Some results on $\sigma$-ideal of $\sigma$-prime ring

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Abstract
Let $R$ be a $\sigma$-prime ring with characteristic not 2, $Z(R)$ be the center of $R$, $I$ be a nonzero $\sigma$-ideal of $R$, $\alpha, \beta : R \to R$ be two automorphisms, $d$ be a nonzero $(\alpha, \beta)$-derivation of $R$ and $h$ be a nonzero derivation of $R$. In this paper, it is shown that (i) If $d(I) \subset C_{\alpha, \beta}$ and $\beta$ commutes with $\sigma$ then $R$ is commutative. (ii) Let $\alpha$ and $\beta$ commute with $\sigma$. If $a \in I \cap S_\sigma(R)$ and $[d(I), a]_{\alpha, \beta} \subset C_{\alpha, \beta}$ then $a \in Z(R)$. (iii) Let $\alpha, \beta$ and $h$ commute with $\sigma$. If $dh(I) \subset C_{\alpha, \beta}$ and $h(I) \subset I$ then $R$ is commutative.

Keywords: $\sigma$-prime ring, $\sigma$-ideal, $(\alpha, \beta)$-derivation

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1. Introduction
Let $R$ be an associative ring with center $Z(R)$. $R$ is said to be 2-torsion free if whenever $2x = 0$ with $x \in R$, then $x = 0$. Recall that a ring $R$ is prime if $aRb = 0$ implies $a = 0$ or $b = 0$. An involution $\sigma$ of a ring $R$ is an additive mapping satisfying $\sigma(xy) = \sigma(y)\sigma(x)$ and $\sigma^2(x) = x$ for all $x, y \in R$. A ring $R$ equipped with an involution $\sigma$ is said to be $\sigma$-prime if $aRb = aR\sigma(b) = 0$ implies $a = 0$ or $b = 0$. Note that every prime ring which has an involution $\sigma$ is a $\sigma$-prime but the converse is in generally not true. An example, due to Shuliang [8], if $R^d$ denotes the opposite ring of a prime ring $R$, then $R \times R^d$ equipped with the exchange involution $\sigma_{ex}$, defined by $\sigma_{ex}(x, y) = (y, x)$, is $\sigma_{ex}$-prime but not prime. An additive subgroup $I$ of $R$ is said to be an ideal of $R$ if $xr, rx \in I$ for all $x \in I$ and $r \in R$. An ideal $I$ which satisfies $\sigma(I) = I$ is called a $\sigma$-ideal of $R$. An example, due to Rehman [8], Set $R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{Z} \right\}$. We define a map $\sigma : R \to R$ as follows: $\sigma \left( \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \right) = \begin{pmatrix} c & -b \\ 0 & a \end{pmatrix}$. It is easy to check that $I = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid b \in \mathbb{Z} \right\}$ is a $\sigma$-ideal of $R$. Note that $I \cap S_{\sigma_{ex}}(R) \neq \{0\}$. For $\alpha, (\alpha, \beta)$-derivation, $d(aRb) = \alpha(d(a))Rb + ad(Rb) + aRd(b)$.

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Consider a map $\sigma : R \to R$ defined by $\sigma((a,b)) = (b,a)$ for all $(a,b) \in R$. For an ideal $I = \mathbb{Z} \times \{0\}$ of $R$, $I$ is not a $\sigma$-ideal of $R$ since $\sigma(I) = \{0\} \neq I$. $S_\sigma (R)$ will denote the set of symmetric and skew symmetric elements of $R$. i.e. $S_\sigma (R) = \{ x \in R \mid \sigma(x) = \pm x \}$.

As usual the commutator $xy - yx$ will be denoted by $[x,y] = xy - yx$. An additive mapping $h : R \to R$ is called a derivation if $h(xy) = h(x)y + xh(y)$ holds for all $x, y \in R$. For a fixed $a \in R$, the mapping $I_a : R \to R$ is given by $I_a (x) = [a,x]$ is a derivation which is said to be an inner derivation which is determined by $a$. Let $\alpha$ and $\beta$ be two maps of $R$. Set $C_{\alpha,\beta} = \{ c \in R \mid \alpha(c) = \beta(c) \}$ for all $r \in R$ and known as $(\alpha, \beta)$-center of $R$. In particular, $C_{1,1} = Z(R)$ is the center of $R$, where $1 : R \to R$ is identity map. As usual the $(\alpha, \beta)$-commutator $a_\alpha(b) - \beta(b)a$ will be denoted by $[a,b]_{\alpha,\beta} = a_\alpha(b) - \beta(b)a$. An additive mapping $d : R \to R$ is called an $(\alpha, \beta)$-derivation if $d(xy) = d(x)\alpha(y) + \beta(x)d(y)$ holds for all $x, y \in R$. For a fixed $a \in R$, the mapping $I_a : R \to R$ is given by $I_a (x) = [a,x]_{\alpha,\beta}$ is an $(\alpha, \beta)$-inner derivation which is determined by $a$.

Many studies have been objected the relationship between commutativity of a ring and the act of derivations defined on this ring. These results have been generalized by many authors in several ways. Herstein [2] proved that if $R$ is a prime ring of characteristic not 2, $d$ is a nonzero derivation of $R$ and $a \in R$ such that $[a,d(R)] = 0$ then $a \in Z(R)$. N. Aydın and K. Kaya [1] proved that if $R$ is a prime ring of characteristic not 2, $I$ is a nonzero right ideal of $R$, $\sigma$ and $\tau$ are two automorphisms of $R$, $d : R \to R$ is a nonzero $(\sigma, \tau)$-derivations of $R$ and $a \in R$ such that (i) $d(I) \subset Z(R)$ then $R$ is commutative. (ii) $[d(R), a]_{\sigma,\tau} \subset C_{\alpha,\beta}$ then $a \in Z(R)$. In [5], this result was extended to find a prime ring of characteristic not 2, $I$ is a nonzero ideal of $R$, $\sigma$ and $\tau$ are two automorphisms of $R$, $d : R \to R$ is a nonzero $(\sigma, \tau)$-derivations of $R$ and $d_1d_2$ is a nonzero derivation of $R$ such that $d_1d_2(I) \subset C_{\alpha,\beta}$ then $R$ is commutative. In [4], Posner’s result was extended to find a nonzero ideal of a prime ring by L. Oukhtite and S. Salhi. Motivated by these results, we follow this line of investigation.

In this paper, our main goal is to extend these results on a $\sigma$-ideal of a $\sigma$-prime ring.

Throughout the present paper, $R$ is a $\sigma$-prime ring, $Z(R)$ is the center of $R$ and $\alpha, \beta$ are two automorphisms of $R$. We use the following basic commutator identities:

\[
\begin{align*}
[x, yz] &= y[x, z] + [x, y]z \\
[xy, z] &= x[y, z] + [x, z]y \\
[xy]_{\alpha,\beta} &= x[y, z]_{\alpha,\beta} + [x, \beta(z)]y = x[y, \alpha(z)] + [x, z]_{\alpha,\beta}y \\
[x, yz]_{\alpha,\beta} &= \beta(y)[x, z]_{\alpha,\beta} + [x, y]_{\alpha,\beta}\alpha(z) \\
[x, y]_{\alpha,\beta}, z_{\alpha,\beta} &= [x, y]_{\alpha,\beta}y + [x, y, z]_{\alpha,\beta}
\end{align*}
\]

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2. Results

For the proof of our theorems, we give the following known Lemmas.

2.1. Lemma. [6, Theorem 2.2] Let $I$ be a nonzero $\sigma$-ideal of $\sigma$-prime ring $R$. If $a, b$ in $R$ are such that $ab = 0 = aI\sigma(b)$ then $a = 0$ or $b = 0$. 

2.2. Lemma. [5, Lemma 4] Let $R$ be a $\sigma$-prime ring with characteristic not two, $d$ be a derivation of $R$ satisfying $d\sigma = \pm \sigma d$ and $I$ be a nonzero $\sigma$-ideal of $R$. If $d^2(I) = 0$ then $d = 0$.

2.3. Lemma. Let $I$ be a nonzero $\sigma$-ideal of $R$ and $a \in R$. If $Ia = 0$ (or $aI = 0$) then $a = 0$.

Proof. Since $I$ is a $\sigma$-ideal, we know that $IR \subseteq I$. By hypothesis, we have $IRa \subseteq Ia = 0$. Thus, we get $IRa = 0$. Moreover, since $I$ is invariant under $\sigma$, we have $\sigma(I)Ra = 0$. It follows that

$$IRa = \sigma(I)Ra = 0$$

Using $\sigma$-primeness of $R$, we get

$$a = 0$$

Similarly, using $RI \subseteq I$, one can show that if $aI = 0$ then $a = 0$. \hfill \Box

2.4. Lemma. Let $a, b \in R$.

i) If $b, ab \in C_{\alpha, \beta}$ and $a$ (or $b$) $\in S_\sigma(R)$ then $a \in Z(R)$ or $b = 0$.

ii) If $a, ab \in C_{\alpha, \beta}$ and $a$ (or $b$) $\in S_\sigma(R)$ then $a = 0$ or $b \in Z(R)$.

Proof. i) By the hypothesis, we have $[ab, r]_{\alpha, \beta} = 0$ for all $r \in R$. Expanding this equation by using $b \in C_{\alpha, \beta}$, holding for all $r \in R$

$$0 = [ab, r]_{\alpha, \beta} = a \begin{pmatrix} b, \alpha (r) \end{pmatrix} + \begin{pmatrix} a, \beta (r) \end{pmatrix} b$$

Since $b \in C_{\alpha, \beta}$, we get

$$[a, R] RB = 0$$

In the event of $a \in S_\sigma(R)$, we derive $\sigma([a, R]) RB = 0$. Using the last obtained equation together with (2.1), we yield

$$[a, R] RB = \sigma([a, R]) RB = 0$$

Applying the $\sigma$-primeness of $R$, we have

$$a \in Z(R) \text{ or } b = 0$$

In case of $b \in S_\sigma(R)$, from (2.1), we get $[a, R] R\sigma(b) = 0$. Using the last obtained equation together with (2.1), we find

$$[a, R] RB = [a, R] R\sigma(b) = 0$$

Applying the $\sigma$-primeness of $R$,

$$a \in Z(R) \text{ or } b = 0$$

is obtained.

ii) Since $ab \in C_{\alpha, \beta}$, we have $[ab, r]_{\alpha, \beta} = 0$ for all $r \in R$. Expanding this equation by using $a \in C_{\alpha, \beta}$, holding for all $r \in R$

$$0 = [ab, r]_{\alpha, \beta} = a \begin{pmatrix} b, \alpha (r) \end{pmatrix} + \begin{pmatrix} a, r \end{pmatrix}_{\alpha, \beta} b$$

Since $a \in C_{\alpha, \beta}$,

$$aR[b, R] = 0$$

is obtained. After here, it is similar as above. \hfill \Box
2.5. Lemma. Let $I$ be a nonzero $\sigma$-ideal of $R$ and $h$ be a nonzero derivation of $R$. If $h(I) \subset Z(R)$ then $R$ is commutative.

Proof. For any $x, y \in I$ and $r \in R$, using hypothesis,

$$0 = [r, h(xy)] = [r, h(x)y + xh(y)]$$

$$= h(x)[r, y] + [r, h(x)]y + [r, h(y)] + [r, x]h(y)$$

$$= h(x)[r, y] + [r, x]h(y)$$

And so,

$$h(x)[r, y] + [r, x]h(y) = 0, \forall x, y \in I, r \in R$$

is obtained. In the last equality, $x$ is taken instead of $r$ and we obtain $h(x)[x, y] = 0$ for all $x, y \in I$. Substituting $y$ by $zy$ where $z \in I$, it holds that

$$(2.2) \quad h(x)[x, y] = 0, \forall x, y \in I$$

It is supposed that $x \in I \cap S_\sigma(R)$. In $(2.2)$, replacing $y$ with $\sigma(y)$, we get $h(x)I\sigma([x, y]) = 0$ for all $y \in I$. According to Lemma 2.1, it is derived that

$$(2.3) \quad h(x) = 0 \text{ or } x \in Z(R), \forall x \in I \cap S_\sigma(R)$$

Assume that $x \in I$. In this case, $x - \sigma(x) \in I \cap S_\sigma(R)$. So, from $(2.3)$, we have $h(x - \sigma(x)) = 0$ or $x - \sigma(x) \in Z(R)$ for all $x \in I$. We set $A = \{x \in I \mid h(x - \sigma(x)) = 0\}$ and $B = \{x \in I \mid x - \sigma(x) \in Z(R)\}$. It is clear that $A$ and $B$ are additive subgroups of $I$ such that $I = A \cup B$. But, a group cannot be an union of two of its proper subgroups. Therefore, it is implied $I = A$ or $I = B$. In the former case, $h(x) = h(\sigma(x))$ for all $x \in I$. In $(2.2)$, replacing $y$ by $\sigma(y)$ and $x$ by $\sigma(x)$, we have $h(x)I\sigma([x, y]) = 0$ for all $x, y \in I$. And so,

$$h(x)[x, y] = h(x)I\sigma([x, y]) = 0, \forall x, y \in I$$

is obtained. By Lemma 2.1, get $h(x) = 0$ or $x \in Z(R)$ for all $x \in I$. In the latter case, $x - \sigma(x) \in Z(R)$ for all $x \in I$. This means $[x, r] = [\sigma(x), r]$ for all $x \in I, r \in R$. In $(2.2)$, taking $\sigma(y)$ instead of $y$, we get $h(x)I\sigma([x, y]) = 0$ for all $x, y \in I$. And so, $h(x)[x, y] = h(x)I\sigma([x, y]) = 0, \forall x, y \in I$ is derived. According to Lemma 2.1, we have $h(x) = 0$ or $x \in Z(R)$ for all $x \in I$. So, both the cases yield either

$$h(x) = 0 \text{ or } x \in Z(R), \forall x \in I$$

Now, we set $K = \{x \in I \mid h(x) = 0\}$ and $L = \{x \in I \mid x \in Z(R)\}$. Each of $K$ and $L$ is an additive subgroup of $I$. Moreover, $I$ is the set-theoretic union of $K$ and $L$. But a group cannot be the set-theoretic union of two proper subgroups, hence $I = K$ or $I = L$. In the former case, $h(I) = 0$. So, we have $h = 0$. But, $h$ is a nonzero derivation of $R$. So, from the latter case, we get $I \subseteq Z(R)$. Therefore, $R$ is commutative. \qed

2.6. Lemma. Let $I$ be a nonzero $\sigma$-ideal of $R$, $d$ be a $(\alpha, \beta)$-derivation of $R$ and $a \in R$. If $ad(I) = \sigma(a)d(I) = 0$ and $\beta$ commutes with $\sigma$ (or $d(I)a = d(I)\sigma(a) = 0$ and $\alpha$ commutes with $\sigma$) then $a = 0$ or $d = 0$.

Proof. For any $x \in I$ and $r \in R$, using $ad(I) = 0$, we get

$$0 = ad(xr) = ad(x)\alpha(r) + a\beta(x)d(r)$$

It becomes

$$a\beta(I)d(r) = 0, \forall r \in R$$
Similarly, using $\sigma(a) d(I) = 0$, we derive
\[
\sigma(a) \beta(I) d(r) = 0, \; \forall r \in R
\]

And so,
\[
a \beta(I) d(r) = \sigma(a) \beta(I) d(r) = 0, \; \forall r \in R
\]
is obtained. Since $\beta$ commutes with $\sigma$, $\beta(I)$ is a nonzero $\sigma$-ideal of $R$. Therefore, according to Lemma 2.1, we have

\[
a = 0 \text{ or } d = 0
\]

Let us consider $d(I) a = d(I) \sigma(a) = 0$ and $\alpha$ commutes with $\sigma$. Since $\alpha(I)$ is a nonzero $\sigma$-ideal of $R$, one can show that $a = 0$ or $d = 0$ similarly as above.

\textbf{2.7. Lemma.} Let $I$ be a nonzero $\sigma$-ideal of $R$ and $d$ be a ($\alpha, \beta$)-derivation of $R$. If $d(I) = 0$ and $\alpha$ (or $\beta$) commutes with $\sigma$ then $d = 0$.

\textbf{Proof.} By hypothesis, it holds that for all $x \in I$ and $r \in R$
\[
0 = d(xr) = d(r) \alpha(x) + \beta(r) d(x)
\]
Thus, we get
\[
d(r) \alpha(I) = 0, \; \forall r \in R
\]
Since $\alpha$ commutes with $\sigma$, $\alpha(I)$ is a nonzero $\sigma$-ideal of $R$. Therefore, by Lemma 2.3, we have $d = 0$.

Suppose that $\beta$ commutes with $\sigma$. For any $x \in I$ and $r \in R$, from the hypothesis, we get
\[
0 = d(xr) = d(x) \alpha(r) + \beta(x) d(r)
\]
So, it yields that
\[
\beta(I) d(r) = 0, \; \forall r \in R
\]

Since $\beta$ commutes with $\sigma$, $\beta(I)$ is a nonzero $\sigma$-ideal of $R$. Therefore, by Lemma 2.3, we have $d = 0$. \hfill \square

\textbf{2.8. Theorem.} Let $R$ be a $\sigma$-prime ring with characteristic not 2, $I$ be a nonzero $\sigma$-ideal of $R$ and $d$ be a nonzero ($\alpha, \beta$)-derivation of $R$ such that $\beta$ commutes with $\sigma$. If $d(I) \subset C_{\alpha, \beta}$ then $R$ is commutative.

\textbf{Proof.} By hypothesis, $d(x^2) = d(x) \alpha(x) + \beta(x) d(x) \in C_{\alpha, \beta}$ for all $x \in I$. Using $d(x) \in C_{\alpha, \beta}$, we get $2 \beta(x) d(x) \in C_{\alpha, \beta}$. Since $\text{char} R \neq 2$, we obtain $\beta(x) d(x) \in C_{\alpha, \beta}$ which means $[\beta(x) d(x), r]_{\alpha, \beta} = 0$ for all $r \in R, x \in I$. Expanding this equation by using $d(x) \in C_{\alpha, \beta}$, we arrive
\[
0 = [\beta(x) d(x), r]_{\alpha, \beta} = \beta(x) [d(x), r]_{\alpha, \beta} + \beta([x, r]) d(x)
\]
Since $d(x) \in C_{\alpha, \beta}$, it follows that
\[
(2.4) \quad \beta([x, r]) Rd(x) = 0, \; \forall x \in I, r \in R
\]
Assume that $x \in I \cap S_{\sigma}(R)$. In (2.4) taking $\sigma(r)$ instead of $r$ and using the fact that $\beta$ commutes with $\sigma$, we have $\sigma(\beta([x, r])) Rd(x) = 0$ for all $x \in I, r \in R$. Since $R$ is $\sigma$-prime, we derive
\[
x \in Z(R) \text{ or } d(x) = 0, \; \forall x \in I \cap S_{\sigma}(R)
\]
Assume that \( x \in I \). In this case, \( x - \sigma(x) \in I \cap S_\sigma(R) \). Therefore, we have \( x - \sigma(x) \in Z(R) \) or \( d(x - \sigma(x)) = 0 \) for all \( x \in I \). Set \( A = \{ x \in I \mid d(x - \sigma(x)) = 0 \} \) and \( B = \{ x \in I \mid x - \sigma(x) \in Z(R) \} \). It is clear that \( A \) and \( B \) are additive subgroups of \( I \) such that \( I = A \cup B \). But, a group cannot be an union of two of its proper subgroups. Therefore, we yield either \( I = A \) or \( I = B \). In the former case, \( d(x) = d(\sigma(x)) \) for all \( x \in I \). In (2.4) substituting \( x \) by \( \sigma(x) \) and \( r \) by \( \sigma(r) \) and using the fact that \( \beta \) commutes with \( \sigma \), we have \( \sigma(\beta([x,r])) R d(x) = 0 \) for all \( x \in I, r \in R \). Since \( R \) is \( \beta \)-prime, we arrive \( x \in Z(R) \) or \( d(x) = 0 \) for all \( x \in I \). In the latter case, \( x - \sigma(x) \in Z(R) \) for all \( x \in I \).

This means, \([x,r] = [\sigma(x),r]\) for all \( r \in R \). In (2.4), replacing \( r \) by \( \sigma(r) \) and using the fact that \( \beta \) commutes with \( \sigma \), we get \( \sigma(\beta([x,r])) R d(x) = 0 \) for all \( x \in I, r \in R \). Since \( R \) is \( \beta \)-prime, we have \( x \in Z(R) \) or \( d(x) = 0 \) for all \( x \in I \). As a result, both the cases yield either

\[ x \in Z(R) \text{ or } d(x) = 0, \forall x \in I \]

Now, we set \( K = \{ x \in I \mid d(x) = 0 \} \) and \( L = \{ x \in I \mid x \in Z(R) \} \). Each of \( K \) and \( L \) is an additive subgroup of \( I \). Moreover, \( I \) is the set-theoretic union of \( K \) and \( L \). But a group cannot be the set-theoretic union of two of its proper subgroups, hence \( I = K \) or \( I = L \).

In the former case, \( d(I) = 0 \). Since \( \beta \) commutes with \( \sigma \), by Lemma 2.7, we obtain \( d = 0 \). But, \( d \) is a nonzero \((\alpha,\beta)\)-derivation of \( R \), then \( I \) must be contained in \( Z(R) \). So, \( R \) is commutative.

**2.9. Lemma.** Let \( R \) be a \( \sigma \)-prime ring with characteristic not 2, \( I \) be a nonzero \( \sigma \)-ideal of \( R \), \( d \) be a \((\alpha,\beta)\)-derivation of \( R \) such that \( \beta \) commutes with \( \sigma \) and \( h \) be a derivation of \( R \) satisfying \( h\sigma = \pm \sigma h \). If \( dh(I) = 0 \) and \( h(I) \subseteq I \) then \( d = 0 \) or \( h = 0 \).

**Proof.** By hypothesis, it holds that for all \( x, y \in I \)

\[
0 = dh(xy)
\]

\[
= dh(x)\alpha(y) + \beta(h(x))d(y) + d(x)\alpha(h(y)) + \beta(x)dh(y)
\]

\[
= \beta(h(x))d(y) + d(x)\alpha(h(y))
\]

And so,

\[
\beta(h(x))d(y) + d(x)\alpha(h(y)) = 0, \forall x, y \in I
\]

Since \( h(I) \subseteq I \), we take \( h(x) \) instead of \( x \). Using the hypothesis, we get

\[
\beta(h^2(x))d(I) = 0, \forall x \in I
\]

Moreover, replacing \( x \) by \( \sigma(x) \) in the above obtained relation and using the fact that \( \beta \) commute with \( \sigma \) and \( h\sigma = \pm \sigma h \), we derive

\[
\sigma(\beta(h^2(x)))d(I) = 0, \forall x \in I
\]

And so,

\[
\beta(h^2(x))d(I) = \sigma(\beta(h^2(x)))d(I) = 0, \forall x \in I
\]

Since \( \beta \) commutes with \( \sigma \), by Lemma 2.6, we yield either \( h^2(I) = 0 \) or \( d = 0 \). Since \( h\sigma = \pm \sigma h \), by Lemma 2.2, we have \( h = 0 \) or \( d = 0 \).

**2.10. Lemma.** Let \( R \) be a \( \sigma \)-prime ring with characteristic not 2, \( I \) be a nonzero \( \sigma \)-ideal of \( R \), \( d \) be a nonzero \((\alpha,\beta)\)-derivation of \( R \) such that \( \beta \) commutes with \( \sigma \). If \( a \in I \cap S_\sigma(R) \) and \([d(I),a]_{\alpha,\beta} = 0\) then \( a \in Z(R) \).
Proof. For any \( x, y \in I \), from the hypothesis, we have \( [d([x, y]), a]_{\alpha, \beta} = 0 \). Since \( d([x, y]) = [d(x), y]_{\alpha, \beta} - [d(y), x]_{\alpha, \beta} \), we get
\[
[d(y), x]_{\alpha, \beta} = [d(x), y]_{\alpha, \beta}, \quad \forall x, y \in I
\]

In the above obtained relation, applying \([a, b]_{\alpha, \beta} c = [a, c]_{\alpha, \beta} b + [a, b]_{\alpha, \beta}\) for all \( a, b, c \in R \) and using the hypothesis, it becomes
\[
[d(y), x]_{\alpha, \beta} = [d(x), y]_{\alpha, \beta} + [d(x), [y, a]]_{\alpha, \beta}
\]

And so,
\[
[d(y), x]_{\alpha, \beta} = [d(x), [y, a]]_{\alpha, \beta}, \quad \forall x, y \in I
\]
is obtained. In the last equation, substituting \( x \) by \( a \) and using the hypothesis, we yield
\[
[d(a), [y, a]]_{\alpha, \beta} = 0, \quad \forall y \in I
\]
The mapping \( I(d(a)) : R \to R \) is given by \( I(d(a)) (r) = [d(a), r]_{\alpha, \beta} \) is a \((\alpha, \beta)\)-derivation which is determined by \( d(a) \) and \( I_a : R \to R \) is given by \( I_a (r) = [r, a] \) is a derivation which is determined by \( a \). So, we derive
\[
(I(d(a)) I_a) (I) = 0
\]
Since \( a \in I \cap S_\sigma (R) \), we have \( I_a \sigma = \pm \sigma I_a \). Therefore, by Lemma 2.9, we have
\[
d(a) \in C_{\alpha, \beta} \text{ or } a \in Z (R)
\]
Assume that \( a \notin Z (R) \), which means that \( d(a) \in C_{\alpha, \beta} \). From the hypothesis, we get
\[
d([x, a]) = [d(x), a]_{\alpha, \beta} - [d(a), x]_{\alpha, \beta} = 0 \quad \text{for all } x \in I. \quad \text{That is,}
\]
(2.5)
\[
d([I, a]) = 0
\]
On the other hand, by hypothesis, we have \( [d(xy), a]_{\alpha, \beta} = 0 \) for \( x, y \in I \). Expanding this equation, it becomes \( d(x) \alpha ([y, a]) + \beta ([x, a]) d(y) = 0 \) for all \( x, y \in I \). Taking \([x, a]\) instead of \( x \) and using (2.5), we derive \( \beta ([x, a], a] d(I) = 0 \) for all \( x \in I \). In this equation, replacing \( x \) by \( \sigma (x) \) and using the fact that \( \beta \) commutes with \( \sigma \), we obtain \( \sigma (\beta ([x, a], a]) d(I) = 0 \) for all \( x \in I \). And so, we yield
\[
\beta ([x, a], a]) d(I) = \sigma (\beta ([x, a], a]) d(I) = 0, \quad \forall x \in I
\]
Since \( \beta \) commutes with \( \sigma \), by Lemma 2.6, it implies that \( d = 0 \) or \([x, a], a] = 0 \) for all \( x \in I \). That is, \( d = 0 \) or \( I_a^2 (I) = 0 \). Since \( a \in I \cap S_\sigma (R) \), we have \( I_a \sigma = \pm \sigma I_a \). So, by Lemma 2.9, we have \( d = 0 \). This is a contradiction which completes the proof. \( \square 

2.11. Theorem. Let \( R \) be a \( \sigma \)-prime ring with characteristic not 2, \( I \) be a nonzero \( \sigma \)-ideal of \( R \), \( d \) be a nonzero \((\alpha, \beta)\)-derivation of \( R \) such that \( \alpha, \beta \) commute with \( \sigma \). If \( a \in I \cap S_\sigma (R) \) and \([d(I), a]_{\alpha, \beta} \subset C_{\alpha, \beta} \) then \( a \in Z (R) \).

Proof. By hypothesis, \([d(a^2), a]_{\alpha, \beta} \subset C_{\alpha, \beta} \). Expanding this, it becomes
\[
[d(a^2), a]_{\alpha, \beta} = [d(a) \alpha (a) + \beta (a) d(a), a]_{\alpha, \beta}
\]
\[
= d(a) \alpha (a, a] + [d(a), a]_{\alpha, \beta} \alpha (a) + \beta (a) [d(a), a]_{\alpha, \beta}
\]
\[
+ \beta ([a, a]) d(a)
\]
\[
= [d(a), a]_{\alpha, \beta} \alpha (a) + \beta (a) [d(a), a]_{\alpha, \beta} \subset C_{\alpha, \beta}
\]
And so,

\[ [d(a), a]_{\alpha, \beta} \alpha (a) + \beta (a) [d(a), a]_{\alpha, \beta} \in C_{\alpha, \beta} \]

is obtained. In the above obtained relation, using \([d(a), a]_{\alpha, \beta} \in C_{\alpha, \beta}\), we have \(2 \beta (a) [d(a), a]_{\alpha, \beta} \in C_{\alpha, \beta}\). Since \(\text{char} R \neq 2\), we get

\[ (2.6) \quad \beta (a) [d(a), a]_{\alpha, \beta} \in C_{\alpha, \beta} \]

Since \(a \in I \cap S_{\sigma} (R)\), it is clear that \(\beta (a) \in S_{\sigma} (R)\). Using the hypothesis together with (2.6), according to Lemma 2.4 (i), we yield either

\[ a \in Z (R) \quad \text{or} \quad [d(a), a]_{\alpha, \beta} = 0 \]

Assume that \(a \notin Z (R)\) which means \([d(a), a]_{\alpha, \beta} = 0\). On the other hand, by hypothesis, it holds that \([d([a, x]), a]_{\alpha, \beta} \in C_{\alpha, \beta}\). So,

\[ [d([a, x]), a]_{\alpha, \beta} = \left[ [d(a), x]_{\alpha, \beta}, a \right]_{\alpha, \beta} - \left[ [d(x), a]_{\alpha, \beta}, a \right]_{\alpha, \beta} \in C_{\alpha, \beta} \]

is obtained. Using the hypothesis, we have

\[ \left[ [d(a), x]_{\alpha, \beta}, a \right]_{\alpha, \beta} \in C_{\alpha, \beta}, \forall x \in I \]

Replacing \(x\) by \(ax\) and using \([d(a), a]_{\alpha, \beta} = 0\), it becomes

\[ \beta (a) \left[ [d(a), x]_{\alpha, \beta}, a \right]_{\alpha, \beta} \in C_{\alpha, \beta}, \forall x \in I \]

We know that \(\beta (a) \in S_{\sigma} (R)\) and \([d(a), x]_{\alpha, \beta}, a \right]_{\alpha, \beta} \in C_{\alpha, \beta}\). Therefore, by Lemma 2.4 (i), we derive \([d(a), x]_{\alpha, \beta}, a \right]_{\alpha, \beta} = 0\) for all \(x \in I\). Applying the identity \([a, b]_{\alpha, \beta}, c]_{\alpha, \beta} = [a, c]_{\alpha, \beta}, b + [a, b, c]_{\alpha, \beta}\) for all \(a, b, c \in R\) and using the assumption, we arrive

\[ [d(a), x]_{\alpha, \beta}, a \right]_{\alpha, \beta} = 0, \forall x \in I \]

The mapping \(I_{d(a)} : R \to R\) is given by \(I_{d(a)} (r) = [d(a), r]_{\alpha, \beta}\) is a \((\alpha, \beta)\)-derivation which is determined by \(d(a)\) and \(I_a : R \to R\) is given by \(I_a (r) = [r, a]\) is a derivation which is determined by \(a\). So,

\[ (I_{d(a)} I_a) (I) = 0 \]

is obtained. Since \(a \in I \cap S_{\sigma} (R)\), we have \(I_a \sigma = \pm \sigma I_a\). According to Lemma 2.9, we yield either

\[ I_{d(a)} = 0 \quad \text{or} \quad I_a = 0 \]

which means \(d(a) \in C_{\alpha, \beta}\). On the other hand, by hypothesis, we have \([d(ax), a]_{\alpha, \beta} \in C_{\alpha, \beta}\) for all \(x \in I\). So, we get

\[ (2.7) \quad d(a) \alpha ([x, a]) + \beta (a) [d(x), a]_{\alpha, \beta} \in C_{\alpha, \beta}, \forall x \in I \]

Commuting (2.7) with \(a\), it follows that

\[ 0 = [d(a) \alpha ([x, a]) + \beta (a) [d(x), a]_{\alpha, \beta}, a]_{\alpha, \beta} \]

\[ = [d(a) \alpha ([x, a]), a]_{\alpha, \beta} + \beta (a) [d(x), a]_{\alpha, \beta}, a]_{\alpha, \beta} \]

\[ = d(a) \alpha ([x, a], a]_{\alpha, \beta} + \beta (a) [d(x), a]_{\alpha, \beta}, a]_{\alpha, \beta} \]

\[ + \beta (a) [d(x), a]_{\alpha, \beta}, a]_{\alpha, \beta} + \beta (a) [d(x), a]_{\alpha, \beta}, a]_{\alpha, \beta} \]

\[ = d(a) \alpha ([x, a]_{\alpha, \beta} + \beta (a) [d(x), a]_{\alpha, \beta}, a]_{\alpha, \beta} \]
And so, it becomes
\[ d(a) \alpha (\alpha ([x, a], a] \alpha (a)) + \beta (a) \bigg[ [d(x), a]_{\alpha, \beta}, a \bigg]_{\alpha, \beta} = 0, \quad \forall x \in I \]

Using \([d(x), a]_{\alpha, \beta} \in C_{\alpha, \beta}\), we have \(d(a) \alpha (\alpha ([x, a], a]) = 0\) for all \(x \in I\). Since \(d(a) \in C_{\alpha, \beta}\),
\[ d(a) R \alpha (\alpha ([x, a], a]) = 0, \quad \forall x \in I \]
is obtained. In the above obtained relation, taking \(\alpha (x)\) instead of \(x\) and using the fact that \(\alpha\) commutes with \(\sigma\), we derive
\[ d(a) R\alpha (\alpha ([x, a], a]) = 0, \quad \forall x \in I \]
And so, we yield
\[ d(a) R\alpha (\alpha ([x, a], a]) = d(a) R\sigma (\alpha ([x, a], a]) = 0, \quad \forall x \in I \]
Since \(R\) is \(\sigma\)-prime, we get \(d(a) = 0\) or \([x, a], a] = 0\) for all \(x \in I\). That is, \(d(a) = 0\) or \(\sigma R = 0\). Since \(I \sigma = \sigma I\), by Lemma 2.9, we have \(d(a) = 0\). In (2.7), using \(d(a) = 0\), it becomes
\[ \beta (a) [d(x), a]_{\alpha, \beta} \in C_{\alpha, \beta}, \forall x \in I \]
We know that \(\beta (a) \in S_{\sigma} (R)\) and \([d(x), a]_{\alpha, \beta} \in C_{\alpha, \beta}\) from the hypothesis. Therefore, according to Lemma 2.4 (i), we have \([d(x), a]_{\alpha, \beta} = 0\) for all \(x \in I\). Since \(a \in I \cap S_{\sigma} (R)\) and \(\beta\) commutes with \(\sigma\), by Lemma 2.10, we derive \(a \in Z(R)\). This is a contradiction which completes the proof. \(\square\)

2.12. Theorem. Let \(R\) be a \(\sigma\)-prime ring with characteristic not 2, \(I\) be a nonzero \(\sigma\)-ideal of \(R\), \(d\) be a nonzero \((\alpha, \beta)\)-derivation of \(R\) such that \(\alpha\) and \(\beta\) commute with \(\sigma\) and \(h\) be a nonzero derivation of \(R\) which commutes with \(\sigma\). If \(dh(I) \subset C_{\alpha, \beta}\) and \(h(I) \subset I\) then \(R\) is commutative.

Proof. For any \(x, y \in I\), from the hypothesis, we have \(dh([x, y]) \in C_{\alpha, \beta}\). Expanding this identity, it follows that
\[
dh([x, y]) = d([h(x), y] + [x, h(y)])
\]
\[
= ([dh)x, y]_{\alpha, \beta} - [d(y), h(x)]_{\alpha, \beta} + [d(x), h(y)]_{\alpha, \beta} - ([dh)x, y]_{\alpha, \beta}
\]
\[
= [d(x), h(y)]_{\alpha, \beta} - [d(y), h(x)]_{\alpha, \beta} \in C_{\alpha, \beta}
\]
And it becomes
\[
[d(x), h(y)]_{\alpha, \beta} - [d(y), h(x)]_{\alpha, \beta} \in C_{\alpha, \beta}, \quad \forall x, y \in I
\]
Since \(h(I) \subset I\), we replace \(y\) by \(h(y)\). So, we arrive \([d(x), h^2(y)]_{\alpha, \beta} \in C_{\alpha, \beta}\) for all \(x, y \in I\). That is,
\[
d^2(I), h^2(I) \subset C_{\alpha, \beta}
\]
Using the fact that \(h(I) \subset I\) and \(h\) commutes with \(\sigma\), we assure \(h^2(I) \subset I \cap S_{\sigma} (R)\). In additional, we know that from the hypothesis \(\alpha\) and \(\beta\) commute with \(\sigma\). Thereby, according to Theorem 2.11, it yields \(h^2(I) \subset Z(R)\). So, for all \(x, y \in I\)
\[
h^2([x, y]) = h([h(x), y] + [x, h(y)])
\]
\[
= [h^2(x), y] + 2[h(x), h(y)] + [x, h^2(y)]
\]
\[
= 2[h(x), h(y)] \in Z(R)
\]
is obtained. Since \(\text{char} R \neq 2\), we have \([h(x), h(y)] \in Z(R)\) for all \(x, y \in I\). Thus,
\[ [h(I), h(I)] \subset Z(R) \]
Using $h(I) \subset I \cap S_\sigma(R)$, by Theorem 2.11, we derive $h(I) \subset Z(R)$. According to Lemma 2.5, it implies that $R$ is commutative.

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References