Coefficient bounds for certain classes of bi-univalent functions

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Abstract

In this paper, we introduce two new subclasses of the function class Σ of bi-univalent functions defined in the open unit disk. Furthermore, we find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in these new subclasses.

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1. Introduction and definitions

Let $A$ denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disc $\mathcal{U} = \{ z : |z| < 1 \}$. Further, by $S$ we shall denote the class of all functions in $A$ which are univalent in $\mathcal{U}$. A function $f(z)$ belonging to $S$ is said to be starlike of order $\alpha$ if it satisfies

$$\Re \left( \frac{zf'(z)}{f(z)} \right) > \alpha \quad (z \in \mathcal{U})$$

for some $\alpha (0 \leq \alpha < 1)$. We denote by $S^*(\alpha)$ the subclass of $S$ consisting of functions which are starlike of order $\alpha$ in $\mathcal{U}$. Also, a function $f(z)$ belonging to $S$ is said to be convex of order $\alpha$ if it satisfies

$$\Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \alpha \quad (z \in \mathcal{U})$$

for some $\alpha (0 \leq \alpha < 1)$. We denote by $K(\alpha)$ the subclass of $S$ consisting of functions which are convex of order $\alpha$ in $\mathcal{U}$.

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Gao and Zhou [5] showed some mapping properties of the following subclass of $A$:

$$\mathcal{R}(\alpha, \beta) = \{ f \in A : \Re((f'(z) + \beta zf''(z))) > \alpha, \beta > 0, 0 \leq \alpha < 1; z \in \mathbb{U} \} .$$

Yang and Liu [12, Theorem 3.1, p.9], proved that the class $\mathcal{R}(\alpha, \beta) \subset S$ iff

$$2(1 - \alpha) \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m^{\beta m+1}} \leq 1 .$$

It is well known that every function $f \in S$ has an inverse $f^{-1}$, defined by

$$f^{-1}(f(z)) = z \quad (z \in \mathbb{U})$$

and

$$f^{-1}(f(w)) = w \quad (|w| < r_0(f); r_0(f) \geq \frac{1}{4})$$

where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots .$$

A function is said to be bi-univalent in $\mathbb{U}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathbb{U}$.

Let $\Sigma$ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1.1). Example of functions in the class $\Sigma$ are

$$z_1 - z, \log \frac{1}{1-z}, \log \sqrt{\frac{1+z}{1-z}} .$$

However, the familiar Koebe function is not a member of $\Sigma$. Other common examples of functions in $\mathbb{U}$ such as

$$\frac{2z - z^2}{2} \quad \text{and} \quad \frac{z}{1 - z^2}$$

are also not members of $\Sigma$.

Lewin [6] investigated the bi-univalent function class $\Sigma$ and showed that $|a_2| < 1.51$. Subsequently, Brannan and Clunie [1] conjectured that $|a_2| < \sqrt{2}$. Netanyahu [7], on the other hand, showed that $\max f \in \Sigma |a_2| = 4/3$.

The coefficient estimate problem for each of the Taylor–Maclaurin coefficients $|a_n|$ ($n \geq 3; n \in \mathbb{N}$) is presumably still an open problem.

Brannan and Taha [2] (see also [10]) introduced certain subclasses of the bi-univalent function class $\Sigma$ similar to the familiar subclasses $S^*(\alpha)$ and $K(\alpha)$ (see [3]). Thus, following Brannan and Taha [2] (see also [10]), a function $f \in A$ is in the class $S^*_C(\alpha)$ of strongly bi-starlike functions of order $\alpha (0 < \alpha \leq 1)$ if each of the following conditions are satisfied:

$$f \in \Sigma \text{ and } \left| \arg \left( \frac{zf'(z)}{f(z)} \right) \right| < \frac{\alpha \pi}{2} \quad (0 < \alpha \leq 1, z \in \mathbb{U})$$

and

$$\left| \arg \left( \frac{zg'(w)}{g(w)} \right) \right| < \frac{\alpha \pi}{2} \quad (0 < \alpha \leq 1, w \in \mathbb{U}) ,$$
where \( g \) is the extension of \( f^{-1} \) to \( \mathbb{U} \). The classes \( \mathcal{S}_\Sigma^*(\alpha) \) and \( \mathcal{K}_\Sigma(\alpha) \) of bi-starlike functions of order \( \alpha \) and bi-convex functions of order \( \alpha \), corresponding (respectively) to the function classes defined by (1.2) and (1.3), were also introduced analogously. For each of the function classes \( \mathcal{S}_\Sigma^*(\alpha) \) and \( \mathcal{K}_\Sigma(\alpha) \), they found non-sharp estimates on the first two Taylor–Maclaurin coefficients \(|a_2|\) and \(|a_3|\) (for details, see [7,8]).

The object of the present paper is to introduce two new subclasses of the function class \( \Sigma \) and find estimates on the coefficients \(|a_2|\) and \(|a_3|\) for functions in these new subclasses of the function class \( \Sigma \) employing the techniques used earlier by Srivastava et al. [9] (see also, [4] and [11]).

In order to derive our main results, we have to recall here the following lemma [8].

1.1. Lemma. If \( h \in \mathcal{P} \) then \(|c_k| \leq 2\) for each \( k \),

where \( \mathcal{P} \) is the family of all functions \( h \) analytic in \( \mathbb{U} \) for which \( \Re h(z) > 0 \),

\[ h(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots \quad \text{for} \quad z \in \mathbb{U}. \]

2. Coefficient bounds for the function class \( \mathcal{H}_\Sigma(\alpha, \beta) \)

2.1. Definition. A function \( f(z) \) given by (1.1) is said to be in the class \( \mathcal{H}_\Sigma(\alpha, \beta) \) if the following conditions are satisfied:

\begin{align*}
(2.1) & \quad f \in \Sigma \text{ and } |\arg (f'(z) + \beta zf''(z))| < \frac{\alpha \pi}{2} \quad (z \in \mathbb{U}) \\
\text{and} \quad (2.2) & \quad |\arg (g'(w) + \beta wg''(w))| < \frac{\alpha \pi}{2} \quad (w \in \mathbb{U}),
\end{align*}

where \( \beta > 0, 0 < \alpha < 1, 2(1 - \alpha) \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{\beta^{m+1}} \leq 1, \) and the function \( g \) is given by

\[ g(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_3^2 - 5a_2a_3 + a_4)w^4 + \cdots. \]

We begin by finding the estimates on the coefficients \(|a_2|\) and \(|a_3|\) for functions in the class \( \mathcal{H}_\Sigma(\alpha, \beta) \).

2.2. Theorem. Let \( f(z) \) given by (1.1) be in the class \( \mathcal{H}_\Sigma(\alpha, \beta) \) where \( \beta > 0, 0 < \alpha < 1, \) and \( 2(1 - \alpha) \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{\beta^{m+1}} \leq 1. \) Then

\begin{align*}
(2.4) & \quad |a_2| \leq \frac{2\alpha}{\sqrt{2(\alpha + 2) + 4\beta(\alpha + \beta + 2 - \alpha\beta)}} \\
\text{and} \quad (2.5) & \quad |a_3| \leq \frac{\alpha^2}{(1 + \beta)^2} + \frac{2\alpha}{3(1 + 2\beta)}.
\end{align*}
Proof. It follows from (2.1) and (2.2) that

\[(2.6) \quad f'(z) + \beta zf''(z) = [p(z)]^\alpha \]

and

\[(2.7) \quad g'(w) + \beta wg''(w) = [q(w)]^\alpha \]

where \(p(z)\) and \(q(w)\) in \(P\) and have the forms

\[(2.8) \quad p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots \]

and

\[(2.9) \quad q(w) = 1 + q_1 w + q_2 w^2 + q_3 w^3 + \cdots . \]

Now, equating the coefficients in (2.6) and (2.7), we get

\[(2.10) \quad 2(1 + \beta)a_2 = \alpha p_1, \]

\[(2.11) \quad 3(1 + 2\beta)a_3 = \alpha p_2 + \frac{\alpha(\alpha - 1)}{2} p_1^2, \]

\[(2.12) \quad -2(1 + \beta)a_2 = \alpha q_1 \]

and

\[(2.13) \quad 3(1 + 2\beta)(2a_2^2 - a_3) = \alpha q_2 + \frac{\alpha(\alpha - 1)}{2} q_1^2. \]

From (2.10) and (2.12), we get

\[(2.14) \quad p_1 = -q_1 \]

and

\[(2.15) \quad 8(1 + \beta)^2 a_2^2 = \alpha^2 (p_1^2 + q_1^2). \]

Now from (2.11), (2.13) and (2.15), we obtain

\[(2.16) \quad 6(1 + 2\beta)a_2^2 = \alpha (p_2 + q_2) + \frac{\alpha(\alpha - 1)}{2} (p_1^2 + q_1^2) \]

\[= \alpha (p_2 + q_2) + \frac{4(\alpha - 1)(1 + \beta)^2}{\alpha} a_2^2. \]

Therefore, we have

\[a_2^2 = \frac{\alpha^2 (p_2 + q_2)}{2(\alpha + 2) + 4\beta(\alpha + \beta + 2 - \alpha\beta)}. \]

Applying Lemma 1.1 for the coefficients \(p_2\) and \(q_2\), we immediately have

\[|a_2| \leq \frac{2\alpha}{\sqrt{2(\alpha + 2) + 4\beta(\alpha + \beta + 2 - \alpha\beta)}}. \]

This gives the bound on \(|a_2|\) as asserted in (2.4).

Next, in order to find the bound on \(|a_3|\), by subtracting (2.13) from (2.11), we get

\[(2.16) \quad 6(1 + 2\beta)a_3 - 6(1 + 2\beta)a_2^2 = \alpha p_2 + \frac{\alpha(\alpha - 1)}{2} p_1^2 - \left( \alpha q_2 + \frac{\alpha(\alpha - 1)}{2} q_1^2 \right). \]
Upon substituting the value of $a_2^2$ from (2.15) and observing that $p_1^2 = q_1^2$, it follows that

$$a_3 = \frac{\alpha^2 p_1^2}{4(1 + \beta)^2} + \frac{\alpha(p_2 - q_2)}{6(1 + 2\beta)}.$$  

Applying Lemma 1.1 once again for the coefficients $p_1, p_2, q_1$ and $q_2$, we readily get

$$|a_3| \leq \frac{\alpha^2}{(1 + \beta)^2} + \frac{2\alpha}{3(1 + 2\beta)}.$$  

This completes the proof of Theorem 2.2. □

Putting $\beta = 1$ in Theorem 2.2, we have

2.3. Corollary. Let $f(z)$ given by (1.1) be in the class $\mathcal{H}_\Sigma(\alpha, 1)$ where $0 < \alpha < 1$, and $2(1 - \alpha) \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m+1} \leq 1$. Then

$$|a_2| \leq \frac{2\alpha}{\sqrt{2(\alpha + 2) + 12}}$$  

and

$$|a_3| \leq \frac{9\alpha^2 + 8\alpha}{36}.$$  

3. Coefficient bounds for the function class $\mathcal{H}_\Sigma(\gamma, \beta)$

3.1. Definition. A function $f(z)$ given by (1.1) is said to be in the class $\mathcal{H}_\Sigma(\gamma, \beta)$ if the following conditions are satisfied:

1. $f \in \Sigma$ and $\Re(f'(z) + \beta zf''(z)) > \gamma \quad (z \in \mathbb{U})$  

and

2. $\Re(g'(w) + \beta wg''(w)) > \gamma \quad (w \in \mathbb{U})$,

where $\beta > 0, 0 \leq \gamma < 1, 2(1 - \gamma) \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{\beta m+1} \leq 1$, and the function $g$ is given by (2.3).

3.2. Theorem. Let $f(z)$ given by (1.1) be in the class $\mathcal{H}_\Sigma(\gamma, \beta)$, where $\beta > 0, 0 \leq \gamma < 1$, and $2(1 - \gamma) \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{\beta m+1} \leq 1$. Then

$$|a_2| \leq \frac{2(1 - \gamma)}{3(1 + 2\beta)}$$  

and

$$|a_3| \leq \frac{(1 - \gamma)^2}{(1 + \beta)^2} + \frac{2(1 - \gamma)}{3(1 + 2\beta)}.$$
Proof. It follows from (3.1) and (3.2) that there exist \( p \) and \( q \in \mathcal{P} \) such that
\[
f'(z) + \beta zf''(z) = \gamma + (1 - \gamma)p(z)
\]
and
\[
g'(w) + \beta wg''(w) = \gamma + (1 - \gamma)q(w)
\]
where \( p(z) \) and \( q(w) \) have the forms (2.8) and (2.9), respectively. Equating coefficients in (3.5) and (3.6) yields
\[
2(1 + \beta)a_2 = (1 - \gamma)p_1,
\]
\[
3(1 + 2\beta)a_3 = (1 - \gamma)p_2,
\]
\[
-2(1 + \beta)a_2 = (1 - \gamma)q_1
\]
and
\[
3(1 + 2\beta)(2a_2^2 - a_3) = (1 - \gamma)q_2
\]
From (3.7) and (3.9), we get
\[
p_1 = -q_1
\]
and
\[
8(1 + \beta)^2a_2^2 = (1 - \gamma)^2(p_2^2 + q_1^2).
\]
Also, from (3.8) and (3.10), we find that
\[
6(1 + 2\beta)a_3^2 = (1 - \gamma)(p_2 + q_2).
\]
Thus, we have
\[
|a_2^2| \leq \frac{(1 - \gamma)}{6(1 + 2\beta)}(|p_2| + |q_2|) = \frac{2(1 - \gamma)}{3(1 + 2\beta)}
\]
which is the bound on \( |a_2^2| \) as given in (3.3).

Next, in order to find the bound on \( |a_3| \), by subtracting (3.10) from (3.8), we get
\[
6(1 + 2\beta)a_3 - 6(1 + 2\beta)a_3^2 = (1 - \gamma)(p_2 - q_2)
\]
or, equivalently,
\[
a_3 = a_2^2 + \frac{(1 - \gamma)(p_2 - q_2)}{6(1 + 2\beta)}.
\]
Upon substituting the value of \( a_2^2 \) from (3.12), we obtain
\[
a_3 = \frac{(1 - \gamma)^2(p_2^2 + q_1^2)}{8(1 + \beta)^2} + \frac{(1 - \gamma)(p_2 - q_2)}{6(1 + 2\beta)}.
\]
Applying Lemma 1.1 for the coefficients \( p_1, p_2, q_1 \) and \( q_2 \), we readily get
\[
|a_3| \leq \frac{(1 - \gamma)^2}{(1 + \beta)^2} + \frac{2(1 - \gamma)}{3(1 + 2\beta)}
\]
which is the bound on $|a_3|$ as asserted in (3.4).

Putting $\beta = 1$ in Theorem 3.2, we have

3.3. Corollary. Let $f(z)$ given by (1.1) be in the class $\mathcal{H}_\Sigma(\gamma, 1)$, where $0 \leq \gamma < 1$, and $2(1-\gamma) \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m+1} \leq 1$.

\begin{align*}
(3.13) \quad |a_2| & \leq \frac{1}{3} \sqrt{2(1-\gamma)} \\
\text{and} \quad (3.14) \quad |a_3| & \leq \frac{(1-\gamma)(9(1-\gamma)+8)}{36}.
\end{align*}

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References