_A C_\beta,
\]

\[
\rho_{\alpha, \beta}(\mu_{\alpha \beta}(n \otimes E m)) = \mu_\alpha(n \otimes E m_{[0, \beta]}) \otimes_A m_{[1, \beta]}.
\]

For \( X \in M^{G, C}_E \), define \( U \) as follows:

\[ U : M^{G, C}_E \rightarrow M_E, \quad U_2(X) = \text{HOM}^{G, C}_E(\mu_\alpha(M), X). \]

Next, we apply Proposition 5.2 to the particular \( G \)-\( A \)-coring \( \{ A \otimes C_\alpha \}_{\alpha \in G} \) arising from \( (A, C) \ast \psi \). Under the assumption that \( A \) is an object of \( U^{G, C}_A(\psi) \) with the structure maps \( m_A \) and \( \rho_A = \{ \rho_A^\alpha \} \), we have a special pair of adjoint functors \((\widetilde{F}, \widetilde{U})\):

\[ \widetilde{F} : M_B \rightarrow U^{G, C}_A(\psi), \quad \widetilde{F}(N) = \{ \mu_\alpha(N \otimes B A) \}_{\alpha \in G}. \]

The coaction maps are

\[
\rho_{\alpha, \beta} : \mu_{\alpha \beta}(N \otimes B A) \rightarrow \mu_\alpha(N \otimes B A) \otimes_A C_\beta,
\]

\[
\rho_{\alpha, \beta}(\mu_{\alpha \beta}(n \otimes B a)) = \mu_\alpha(n \otimes B 1_{A[0, \beta]} \psi_{\alpha \beta}) \otimes_A 1_{A[1, \beta]} \psi_\beta.
\]

For \( X \in U^{G, C}_A(\psi) \), define \( \widetilde{U} \) as follows:

\[ \widetilde{U} : U^{G, C}_A(\psi) \rightarrow M_B, \quad \widetilde{U}(X) = \text{HOM}^{G, C}_A(\mu_\alpha(A), X). \]

For \( M \in U^{G, C}_A(\psi) \), we define

\[ M^\alpha = \{ m = (m_\alpha)_{\alpha \in G} \in \prod_{\alpha \in G} M_{\alpha} | m_{[0, \alpha]} \otimes m_{[1, \beta]} = m \cdot 1_{A[0, \beta]} \otimes 1_{A[1, \beta]} \}. \]

Then we have

**5.3. Lemma.** There exists an isomorphism

\[ \text{HOM}^{G, C}_A(\mu_\alpha(A), M) \cong M^\alpha \]

as right \( B \)-modules.
Proof. For any $f \in \text{HOM}_{C}(\mu_{a}(A), M)$, since $f$ is a right $C$-comodule, then we have
\[ \rho_{\alpha}^{M}(f_{\alpha\beta}(\mu_{a}(1_{A}))) = f_{\alpha}(\mu_{a}(1_{A}[0,\beta])) \otimes 1_{A[1,\beta]} \].
Set $m = (m_{a})_{a \in G}$, where $m_{a} = f_{a}(\mu_{a}(1_{A}))$. Straightforward calculation can show that $m \in \underline{M}$.
Thus we define a map
\[ \hat{\Phi} : \text{HOM}_{C}(\mu_{a}(A), M) \to \underline{M}, \quad \hat{\Phi}(f) = (f_{a}(\mu_{a}(1_{A})))_{a \in G} \].
Take $m \in \underline{M}$, we define a map
\[ \tilde{\Psi} : \underline{M} \to \text{HOM}_{C}(\mu_{a}(A), M), \quad \tilde{\Psi}(m_{a}(\mu_{a}(a))) = m_{a} \cdot a. \]
It follow easily that $\hat{\Phi}$ and $\tilde{\Psi}$ are both $B$-linear and mutually inverses.

From Lemma 5.3 and what was discussed above, we have a pair of adjoint functors $(\tilde{\Phi}, \tilde{\Psi})$:
\[ \tilde{\Phi} : M_{B} \to \underline{U}_{A}^{C}(\psi), \quad \tilde{\Phi}(N) = \{\mu_{a}(N \otimes_{B} A)\}_{a \in G}. \]
\[ \tilde{\Psi} : \underline{U}_{A}^{C}(\psi) \to M_{B}, \quad \tilde{\Psi}(X) = X^{\mathcal{C}}. \]

By the discussion as above, and [7, Theorem 9.2], we can achieve the main goal in this section.

5.4. Theorem. Let $(A, C)$ be a $G$-entwined structure. Suppose that $A \in \underline{U}_{C}(\psi)$. Consider the map
\[ \text{can} : (A \otimes_{B} A)(\psi) \to A \otimes_{C} C, \quad \text{can}_{\alpha}(a \otimes_{B} b) = a1_{A[0,\alpha]}b_{\beta} \otimes 1_{A[1,\alpha]}\nu_{\beta}. \]

Then the following statements are equivalent:

1. $\text{can}$ is an isomorphism of group corings, and $A$ is faithfully flat as a left $B$-module,
2. *can is an isomorphism of graded rings and $A$ is a left $B$-progenerator,
3. The graded Morita context $GM = (G \ast S, \sharp(A), A\{G\}, QG, \omega, \nu')$ is strict,
4. $(\tilde{\Phi}, \tilde{\Psi})$ is an equivalence of categories.

As the end of this paper, we discuss the $(H, A)$-Hopf module for an $H$-comodule algebra $A$ over a Hopf $G$-coalgebra $H$.

Let $H = (\{H_{a}, m_{a}, 1_{a}, \Delta, \varepsilon\})$ be a Hopf $G$-coalgebra in the sense of [16] and $A$ an $H$-comodule. We recall from [16] that a right $H$-comodule algebra is a right $H$-comodule $(A, \rho^{H} = \{\rho_{\alpha}^{H}\})$, such that the following conditions are satisfied:

- $\rho_{\alpha}(ab) = a_{[0,\alpha]}b_{[0,\alpha]} \otimes a_{[1,\alpha]}b_{[1,\alpha]}$ for all $a, b \in A$ and $\alpha \in G$, 
- $\rho_{\alpha}^{H}(1) = 1_{A} \otimes 1_{a}$ for all $\alpha \in G$.

Given an $H$-comodule algebra $A$, we have a $G$-entwined structure $\psi_{\alpha} : H_{a} \otimes A \to A \otimes H_{a}$, $\psi_{\alpha}(h \otimes a) = a_{[0,\alpha]} \otimes ha_{[1,\alpha]}$. We call a special $(A, C)_{\psi}$-module a $(right-right)$ $(H, A)$-Hopf module and denote the category of $(H, A)$-Hopf modules by $\text{HOM}_{A}^{H}$. It is easy to see that $A \in \underline{M}_{A}^{H}$. Let us take the grouplike family $\{1_{a}\}_{a \in G}$. Then we have a graded Morita context $GM = (G \ast S, \sharp(H, A), A\{G\}, QG, \omega, \nu')$ with connecting map $\omega'$ and $\nu'$ given by the formulas
\[ \omega' : A\{G\} \otimes_{\sharp(H, A)} QG \to G \ast S, \]
\[ \nu'(\mu_{a}(a) \otimes_{\sharp(H, A)} \omega_{\sigma}(q)) = \mu_{a}(a_{[0,\sigma\beta]^{-1}})q_{\sigma\beta}(a_{[1,\sigma\beta]^{-1}})), \]
\[ \nu' : QG \otimes_{G \ast S} A\{G\} \to \sharp(H, A), \]
\[ \nu'(\omega_{\sigma}(q) \otimes_{G \ast S} \mu_{a}(a))(c) = q_{\sigma\alpha}(c)a, \forall c \in C_{(\sigma\alpha)}^{-1}. \]
where
\[
Q = \{ q = (q_\alpha)_{\alpha \in G} \in \prod_{\alpha \in G} \sharp (H_{\alpha^{-1}}, A)_\alpha \ |
q_\alpha (h_{(2, \alpha^{-1})})_{[0, \beta-1]} \otimes h_{(1, \beta-1)} q_\alpha (h_{(2, \alpha^{-1})})_{[1, \beta-1]} = q_{\alpha \beta} (h) \otimes 1_{\beta^{-1}}, h \in H_{(\alpha \beta)^{-1}} \}
\]
and
\[
S = \{ h = (b_\alpha)_{\alpha \in G} \in \prod_{\alpha \in G} A [b_\alpha_{[0, \beta-1]} \otimes b_\alpha_{[1, \beta-1]} = b_{\alpha \beta} \otimes 1_{\beta^{-1}}] \}.
\]

5.5. Remark. If \( \pi \) is a trivial group, then \( S = A^{\text{co}H} \) and
\[
Q = \{ q \in \sharp (H, A)| q(h_{(2)})_{[0]} \otimes h_{(1)} q(h)_{[1]} = q(h) \otimes 1_H, h \in H \}.
\]
Hence, the graded Morita context \( GM = (G * S, \sharp (H, A), A(G), QG, \omega', \nu') \) is just the Morita context of Doi in [12].

Acknowledgement
The authors sincerely thank the referee for his/her numerous very valuable comments and suggestions on this article. This work was supported by the National Natural Science Foundation of China (No.11261063 and 11471186) and the Foundation for Excellent Youth Science and Technology Innovation Talents of Xin Jiang Uygur Autonomous Region (No.2013 721043)

References