Some identities and recurrences relations for the
$q$-Bernoulli and $q$-Euler polynomials

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Abstract
In this article we prove some relations between two-variable $q$-Bernoulli polynomials and two-variable $q$-Euler polynomials. By using the equality $e_q(z) E_q(-z) = 1$, we give an identity for the two-variable $q$-Genocchi polynomials. Also, we obtain an identity for the two-variable $q$-Bernoulli polynomials. Furthermore, we prove two theorems which are analogues of the $q$-extension Srivastava-Pinter additional theorem.

Keywords: Bernoulli numbers and polynomials, Euler numbers and polynomials, Genocchi polynomials, generating functions, generalized Bernoulli polynomials, generalized Genocchi polynomials, $q$-Bernoulli polynomials, $q$-Euler polynomials, $q$-Genocchi polynomials.

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1. Introduction Definition and Notation
The classical Bernoulli polynomials $B_n(x)$ and the classical Euler polynomials $E_n(x)$ are usually defined by means of the following generating functions;

$$\frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \quad |t| < 2\pi$$

and

$$\frac{2}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \quad |t| < \pi,$$

respectively. The corresponding Bernoulli numbers $B_n$ and Euler numbers $E_n$ are given by

$$B_n := B_n(0) = (-1)^n B_n(1) = (2^{1-n} - 1)^{-1} B_n \left( \frac{1}{2} \right)$$

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and
\[ E_n := 2^n E_n \left( \frac{1}{2} \right), \quad n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \]
respectively.

Many mathematicians investigated these polynomials in ([2]-[17]). They proved some theorems and gave some interesting recurrences relations. Firstly, Carlitz in [2] gave \( q \)-Bernoulli polynomials.

In this work we give some recurrences relations and properties for two-variable \( q \)-Bernoulli polynomials and \( q \)-Euler polynomials.

Throughout this paper, we make use of the following notations; \( \mathbb{N} \) denotes the set of natural numbers, \( \mathbb{C} \) denotes the set of complex numbers and \( q \in \mathbb{C} \) with \( |q| < 1 \). The \( q \)-basic numbers and \( q \)-factorials are defined ([2], [7]-[15]) by
\[
[q]_q = \frac{1 - q^a}{1 - q}, \quad \text{and} \quad [n]_q! = [n]_q [n - 1]_q \ldots [2]_q [1]_q,
\]
respectively, where \([0]_q! = 1\) and \( n \in \mathbb{N}, a \in \mathbb{C} \).

The \( q \)-binomial formula is defined ([8], [14]) by
\[
\left( x + y \right)_q^n = \sum_{k=0}^{n} \frac{n}{k} \left( \begin{array}{c} n \\ k \end{array} \right)_q \left( q; q \right)_n (q^{k+1} - 1) x^{n-k} y^k,
\]
where \( \left[ \begin{array}{c} n \\ k \end{array} \right]_q \) is the \( q \)-binomial coefficient (or Gaussian binomial coefficient) given by
\[
\left[ \begin{array}{c} n \\ k \end{array} \right]_q = \frac{\left( q; q \right)_n \left( q; q \right)_{n-k} \left( q; q \right)_k}{[n]_q! \left( q; q \right)_k}.
\]

The \( q \)-exponential functions are given ([1], [8], [12], [13]) by
\[
c_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!} = \prod_{k=0}^{\infty} \frac{1}{1 - (1-q) q^k z}, \quad 0 < |q| < 1, \quad |z| < \frac{1}{|1-q|}
\]
and
\[
E_q(z) = \sum_{n=0}^{\infty} q^n \frac{z^n}{[n]_q!} = \prod_{k=0}^{\infty} \left( 1 + (1-q) q^k z \right), \quad 0 < |q| < 1, \quad z \in \mathbb{C}.
\]
From the last equations, we can easily see that \( c_q(z) E_q(-z) = 1 \).

The Jack-derivative \( D_q \) is defined ([7], [10], [13], [14]) by
\[
D_q f(z) = \frac{f(qz) - f(z)}{qz - z}, \quad 0 < |q| < 1, \quad 0 \neq z \in \mathbb{C}.
\]
The derivative of the product of two functions and the derivative of the division of two functions are given by the following equations in [7], respectively.

(1.1) \[
D_q \left( \frac{f(z)}{g(z)} \right) = \frac{g(qz) D_q f(z) - f(qz) D_q g(z)}{g(z) g(qz)},
\]
\[
D_q \left( f(z) g(z) \right) = f(qz) D_q g(z) + g(z) D_q f(z).
\]
Carlitz was the first to extend the classical Bernoulli numbers and polynomials, Euler numbers and polynomials ([2], [3]). Cheon in [5] gave explicit expansions for the classical Bernoulli polynomials and the classical Euler polynomials. Srivastava et al [16] proved some formulae for the Bernoulli polynomials and the Euler polynomials. Also, they gave the addition-formulae between the Bernoulli polynomials and the Euler polynomials. There are numerous recent investigations on the \( q \)-Bernoulli polynomials and \( q \)-Euler
polynomials by many mathematicians, including as Cenkci et al [4], Choi et al [6], Kim ([8], [9]), Kim et al [10], Luo [11], Luo and Srivastava [12], Srivastava et al ([16], [17]), Tremblay et al [18] and Mahmudov ([13], [14]).

Mahmudov defined and studied properties of the following generalized $q$-Bernoulli polynomials $B_{n,q}^{(a)}(x,y)$ of order $a$ and $q$-Euler polynomials $E_{n,q}^{(a)}(x,y)$ of order $a$ as follows ([13], [14]).

Let $q \in \mathbb{C}$, $a \in \mathbb{N}$ and $0 < |q| < 1$. The $q$-Bernoulli numbers $B_{n,q}^{(a)}$ and polynomials $B_{n,q}^{(a)}(x,y)$ in $x, y$ of order $a$ are defined by means of the generating functions:

\begin{equation}
\sum_{n=0}^{\infty} \frac{B_{n,q}^{(a)}}{[n]_q} t^n = \left( \frac{t}{e_q(t) - 1} \right)^a, \quad |t| < 2\pi
\end{equation}

and

\begin{equation}
\sum_{n=0}^{\infty} B_{n,q}^{(a)}(x,y) \frac{t^n}{[n]_q} = \left( \frac{t}{e_q(t) - 1} \right)^a e_q(x t) E_q(y t), \quad |t| < 2\pi.
\end{equation}

The $q$-Euler numbers $E_{n,q}^{(a)}$ and polynomials $E_{n,q}^{(a)}(x,y)$ in $x, y$ of order $a$ are defined by means of the generating functions:

\begin{equation}
\sum_{n=0}^{\infty} E_{n,q}^{(a)} \frac{t^n}{[n]_q} = \left( \frac{2}{e_q(t) + 1} \right)^a, \quad |t| < \pi
\end{equation}

and

\begin{equation}
\sum_{n=0}^{\infty} E_{n,q}^{(a)}(x,y) \frac{t^n}{[n]_q} = \left( \frac{2}{e_q(t) + 1} \right)^a e_q(x t) E_q(y t), \quad |t| < \pi.
\end{equation}

The $q$-Genocchi numbers $G_{n,q}^{(a)}$ and polynomials $G_{n,q}^{(a)}(x,y)$ in $x, y$ of order $a$ are defined by means of the generating functions:

\begin{equation}
\sum_{n=0}^{\infty} G_{n,q}^{(a)} \frac{t^n}{[n]_q} = \left( \frac{2t}{e_q(t) + 1} \right)^a, \quad |t| < \pi
\end{equation}

and

\begin{equation}
\sum_{n=0}^{\infty} G_{n,q}^{(a)}(x,y) \frac{t^n}{[n]_q} = \left( \frac{2t}{e_q(t) + 1} \right)^a e_q(x t) E_q(y t), \quad |t| < \pi.
\end{equation}

It is obvious that

\[
\lim_{q \to 1^-} B_{n,q}^{(a)}(x,y) = B_{n}^{(a)} (x+y),
\]

\[
\lim_{q \to 1^-} E_{n,q}^{(a)}(x,y) = E_{n}^{(a)} (x+y),
\]

\[
\lim_{q \to 1^-} G_{n,q}^{(a)}(x,y) = G_{n}^{(a)} (x+y)
\]

and

\[
D_{q,x}^{(a)} B_{n,q}^{(a)}(x,y) = [n]_q B_{n-1,q}^{(a)}(x,y), \quad D_{q,y} B_{n,q}^{(a)}(x,y) = [n]_q B_{n,q-1}^{(a)}(x,qy),
\]

\[
D_{q,x} e_q(x t) = x e_q(x t), \quad D_{q,y} E_q(y t) = y E_q(qyt).
\]
2. Main Theorems

In this section, we give some relations for $q$-Bernoulli polynomials $\mathfrak{B}_{n,q}(x,y)$ and $q$-Euler polynomials $\mathcal{E}_{n,q}^{(\alpha)}(x,y)$. By applying the derivative operator to $q$-Bernoulli polynomials and $q$-Euler polynomials, we have recurrence relations for these polynomials.

2.1. Proposition. The generalized $q$-Bernoulli polynomials satisfy the following relation.

\begin{equation}
\sum_{l=0}^{n} \left[ \sum_{k=0}^{\lfloor l/2 \rfloor} \binom{n}{k} \mathfrak{B}_{n-l,q}(x,y) - \mathfrak{B}_{n,q}(x,y) \right] = [n]_q \mathfrak{B}_{n-1,q}(x,y).
\end{equation}

Proof. From (1.3), we have

\begin{equation}
\sum_{n=0}^{\infty} \mathfrak{B}_{n,q}(x,y) \frac{t^n}{[n]_q!} = \left( \frac{t}{e_q(t) - 1} \right)^\alpha e_q(x)t^n e_q(yt).
\end{equation}

By using Cauchy product and comparing the coefficient of $\frac{t^n}{[n]_q!}$, we have (2.1).

The following equations can be obtained easily from (1.2)-(1.5).

\begin{align}
(2.2) \quad \mathfrak{B}_{n,q}^{(\alpha-\beta)}(x,y) & = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{k} \mathfrak{B}_{k,q}^{(\alpha-\beta)}(0,0)(x+y)^n k,

(2.3) \quad \mathfrak{B}_{n,q}^{(\alpha-\beta)}(x,y) & = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{k} \mathfrak{B}_{k,q}^{(\alpha)}(x,0) \mathfrak{B}_{n-k,q}^{(-\beta)}(0,y),

(2.4) \quad (x+y)_n \quad = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{k} \mathcal{E}_{n-k,q}^{(\alpha)}(x,y) \mathcal{E}_{k,q}^{(-\alpha)}(0,0),

(2.5) \quad 2\mathcal{E}_{n,q}^{(\alpha-1)}(x,y) & = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{k} \mathcal{E}_{n-k,q}^{(\alpha)}(x,y) + \mathcal{E}_{n,q}^{(\alpha)}(x,y),
\end{align}

where $\alpha, \beta \in \mathbb{N}$.

2.2. Theorem. The generalized $q$-Bernoulli polynomials satisfy the following recurrence relation.

\begin{equation}
\mathfrak{B}_{n+1,q}(x,y) = \mathfrak{B}_{n,q}(x,y) + [n+1]_q \left\{ qy\mathfrak{B}_{n,q}(x,y) + qx\mathfrak{B}_{n,q}(x,y) \right\}
- \sum_{k=0}^{n+1} \binom{n+1}{k} \mathfrak{B}_{k,q}(x,y) q^k \mathfrak{B}_{n+1-k,q}(1,0).
\end{equation}

Proof. In (1.3), for $\alpha = 1$, we take the $q$-Jackson derivative of the generalized $q$-Bernoulli polynomials $\mathfrak{B}_{n,q}(x,y)$ according to $t$, then we have

\begin{equation}
\sum_{n=0}^{\infty} D_{q,t} \mathfrak{B}_{n,q}(x,y) \frac{t^n}{[n]_q!} = D_{q,t} \left( \frac{te_q(x)E_q(yt)}{e_q(t) - 1} \right).
\end{equation}
Proof. In (1.5), for \( t \) there is the following relation.

\[
(2.7)
\]

\[
\sum_{n=0}^{\infty} D_{q,t} B_{n,q} (x, y) \frac{t^n}{[n]_q} = \frac{(e_q (qxt) - 1) D_{q,t} [te_q (xt) E_q (yt)] - qte_q (qxt) E_q (qyt) D_{q,t} [e_q (t) - 1]}{(e_q (t) - 1) (e_q (qt) - 1)}.
\]

By using the equalities (1.1) in the last expression we have

\[
\sum_{n=0}^{\infty} \frac{1}{[n + 1]_q} B_{n+1,q} (x, y) \frac{t^n}{[n]_q} = \sum_{n=0}^{\infty} q^n B_{n,q} (qx, qy) + x B_{n,q} (x, y) \frac{t^n}{[n]_q} \left( \sum_{k=0}^{n+1} \frac{1}{k} \right) B_{k,q} (x, y) q^k B_{n+1-k,q} (1, 0) t^n \left( \frac{t}{[n]_q} \right).
\]

Comparing the coefficient of \( \frac{t^n}{[n]_q} \) we obtain (2.6). 

2.3. Theorem. The generalized \( q \)-Euler polynomials \( E_{n,q} (x, y) \) satisfy the following relation.

\[
E_{n+1,q} (x, y) = [n + 1]_q \left( y E_{n,q} (qx, qy) + x E_{n,q} (x, y) - \frac{1}{4} \sum_{k=0}^{n} \begin{bmatrix} n+1 \end{bmatrix}_q E_{k,q} (x, y) q^k E_{n-k,q} (1, 0) \right)
\]

Proof. In (1.5), for \( \alpha = 1 \), by using the equalities (1.1), the proof can be obtained. 

2.4. Theorem. There is the following relation.

\[
(2.7)
\]

\[
B_{n,q} (x, y) = \frac{m^{-n}}{m+1} \sum_{n=0}^{\infty} \left( \frac{1}{k} \right) B_{k,q} \left( \frac{1}{m}, 0 \right) \left( B_{n+1-k,q} (x, y) m^k \right).
\]

Proof. From (1.2), we have

\[
\sum_{n=0}^{\infty} B_{n,q} \frac{t^n}{[n]_q} = \frac{t}{e_q (t) - 1} \frac{t}{e_q (\frac{t}{m})} - \frac{t}{e_q (\frac{t}{m})} - \frac{t}{e_q (\frac{t}{m})} - \frac{t}{e_q (\frac{t}{m})} - \frac{t}{e_q (\frac{t}{m})}.
\]

\[
\sum_{n=0}^{\infty} B_{n,q} \left( \frac{1}{m}, 0 \right) - B_{n,q} (0, 0) \frac{t^n}{[n]_q} \sum_{n=0}^{\infty} B_{n,q} (0, 0) m^n \left( \frac{t}{[n]_q} \right).
\]

Using the Cauchy product and comparing the coefficient of \( \frac{t^n}{[n]_q} \) we obtain (2.7). 

2.5. Theorem. The generalized \( q \)-Euler numbers \( E_{n,q}^{(\alpha)} (0, 0) \) satisfy the following relation.

\[
E_{n,q}^{(\alpha)} = \frac{1}{2 [n + 1]_q} \sum_{k=0}^{n+1} \begin{bmatrix} n + 1 \end{bmatrix}_q \left( E_{k,q}^{(\alpha)} \left( \frac{1}{m}, 0 \right) + E_{k,q}^{(\alpha)} (0, 0) \right) E_{n+1-k,q} (0, 0) m^{k-n}.
\]
3. Some Relations Between the q-Bernoulli Polynomials and q-Euler Polynomials

In this section, we prove an interesting relationship between the q-Bernoulli polynomials $\mathfrak{B}_{n,q}^{(\alpha)}(x,y)$ of order $\alpha$ and q-Euler polynomials $\mathfrak{E}_{n,q}^{(\alpha)}(x,y)$ of order $\alpha$.

3.1. Theorem. There is the following relation between the q-Euler polynomials and q-Bernoulli polynomials.

\[
\mathfrak{E}_{n,q}^{(\alpha)}(x,y) = \frac{1}{2} \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right]_q \left\{ \sum_{r=0}^{p} \left[ \begin{array}{c} p \\ r \end{array} \right]_q \mathfrak{B}_{r,q}^{(\alpha)}(x,0) m^{r-n} + \mathfrak{B}_{n-k,q}^{(\alpha)}(x,0) m^{-k} \right\} \mathfrak{E}_{k,q}(0,my).
\]

Proof. From (1.3), we have

\[
\sum_{n=0}^{\infty} \mathfrak{E}_{n,q}^{(\alpha)}(x,y) \frac{t^n}{[n]_q!} = \frac{2}{e_q(t)+1} E_q \left( \frac{my}{m} \right) \frac{1}{2} e_q \left( \frac{t}{m} \right) \left( e_q \left( \frac{t}{m} \right) - 1 \right)^\alpha e_q (xt)
\]

\[
= \frac{1}{2} e_q \left( \frac{t}{m} \right) + 1 E_q \left( \frac{my}{m} \right) \frac{1}{2} e_q \left( \frac{t}{m} \right) \left( e_q \left( \frac{t}{m} \right) - 1 \right)^\alpha e_q (xt)
\]

\[
= \frac{1}{2} \sum_{n=0}^{\infty} \mathfrak{E}_{n,q}(0,my) \frac{t^n}{[n]_q!} \sum_{n=0}^{\infty} \mathfrak{B}_{n,q}^{(\alpha)}(x,0) \frac{t^n}{[n]_q!}
\]

\[
= \frac{1}{2} \left\{ \sum_{n=0}^{\infty} \mathfrak{E}_{n,q}(0,my) \frac{t^n}{[n]_q!} \right\} \left( \sum_{n=0}^{\infty} \mathfrak{B}_{n,q}^{(\alpha)}(x,0) \frac{t^n}{[n]_q!} \right)
\]

Comparing the coefficient of $\frac{t^n}{[n]_q!}$ we obtain

\[
\mathfrak{E}_{n,q}^{(\alpha)}(x,y) = \frac{1}{2} \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right]_q \left\{ \sum_{r=0}^{p} \left[ \begin{array}{c} p \\ r \end{array} \right]_q \mathfrak{B}_{r,q}^{(\alpha)}(x,0) m^{r-n} + \mathfrak{B}_{n-k,q}^{(\alpha)}(x,0) m^{-k} \right\} \mathfrak{E}_{k,q}(0,my).
\]

3.2. Theorem. There is the following relation between the q-Bernoulli polynomials and q-Euler polynomials.

\[
\mathfrak{E}_{n+k,q}(x,y) = \frac{m}{[n+1]_q} \sum_{k=0}^{n+1} \left[ \begin{array}{c} n+1 \\ k \end{array} \right]_q \left\{ \sum_{r=0}^{n+1-k} \left[ \begin{array}{c} n+1-k \\ r \end{array} \right]_q \mathfrak{B}_{r,q}^{(\alpha)}(x,0) m^{r-n-1} - \mathfrak{E}_{n+1-k,q}^{(\alpha)}(x,0) m^{-k} \right\} \mathfrak{E}_{k,q}(0,my).
\]
Proof. From (1.5), we write
\[
\sum_{n=0}^{\infty} e_{n,q}^{(\alpha)}(x,y) \frac{t^n}{[n]_q!} = \frac{t}{m} e_q \left( \frac{t}{m} \right) \sum_{n=0}^{\infty} \frac{e_t}{[n]_q} \left( \frac{t}{m} \right) - \left( \frac{2}{c} + 1 \right) e_q (x(t))
\]
\[
= \frac{m}{t} \sum_{n=0}^{\infty} \mathcal{B}_{n,q} (0,my) \frac{t^n}{m^n [n]_q!} \sum_{n=0}^{\infty} m^n [n]_q! \sum_{n=0}^{\infty} e_{n,q}^{(\alpha)} (x,0) \frac{t^n}{[n]_q!}.
\]
\[
= \frac{m}{t} \sum_{k=0}^{\infty} \mathcal{B}_{k,q} (0,my) \frac{t^k}{m^k [k]_q!} \left\{ \sum_{p=0}^{\infty} \sum_{r=0}^{\infty} p \frac{t^r}{r!} \mathcal{E}_{r,q}^{(\alpha)} (x,0) m^{r-p} - \mathcal{E}_{r,q}^{(\alpha)} (x,0) \right\} \frac{t^p}{[p]_q!}.
\]
Using the Cauchy product and comparing the the coefficient of \( \frac{t^n}{[n]_q!} \) we obtain (3.2).  

\[ \square \]

3.3. Corollary. The following relations holds
\[ (3.3) \]
\[
\mathcal{B}_{n,q}^{(\alpha)} = \frac{1}{[n+1]_q} \sum_{k=0}^{n+1} \left[ \frac{n+1}{k} \right] m^{k-n} \left\{ \mathcal{B}_{k,q}^{(\alpha)} \left( \frac{1}{m} \right) 0 + \mathcal{B}_{k,q}^{(\alpha)} (0,0) \right\} \mathcal{E}_{n+1-k,q} (0,0)
\]
and
\[ (3.4) \]
\[
\mathcal{E}_{n,q}^{(\alpha)} = \frac{1}{[n+1]_q} \sum_{k=0}^{n+1} \left[ \frac{n+1}{k} \right] m^{k-n} \left\{ \mathcal{E}_{k,q}^{(\alpha)} \left( \frac{1}{m} \right) 0 - \mathcal{E}_{k,q}^{(\alpha)} (0,0) \right\} \mathcal{B}_{n+1-k,q} (0,0).
\]

3.4. Corollary. From (3.3) and (3.4), we have
\[
\left\{ \mathcal{B}_{k,q}^{(\alpha)} \left( \frac{1}{m} \right) 0 + \mathcal{B}_{k,q}^{(\alpha)} (0,0) \right\} \mathcal{E}_{n+1-k,q} (0,0) \mathcal{E}_{n,q}^{(\alpha)} (0,0)
\]
\[
= \left\{ \mathcal{E}_{k,q}^{(\alpha)} \left( \frac{1}{m} \right) 0 - \mathcal{E}_{k,q}^{(\alpha)} (0,0) \right\} \mathcal{B}_{n+1-k,q} (0,0) \mathcal{B}_{n,q}^{(\alpha)} (0,0).
\]

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References


