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INTERNAL STATE VARIABLES IN DIPOLAR THERMOELASTIC BODIES

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Abstract

The aim of our study is prove that the presence of the internal state variables in a thermoelastic dipolar body do not influence the uniqueness of solution. After the mixed initial boundary value problem in this context is formulated, we use the Gronwall's inequality to prove the uniqueness of solution of this problem.

Keywords: thermoelastic, dipolar, internal state variables, uniqueness, Gronwall's inequality

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1. Introduction

Interest to consider the internal state variables as a means to estimate mechanical properties has grown rapidly in recent years.

The theories of internal state variables in different kind of materials represent a material length scale and are quite sufficient for a large number of the solid mechanics applications.

The internal state variables are the smallest possible subset of system variables that can represent the entire state of the system at any given time. The minimum number of state variables required to represent a given system, n, is usually equal to the order of the differential equations system's defining. If the system is represented in the transfer function form, the minimum number of state variables is equal to the order of the transfer function's denominator after it has been reduced to a proper fraction. It is important to understand that converting a state space realization to a transfer function form may lose some internal information about the system, and may provide a description of a system which is stable, when the state-space realization is unstable at certain points. For instance, in the electric circuits, the number of state variables is often, though not

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always, the same as the number of energy storage elements in the circuit such as capacitors and inductors.

The theory of bodies with internal state variables has been first formulated for the thermo-viscoelastic materials (see, for instance Chirita [3]). Then the internal state variables has been considered for different kind of materials.

The study [9] of Nachlinger and Nunziato is dedicated to the internal state variables approach of finite deformations without heat conduction in the one-dimensional case.

In the paper [12] the authors describe how the so-called Bammann internal state variable constitutive approach, which has proven highly successful in modelling deformation processes in metals, can be applied with great benefit to silicate rocks and other geological materials in modelling their deformation dynamics. In its essence, the internal state variables theory provides a constitutive framework to account for changing history states that arise from inelastic dissipative microstructural evolution of a polycrystalline solid.

A thermodynamically consistent framework is proposed for modeling the hysteresis of capillarity in partially saturated porous media in the paper [14]. Capillary hysteresis is viewed as an intrinsic dissipation mechanism, which can be characterized by a set of internal state variables. The volume fractions of pore fluids are assumed to be additively decomposed into a reversible part and an irreversible part. The irreversible part of the volumetric moisture content is introduced as one of the internal variables. It is shown that the pumping effect occurring in a porous medium experiencing a wetting/drying cycle is thermodynamically admissible.

The paper [2] presents the formulation of a constitutive model for amorphous thermoplastics using a thermodynamic approach with physically motivated internal state variables. The formulation follows current internal state variable methodologies used for metals and departs from the spring-dashpot representation generally used to characterize the mechanical behavior of polymers.

Anand and Gurtin develop in the paper [1] a continuum theory for the elastic-viscoplastic deformation of amorphous solids such as polymeric and metallic glasses. Introducing an internal-state variable that represents the local free-volume associated with certain metastable states, the authors are able to capture the highly non-linear stress-strain behavior that precedes the yield-peak and gives rise to post-yield strain softening.

In the study [13], is presented a formulation of state variable based gradient theory to model damage evolution and alleviate numerical instability associated within the post-bifurcation regime. This proposed theory is developed using basic microforce balance laws and appropriate state variables within a consistent thermodynamic framework. The proposed theory provides a strong coupling and consistent framework to prescribe energy storage and dissipation associated with internal damage. For other paper in this topic, see [10], [11].

Other results on some generalizations of thermoelastic bodies can be found in the papers [4]-[8].

2. Basic equations

Let us consider B be an open region of three-dimensional Euclidean space R^3 occupied, at time $t = 0$, by the reference configuration of a thermoelastic dipolar body with internal state variables.

We assume that the boundary of the domain B, denoted by ∂B, is a closed, bounded and piece-wise smooth surface which allows us the application of the divergence theorem. A fixed system of rectangular Cartesian axes is used and we adopt the Cartesian tensor notations. The points in B are denoted by (x_i) or (x). The variable t is the time and $t \in [0, t_0)$. We shall employ the usual summation over repeated subscripts while subsripts preceded by a comma denote the partial differentiation with respect to the spatial argument. Also, we use a superposed dot to denote the partial differentiation with respect to t. The Latin indices are understood to range over the integers (1, 2, 3), while the Greek subsripts have the range $1,\ 2, \ldots, n$.

In the following we designate by n_i the components of the outward unit normal to the surface ∂B. The closure of the domain B, denoted by $\bar{B}$, means $\bar{B} = B \cup \partial B$.

Also, the spatial argument and the time argument of a function will be ommited when there is no likelihood of confusion.

The behaviour of a thermoelastic dipolar body is characterized by the following kinematic variables:

$$u_i = u_i(x,t),\ \varphi_{jk} = \varphi_{jk}(x,t),\ (x,t) \in B \times [0,t_0)$$

where u_i are the components of the displacement field and φ_{jk} - the components of the dipolar displacement field.

The fundamental system of field equations, in the theory of dipolar thermoelastic bodies with internal state variables, consists of:

- the equations of motion:

$$\begin{aligned} &(\tau_{ij} + \eta_{ij})_{,j} + \varrho F_i = \varrho \ddot{u}_i, \\ &\mu_{ijk,i} + \sigma_{jk} + \varrho G_{jk} = I_{kr} \ddot{\varphi}_{jr}; \end{aligned} \tag{2.1}$$

- the energy equation:

$$T_0 \dot{\eta} = q_{i,i} + \varrho r; \tag{2.2}$$

- the constitutive equations:

$$\begin{aligned} \tau_{ij} &= C_{ijmn}\, \varepsilon_{mn} + G_{mnij}\, \gamma_{mn} + F_{mnrij}\, \kappa_{mnr} - B_{ij}\, \theta + B_{ij\alpha}\, \omega_\alpha, \\ \sigma_{ij} &= G_{ijmn}\, \varepsilon_{mn} + B_{ijmn}\, \gamma_{mn} + D_{ijmnr}\, \kappa_{mnr} - D_{ij}\, \theta + D_{ij\alpha}\, \omega_\alpha, \\ \mu_{ijk} &= F_{ijkmn}\, \varepsilon_{mn} + D_{mnijk}\, \gamma_{mn} + A_{mnrijk}\, \kappa_{mnr} - F_{ijk}\, \theta + F_{ijk\alpha}\, \omega_\alpha, \\ \eta &= B_{ij}\, \varepsilon_{ij} + D_{ij}\, \gamma_{ij} + F_{ijs}\, \kappa_{ijs} - a\, \theta - G_\alpha\, \omega_\alpha, \\ q_i &= a_{ijk}\, \varepsilon_{jk} + b_{ijk}\, \gamma_{jk} + c_{ijsm}\, \kappa_{jsm} + d_i\, \theta + f_{i\alpha}\, \omega_\alpha + K_{ij}\, \theta_{,\,j}; \end{aligned} \tag{2.3}$$

- the geometric equations:

$$\begin{aligned} &\varepsilon_{ij} = \frac{1}{2}\left(u_{j,i} + u_{i,j}\right), \ \ \gamma_{ij} = u_{j,i} - \varphi_{ij}, \\ &\kappa_{ijk} = \varphi_{jk,i}. \end{aligned} \tag{2.4}$$

Usually, the internal state variables are denoted by $\xi_\alpha,\ \alpha = 1,\ 2,\ \ldots,\ n$. In the linear theory, we denote by ω_α the internal state variables measured from the internal state variables ξ^0_α of the initial state. Also, the temperature θ represents the difference between the absolute temperature T and the temperature T_0, $T_0 > 0$, of the initial state. Thus we have:

$$\xi_\alpha = \xi^0_\alpha + \omega_\alpha,\ T = T_0 + \theta. \tag{2.5}$$

Within the linear approximation, from the entropy production inequality, it follows (see, for instance, [1]):

$$\dot{\omega}_\alpha = f_\alpha, \tag{2.6}$$

where

$$f_\alpha = g_{ij\alpha} \varepsilon_{ij} + h_{ij\alpha} \gamma_{ij} + l_{ijk\alpha} \kappa_{ijk} + p_\alpha \theta + q_{\alpha\beta} \omega_\beta + r_{i\alpha} \theta_{,\,i}. \tag{2.7}$$

The other notations used in the above equations have the following meanings:

- ϱ - the constant mass density;
- $\tau_{ij},\ \sigma_{ij},\ \mu_{ijk}$ - the components of the stress tensors;

- I_{ij} - the coefficients of inertia;
- F_i - the components of body force per unit mass;
- G_{jk} - the components of dipolar body force per unit mass;
- r - the heat supply per unit mass and unit time;
- η - the entropy per unit mass;
- q_i - the components of the heat flux;
- $\varepsilon_{ij}, \gamma_{ij}, \kappa_{ijk}$ - the kinematic characteristics of the strain tensors.

The above coefficients $C_{ijmn},\ B_{ijmn}, \ldots,\ D_{ijm},\ E_{ijm}, \ldots, a_{ijk},\ \ldots,\ g_{ij\alpha}, \ldots,\ r_{i\alpha}$ are functions of x and characterize the thermoelastic properties of the material with internal state variable (the constitutive coefficients). For a homogeneous medium these quantities are constants. The constitutive coefficients obey to the following symmetry relations

$$\begin{aligned}
&C_{ijmn} = C_{mnij} = C_{ijnm},\ B_{ijmn} = B_{mnij},\\
&G_{ijmn} = G_{ijnm}, F_{ijkmn} = F_{ijknm},\ A_{ijkmnr} = A_{mnrijk},\\
&B_{ij} = B_{ji},\ a_{ijk} = a_{ikj},\ K_{ij} = K_{ji},\ g_{ij\alpha} = g_{ji\alpha}.
\end{aligned} \tag{2.8}$$

We supplement the above equations with the following initial conditions

$$\begin{aligned}
&u_i\left(x_s, 0\right) = u_{0i}\left(x_s\right),\ \dot{u}_i\left(x_s, 0\right) = u_{1i}\left(x_s\right),\\
&\varphi_{ij}\left(x_s, 0\right) = \varphi_{0ij}\left(x_s\right),\ \dot{\varphi}_{ij}\left(x_s, 0\right) = \varphi_{1ij}\left(x_s\right),\\
&\theta\left(x_s, 0\right) = \theta_0\left(x_s\right),\ \omega_\alpha\left(x_s, 0\right) = \omega_{0\alpha}\left(x_s\right),\ \left(x_s\right) \in B
\end{aligned} \tag{2.9}$$

and the prescribed boundary conditions

$$\begin{aligned}
&u_i = \tilde{u_i},\ \text{on}\ \overline{\partial B_1} \times [0, t_0],\ t_i \equiv \left(\tau_{ij} + \sigma_{ij}\right) n_j = \tilde{t_i},\ \text{on}\ \partial B_2 \times [0, t_0],\\
&\varphi_{ij} = \tilde{\varphi}_{ij},\ \text{on}\ \overline{\partial B_3} \times [0, t_0],\ \mu_{jk} \equiv \mu_{ijk} n_i = \tilde{\mu}_{jk},\ \text{on}\ \partial B_4 \times [0, t_0],\\
&\theta = \tilde{\theta},\ \text{on}\ \overline{\partial B_5} \times [0, t_0],\ q \equiv q_i n_i = \tilde{q},\ \text{on}\ \partial B_6 \times [0, t_0].
\end{aligned} \tag{2.10}$$

Here $\overline{\partial B_1}$, $\overline{\partial B_3}$, $\overline{\partial B_5}$ and ∂B_2, ∂B_4, ∂B_6 are subsets of the boundary ∂B which satisfay the relations

$$\begin{aligned}
&\overline{\partial B_1} \cup \partial B_2 = \overline{\partial B_3} \cup \partial B_4 = \overline{\partial B_5} \cup \partial B_6 = \partial B\\
&\partial B_1 \cap \partial B_2 = \partial B_3 \cap \partial B_4 = \partial B_5 \cap \partial B_6 = \emptyset
\end{aligned}$$

In the above conditions 2.9 and 2.10, the functions u_{0i}, u_{1i}, φ_{0ij}, φ_{1ij}, θ_0 $\omega_{0\alpha}$, $\tilde{u_i}$, $\tilde{t_i}$, $\tilde{\varphi_{ij}}$, $\tilde{\mu_{jk}}$, $\tilde{\theta}$ and $\tilde{q}$ are prescribed in their domain of definition.

In conclusion, the mixed initial boundary value problem of the thermoelasticity of dipolar bodies with internal variables consists of the equations (2.1), (2.2) and (2.6), the initial conditions (2.9) and the boundary conditions (2.10).

By a solution of this problem we mean a state of deformation (u_i, φ_{ij}, θ, ω_α) satisfying the Eqns. (2.1), (2.2) and (2.6) and the conditions (2.9) and (2.10).

3. Main results

In the main section of our paper we will deduce some estimations and then, as a consequence, we obtain in simple manner the uniqueness theorem of the solution of the above problem.

In order to prove these results, we shall need the following assumptions

- (i) the mass density ϱ is strictly positive, i.e.

$$\varrho\left(x_s\right) \geq \varrho_0 > 0,\ \text{on}\ B;$$

- (ii) there exists a positive constant λ_1 such that

$$I_{ij}\xi_i\xi_j \geq \lambda_1\xi_i\xi_i,\ \forall\ \xi_i;$$

- (iii) the specific heat a from $(3)_4$ is strictly positive, i.e.

$$a\,(x_s) \geq a_0 > 0, \text{ on } B;$$

- (iv) the constitutive tensors C_{ijmn}, B_{ijmn} and A_{ijkmnr} are positive definite:

$$\int_B C_{ijmn}\,\xi_{ij}\,\xi_{mn}\,dv \geq \lambda_2 \int_B \xi_{ij}\,\xi_{ij}\,dv,\ \forall\,\xi_{ij}$$
$$\int_B B_{ijmn}\,\xi_{ij}\,\xi_{mn}\,dv \geq \lambda_3 \int_B \xi_{ij}\,\xi_{ij}\,dv,\ \forall\,\xi_{ij}$$
$$\int_B A_{ijkmnr}\,\xi_{ijk}\,\xi_{mnr}\,dv \geq \lambda_4 \int_B \xi_{ijk}\,\xi_{ijk}\,dv,\ \forall\,\xi_{ijk}$$

where λ_2, λ_3 and λ_4 are positive constants;

- (v) the symmetric part $\tilde{K}_{ij}$ of the thermal conductivity tensor K_{ij} is positive definite, in the sense that there exists a positive constant μ such that

$$\int_B \tilde{K}_{ij}\,\xi_i\,\xi_j\,dv \geq \mu \int_B \xi_i\,\xi_i\,dv, \text{ for all vectors } \xi_i.$$

Let us consider

$$\left(u_i^{(\nu)},\ \varphi_{ij}^{(\nu)},\ \theta^{(\nu)},\ \omega_\alpha^{(\nu)}\right),\ \nu = 1,\ 2$$

two solutions of our initial boundary value problem.

Because of the linearity of the problem, their difference is also solution of the problem. We denote by $(v_i,\ \psi_{ij}, \kappa, w_\alpha)$ the differences,

$$v_i = u_i^{(2)} - u_i^{(1)},\ \psi_i = \varphi_{ij}^{(2)} - \varphi_{ij}^{(1)},\ \kappa = \theta^{(2)} - \theta^{(1)},\ w_\alpha = \omega_\alpha^{(2)} - \omega_\alpha^{(1)}$$

In order to prove the desired uniquness theorem, it suffice to prove that the above considered problem, consists of the equations (2.1), (2.2) and (2.6) and the conditions (2.9) and (2.10), in which

$$F_i = G_{jk} = r = 0$$
$$u_{0i} = u_{1i} = \varphi_{0ij} = \varphi_{1ij} = \theta_0 = \omega_{0\alpha} = 0$$

and

$$\tilde{u}_i = \tilde{t}_i\tilde{\varphi}_{ij} = \tilde{\mu}_{ij} = \tilde{\theta} = \tilde{q} = 0$$

imply that

$$u_i = \varphi_{ij} = \theta = \omega_\alpha = 0,$$

in $B \times [0, t_0]$, provided that the hypotheses (i) - (v) hold.
Therefore, we consider the new problem P_0 defined by the following equations

$$(\tau_{ij} + \sigma_{ij})_{,j} = \varrho\ddot{u}_i,$$
$$\mu_{ijk,i} + \sigma_{jk} = I_{kr}\ddot{\varphi}_{jr} \tag{3.1}$$

$$T_0\dot{\eta} = q_{i,i} \tag{3.2}$$

$$\dot{\omega}_\alpha = f_\alpha, \tag{3.3}$$

with the initial conditions

$$u_i\,(x_s, 0) = 0,\ \dot{u}_i\,(x_s, 0) = 0,\ \varphi_{ij}\,(x_s, 0) = 0,$$
$$\dot{\varphi}_{ij}\,(x_s, 0) = 0,\ \theta\,(x_s, 0) = 0,\ \omega_\alpha\,(x_s, 0) = 0,\ (x_s) \in B \tag{3.4}$$

and the boundary conditions

$$
\begin{aligned}
&u_i = 0, \text{ on } \overline{\partial B_1} \times [0, t_0], \ t_i \equiv (\tau_{ij} + \sigma_{ij})\, n_j = 0, \text{ on } \partial B_2 \times [0, t_0], \\
&\varphi_{ij} = 0, \text{ on } \overline{\partial B_3} \times [0, t_0], \ \mu_{jk} \equiv \mu_{ijk} n_i = 0, \text{ on } \partial B_4 \times [0, t_0], \\
&\theta = 0, \text{ on } \overline{\partial B_5} \times [0, t_0], \ q \equiv q_i n_i = 0, \text{ on } \partial B_6 \times [0, t_0].
\end{aligned} \tag{3.5}
$$

To these equations and conditions we adjoin the constitutive relations (2.3) and (2.7). In order to prove that the problem P_0 admits the null solution, we will show that the function $y(t)$ defined by

$$
y(t) = \int_B \left(\dot{u}_i \dot{u}_i + \dot{\varphi}_{ij} \dot{\varphi}_{ij} + \varepsilon_{ij} \varepsilon_{ij} + \gamma_{ij} \gamma_{ij} + \kappa_{ijr} \kappa_{ijr} + \theta^2 + \omega_\alpha \omega_\alpha \right) dV
$$

vanishes on $[0, t_0]$.
To this aim, we first prove some useful estimations.

3.1. Theorem. *If the ordered array* $(u_i,\ \varphi_{ij},\ \theta,\ \omega_\alpha)$ *is a solution of the problem* P_0*, then the following relation hold*

$$
\begin{aligned}
&\frac{1}{2} \int_B \big(C_{ijmn} \varepsilon_{ij} \varepsilon_{mn} + 2 G_{ijmn} \varepsilon_{ij} \gamma_{mn} + 2 F_{mnrij} \varepsilon_{ij} \kappa_{mnr} + \\
&+ B_{ijmn} \gamma_{ij} \gamma_{mn} + A_{ijsmnr} \kappa_{ijs} \kappa_{mnr} + 2 D_{ijmnr} \gamma_{ij} \kappa_{mnr} + 2 B_{ij\alpha} \varepsilon_{ij} \omega_\alpha + \\
&+ 2 D_{ij\alpha} \gamma_{ij} \omega_\alpha + 2 F_{ijr\alpha} \kappa_{ijr} \omega_\alpha + a\theta^2 + \varrho \dot{u}_i \dot{u}_i + I_{kr} \dot{\varphi}_{jr} \dot{\varphi}_{jk} \big)\, dV = \\
&\int_0^t \int_B \left[\left(B_{ij\alpha} \varepsilon_{ij} + D_{ij\alpha} \gamma_{ij} + F_{ijr\alpha} \kappa_{ijr} \right) \dot{\omega}_\alpha - \frac{1}{T_0} q_i \theta_{,i} \right] dV ds.
\end{aligned} \tag{3.6}
$$

Proof. By using the constitutive equations (2.3) and the symmetry relations (2.8), we obtain

$$
\begin{aligned}
&\tau_{ij} \dot{u}_{j,i} + \sigma_{ij} \dot{\gamma}_{ij} + \mu_{ijs} \dot{\kappa}_{ijs} = \\
&\frac{1}{2} \frac{\partial}{\partial t} \big(C_{ijmn} \varepsilon_{ij} \varepsilon_{mn} + 2 G_{mnij} \varepsilon_{ij} \gamma_{mn} + 2 F_{mnrij} \varepsilon_{ij} \kappa_{mnr} + \\
&+ B_{ijmn} \gamma_{ij} \gamma_{mn} + A_{ijsmnr} \kappa_{ijs} \kappa_{mnr} + 2 D_{ijmnr} \gamma_{ij} \kappa_{mnr} + \\
&+ 2 B_{ij\alpha} \varepsilon_{ij} \omega_\alpha + 2 D_{ij\alpha} \gamma_{ij} \omega_\alpha + 2 F_{ijs\alpha} \kappa_{ijs} \omega_\alpha + a\theta^2 \big) - \\
&- B_{ij\alpha} \varepsilon_{ij} \dot{\omega}_\alpha - D_{ij\alpha} \gamma_{ij} \dot{\omega}_\alpha - F_{ijs\alpha} \kappa_{ijs} \dot{\omega}_\alpha - G_\alpha \theta \dot{\omega}_\alpha.
\end{aligned} \tag{3.7}
$$

On the other hand, in view of (3.1) and (3.2) we deduce:

$$
\begin{aligned}
&\tau_{ij} \dot{u}_{j,i} + \sigma_{ij} \dot{\gamma}_{ij} + \mu_{ijs} \dot{\kappa}_{ijs} = \\
&= \left[(\tau_{ij} + \sigma_{ij})\, \dot{u}_j + \mu_{ijs} \dot{\varphi}_{js} + \frac{1}{T_0} q_i \theta \right]_{,i} - \\
&- \frac{1}{2} \frac{\partial}{\partial t} \left(\varrho \dot{u}_i \dot{u}_i + I_{kr} \dot{\varphi}_{jr} \dot{\varphi}_{jk} \right) - \frac{1}{T_0} q_i \theta_{,i}
\end{aligned} \tag{3.8}
$$

From the equalities (3.7) and (3.8) we have

$$\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}\left(C_{ijmn}\varepsilon_{ij}\varepsilon_{mn} + 2G_{mnij}\varepsilon_{ij}\gamma_{mn} + 2F_{mnrij}\varepsilon_{ij}\kappa_{mnr}+\right.\\
&+B_{ijmn}\gamma_{ij}\gamma_{mn} + A_{ijsmnr}\kappa_{ijs}\kappa_{mnr} + 2D_{ijmnr}\gamma_{ij}\kappa_{mnr}+\\
&+2B_{ij\alpha}\varepsilon_{ij}\omega_\alpha + 2D_{ij\alpha}\gamma_{ij}\omega_\alpha + 2F_{ijs\alpha}\kappa_{ijs}\omega_\alpha+\\
&\left.+a\theta^2 + \varrho\dot{u}_i\dot{u}_i + I_{kr}\dot{\varphi}_{jr}\dot{\varphi}_{jk}\right) =\\
&= \left[(\tau_{ij}+\sigma_{ij})\,\dot{u}_j + \mu_{ijs}\dot{\varphi}_{js} + \frac{1}{T_0}q_i\theta\right]_{,\,i} - \frac{1}{T_0}q_i\theta_{,\,i}+\\
&+\left(B_{ij\alpha}\varepsilon_{ij} + D_{ij\alpha}\gamma_{ij} + F_{ijs\alpha}\kappa_{ijs} + G_\alpha\theta\right)\dot{\omega}_\alpha
\end{aligned} \tag{3.9}$$

Now, we integrate relation (3.9) over the domain B. By using the divergence theorem and the boundary conditions (3.5), we conclude that

$$\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}\int_B\left(C_{ijmn}\varepsilon_{ij}\varepsilon_{mn} + 2G_{mnij}\varepsilon_{ij}\gamma_{mn} + 2F_{mnrij}\varepsilon_{ij}\kappa_{mnr}+\right.\\
&B_{ijmn}\gamma_{ij}\gamma_{mn} + A_{ijsmnr}\kappa_{ijs}\kappa_{mnr} + 2D_{ijmnr}\gamma_{ij}\kappa_{mnr}+\\
&2B_{ij\alpha}\varepsilon_{ij}\omega_\alpha + 2D_{ij\alpha}\gamma_{ij}\omega_\alpha + 2F_{ijs\alpha}\kappa_{ijs}\omega_\alpha+\\
&\left.+a\theta^2 + \varrho\dot{u}_i\dot{u}_i + I_{kr}\dot{\varphi}_{jr}\dot{\varphi}_{jk}\right)dV =\\
&\int_B\left[\left(B_{ij\alpha}\varepsilon_{ij} + D_{ij\alpha}\gamma_{ij} + F_{ijs\alpha}\kappa_{ijs} + G_\alpha\theta\right)\dot{\omega}_\alpha - \frac{1}{T_0}q_i\theta_{,\,i}\right]dV.
\end{aligned} \tag{3.10}$$

Finally, we integrate the equality (20) from 0 to t and, by using the initial condition (3.4), we arrive at the desired result (3.6). □

3.2. Theorem. *Let* $(u_i,\ \varphi_{ij},\ \theta,\ \omega_\alpha)$ *be a solution of the problem* P_0*. Then there exists the positive constants* m_1 *and* m_2 *such that the following relation hold*

$$\begin{aligned}
&\int_B\left[\left(B_{ij\alpha}\varepsilon_{ij} + D_{ij\alpha}\gamma_{ij} + F_{ijs\alpha}\kappa_{ijs} + G_\alpha\theta\right)\dot{\omega}_\alpha - \frac{1}{T_0}q_i\theta_{,\,i}\right]dV \leq\\
&\leq -m_1\int_B\theta_{,\,i}\theta_{,\,j}dV + m_2\int_B\left(\varepsilon_{ij}\varepsilon_{ij} + \gamma_{ij}\gamma_{ij} + \kappa_{ijs}\kappa_{ijs} + \theta^2 + \omega_\alpha\omega_\alpha\right)dV.
\end{aligned} \tag{3.11}$$

Proof. Taking into account the relations (2.6), (2.7) and $(2.3)_5$, we can write:

$$\begin{aligned}
&\int_B\left[\left(B_{ij\alpha}\varepsilon_{ij} + D_{ij\alpha}\gamma_{ij} + F_{ijs\alpha}\kappa_{ijs} + G_\alpha\theta\right)\dot{\omega}_\alpha - \frac{1}{T_0}q_i\theta_{,\,i}\right]dV =\\
&\int_B\left[\left(B_{ij\alpha}\varepsilon_{ij} + D_{ij\alpha}\gamma_{ij} + F_{ijs\alpha}\kappa_{ijs} + G_\alpha\theta\right)\left(g_{ij\alpha}\varepsilon_{ij} + h_{ij\alpha}\gamma_{ij}+\right.\right.\\
&\left.+l_{ijs\alpha}\kappa_{ijs} + p_\alpha\theta + q_{\alpha\beta}\omega_\beta + r_{i\alpha}\theta_{,\,i}\right)-\\
&\left.-\frac{1}{T_0}\left(a_{ijk}\varepsilon_{jk} + b_{ij}\gamma_{jk} + c_{ijsm}\kappa_{jsm} + d_i\theta + f_{i\alpha}\omega_\alpha + K_{ij}\theta_{,\,j}\right)\theta_{,\,i}\right]dV =\\
&-\int_B\frac{1}{T_0}K_{ij}\theta_{,\,i}\theta_{,\,j}dV + \int_B\left(\mathcal{B}_{ij}\varepsilon_{ij}\theta + \mathcal{D}_{ij}\gamma_{ij}\theta + \mathcal{F}_{ijs}\kappa_{ijs}\theta+\right.\\
&\mathcal{M}\theta^2 + \mathcal{L}_\alpha\omega_\alpha\theta + \mathcal{D}_i\theta\theta_{,\,i} + \mathcal{C}_{ijmn}\varepsilon_{ij}\varepsilon_{mn} + \mathcal{D}_{ijmn}\varepsilon_{ij}\gamma_{mn}+\\
&\mathcal{F}_{ijmnr}\varepsilon_{ij}\kappa_{mnr} + \mathcal{B}_{ij\alpha}\varepsilon_{ij}\omega_\alpha + \mathcal{B}_{ijk}\varepsilon_{ij}\theta_{,\,k} + \mathcal{B}_{ijmn}\gamma_{ij}\gamma_{mn}+\\
&\mathcal{D}_{ijmnr}\gamma_{ij}\kappa_{mnr} + \mathcal{D}_{ij\alpha}\gamma_{ij}\omega_\alpha + \mathcal{D}_{ijk}\gamma_{ij}\theta_{,\,k} + \mathcal{A}_{ijsmnr}\kappa_{ijs}\kappa_{mnr}+\\
&\left.+\mathcal{F}_{ijs\alpha}\kappa_{ijs}\omega_\alpha + \mathcal{F}_{ijsm}\kappa_{ijs}\theta_{,\,m} + \mathcal{P}_{i\alpha}\omega_\alpha\theta_{,\,i}\right)dV,
\end{aligned} \tag{3.12}$$

where we have used the following notations

$$\begin{aligned}
&\mathcal{A}_{ijsmnr}=\frac{1}{2}\left(F_{ijk\alpha}l_{mnr\alpha}+F_{mnr\alpha}l_{ijk\alpha}\right),\ \mathcal{C}_{ijmn}=\frac{1}{2}\left(B_{ij\alpha}g_{mn\alpha}+B_{mn\alpha}g_{ij\alpha}\right),\\
&\mathcal{B}_{ij}=B_{ij\alpha}p_\alpha+G_\alpha g_{ij\alpha},\ \mathcal{B}_{ij\alpha}=B_{ij\beta}q_{\beta\alpha},\ \mathcal{B}_{ijk}=B_{ij\alpha}\gamma_{k\alpha}-\frac{1}{T_0}a_{kji}\\
&\mathcal{D}_{ij}=D_{ij\alpha}p_\alpha+G_\alpha h_{ij\alpha},\ \mathcal{D}_i=G_\alpha r_{i\alpha}-\frac{1}{T_0}d_i,\ \mathcal{D}_{ij\alpha}=D_{ij\beta}q_{\beta\alpha},\\
&\mathcal{D}_{ijk}=D_{ij\alpha}r_{k\alpha}-\frac{1}{T_0}b_{kij},\ \mathcal{F}_{ijmnr}=D_{ij\alpha}l_{mnr\alpha}+F_{ijk\alpha}h_{mn\alpha},\\
&\mathcal{F}_{ijk}=G_\alpha l_{ijk\alpha}+F_{ijk\alpha}p_\alpha,\ \mathcal{F}_{ijk\alpha}=F_{ijk\beta}q_{\beta\alpha},\ \mathcal{F}_{ijkm}=F_{ijk\alpha}r_{m\alpha}-\frac{1}{T_0}c_{mijk},\\
&\mathcal{D}_{ijmn}=B_{ij\alpha}h_{mn\alpha}+D_{mn\alpha}g_{ij\alpha},\ \mathcal{L}_\alpha=G_\beta q_{\beta\alpha},\ \mathcal{M}=G_\alpha p_\alpha,\ \mathcal{P}_{i\alpha}=-\frac{1}{T_0}f_{i\alpha}.
\end{aligned}\tag{3.13}$$

By using the Schwarz's inequality and the arithmetic - geometric mean inequality

$$ab\le\frac{1}{2}\left(\frac{a^2}{\pi^2}+b^2\pi^2\right)\tag{3.14}$$

to the last term in the relation (3.12), we are lead to

$$\begin{aligned}
&\int_B\left[\left(B_{ij\alpha}\,\varepsilon_{ij}+D_{ij\alpha}\,\gamma_{ij}+F_{ijs\alpha}\,\kappa_{ijs}+G_\alpha\,\theta\right)\dot{\omega}_\alpha-\frac{1}{T_0}\,q_i\,\theta_{,\,i}\right]dV\le\\
&\le\left(-2\mu+\pi_1^2+\pi_2^2+\pi_3^2+\pi_4^2+\pi_5^2\right)\int_B\theta_{,\,i}\,\theta_{,\,i}dV+\\
&\left(\frac{M_2^2}{\pi_2^2}+M_6^2+M_{11}^2+M_{12}^2+M_{13}^2+M_{14}^2\right)\int_B\varepsilon_{ij}\,\varepsilon_{ij}dV+\\
&\left(\frac{M_3^2}{\pi_3^2}+M_7^2+M_{15}^2+M_{16}^2+M_{17}^2+1\right)\int_B\gamma_{ij}\,\gamma_{ij}dV+\\
&\left(\frac{M_4^2}{\pi_4^2}+M_8^2+M_{18}^2+M_{19}^2+2\right)\int_B\kappa_{ijs}\,\kappa_{ijs}dV+\\
&\left(\frac{M_5^2}{\pi_5^2}+M_{10}^2+3\right)\int_B\omega_\alpha\,\omega_\alpha dV+\left(\frac{M_1^2}{\pi_1^2}+M_9^2+4\right)\int_B\theta^2dV,
\end{aligned}\tag{3.15}$$

where π_1, π_2, π_3, π_4 and π_5 are arbitrary positive constants. Also, in the inequality (3.15) we have used the notations

$$\begin{aligned}
&M_1^2=\max\left(\mathcal{D}_i\,\mathcal{D}_i\right)(x_s),\ M_2^2=\max\left(\mathcal{B}_{ijk}\,\mathcal{B}_{ijk}\right)(x_s),\\
&M_3^2=\max\left(\mathcal{D}_{ijk}\,\mathcal{D}_{ijk}\right)(x_s),\ M_4^2=\max\left(\mathcal{F}_{ijkm}\,\mathcal{F}_{ijkm}\right)(x_s),\\
&M_5^2=\max\left(\mathcal{P}_{i\alpha}\,\mathcal{P}_{i\alpha}\right)(x_s),\ M_6^2=\max\left(\mathcal{B}_{ij}\,\mathcal{B}_{ij}\right)(x_s),\\
&M_7^2=\max\left(\mathcal{D}_{ij}\,\mathcal{D}_{ij}\right)(x_s),\ M_8^2=\max\left(\mathcal{F}_{ijk}\,\mathcal{F}_{ijk}\right)(x_s),\\
&M_9^2=2\ \max\left|\mathcal{M}(x_s)\right|,\ M_{10}^2=\max\left(\mathcal{L}_\alpha\,\mathcal{L}_\alpha\right)(x_s),\\
&M_{11}^2=2\ \max\left[\left(\mathcal{C}_{ijmn}\mathcal{C}_{ijmn}\right)(x_s)\right]^{1/2},\ M_{12}^2=\max\left(\mathcal{D}_{ijmn}\,\mathcal{D}_{ijmn}\right)(x_s),\\
&M_{13}^2=\max\left(\mathcal{D}_{ijmnr}\,\mathcal{D}_{ijmnr}\right)(x_s),\ M_{14}^2=\max\left(\mathcal{B}_{ij\alpha}\,\mathcal{B}_{ij\alpha}\right)(x_s),\\
&M_{15}^2=2\ \max\left[\left(\mathcal{B}_{ijmn}\mathcal{B}_{ijmn}\right)(x_s)\right]^{1/2},M_{16}^2=\max\left(\mathcal{F}_{ijmnr}\,\mathcal{F}_{ijmnr}\right)(x_s),\\
&M_{17}^2=\max\left(\mathcal{D}_{ij\alpha}\,\mathcal{D}_{ij\alpha}\right)(x_s),\ M_{18}^2=2\ \max\left[\left(\mathcal{A}_{ijkmnr}\mathcal{A}_{ijkmnr}\right)(x_s)\right]^{1/2},\\
&M_{19}^2=\max\left(\mathcal{F}_{ijk\alpha}\,\mathcal{F}_{ijk\alpha}\right)(x_s).
\end{aligned}\tag{3.16}$$

We choose the arbitrary constants π_1, π_2, π_3, π_4 and π_5 so that the quantity m_1 defined by

$$m_1 = \mu - \frac{1}{2}\left(\pi_1^2 + \pi_2^2 + \pi_3^2 + \pi_4^2 + \pi_5^2\right)$$

is strictly positive. Next, if we choose the constant m_2 as follows

$$m_2 = \frac{1}{2}\max\left\{\frac{M_2^2}{\pi_2^2} + M_6^2 + M_{11}^2 + M_{12}^2 + M_{13}^2 + M_{14}^2,\right.$$
$$\frac{M_3^2}{\pi_3^2} + M_7^2 + M_{15}^2 + M_{16}^2 + M_{17}^2 + 1,$$
$$\frac{M_4^2}{\pi_4^2} + M_8^2 + M_{18}^2 + M_{19}^2 + 2,$$
$$\left.\frac{M_5^2}{\pi_5^2} + M_{10}^2 + 3,\ \frac{M_1^2}{\pi_1^2} + M_9^2 + 4\right\}$$

then we arrive to the estimate (21) and this conclude the proof of Theorem 3.2. □

3.3. Theorem. *Let $(u_i,\ \varphi_{ij},\ \theta,\ \omega_\alpha)$ be a solution of the problem P_0 and suppose that the assumptions (i) - (v) are satisfied. Then there exists a positive constant m_3 such that we have the following inequality*

$$\int_B \left(\dot{u}_i\ \dot{u}_i + \dot{\varphi}_{ij}\ \dot{\varphi}_{ij} + \varepsilon_{ij}\ \varepsilon_{ij} + \gamma_{ij}\gamma_{ij} + \kappa_{ijk}\kappa_{ijk} + \theta^2 + \omega_\alpha\omega_\alpha\right) dV \leq$$
$$(3.17)\qquad m_3\int_0^t\int_B \left(\dot{u}_i\ \dot{u}_i + \dot{\varphi}_{ij}\ \dot{\varphi}_{ij} + \varepsilon_{ij}\ \varepsilon_{ij} + \gamma_{ij}\gamma_{ij} + \kappa_{ijk}\kappa_{ijk} + \theta^2 + \omega_\alpha\omega_\alpha\right) dV ds$$

for any $t \in [0, t_0]$.

Proof. First, taking into account the hypotheses (i) - (v), we have

$$m_0\int_B \left(\dot{u}_i\ \dot{u}_i + \dot{\varphi}_{ij}\ \dot{\varphi}_{ij} + \varepsilon_{ij}\ \varepsilon_{ij} + \gamma_{ij}\gamma_{ij} + \kappa_{ijs}\kappa_{ijs} + \theta^2\right) dV \leq$$
$$(3.18)\qquad \int_B (C_{ijmn}\varepsilon_{ij}\varepsilon_{mn} + B_{ijmn}\gamma_{ij}\gamma_{mn} + A_{ijsmnr}\kappa_{ijs}\kappa_{mnr} +$$
$$a\ \theta^2 + \varrho\ \dot{u}_i\ \dot{u}_i + I_{kr}\ \dot{\varphi}_{jr}\ \dot{\varphi}_{jk}\big)\, dV,$$

where we have used the notation

$$m_0 = \min\{\varrho,\ a,\ \lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4\}$$

Next, we use the Schwarz's inequality and the arithmetic - geometric mean inequality (3.14) to the left side of the relation (3.18). So, we are lead to the inequality

$$m_0\int_B \left(\dot{u}_i\ \dot{u}_i + \dot{\varphi}_{ij}\ \dot{\varphi}_{ij} + \varepsilon_{ij}\ \varepsilon_{ij} + \gamma_{ij}\gamma_{ij} + \kappa_{ijs}\kappa_{ijs} + \theta^2\right) dV \leq$$
$$\leq \left(\pi_6^2 + N_4^2 + N_5^2\right)\int_B \varepsilon_{ij}\varepsilon_{ij} dV + \left(\pi_7^2 + N_6^2 + 2\right)\int_B \gamma_{ij}\gamma_{ij} dV +$$
$$(3.19)\qquad + \left(\pi_8^2 + 3\right)\int_B \kappa_{ijs}\kappa_{ijs} dV + \left(\frac{N_1^2}{\pi_6^2} + \frac{N_2^2}{\pi_7^2} + \frac{N_3^2}{\pi_8^2}\right)\int_B \omega_\alpha\omega_\alpha dV -$$
$$+ m_2\int_0^t\int_B \left(\varepsilon_{ij}\ \varepsilon_{ij} + \gamma_{ij}\gamma_{ij} + \kappa_{ijs}\kappa_{ijs} + \theta^2 + \omega_\alpha\omega_\alpha\right) dV ds$$
$$- m_1\int_0^t\int_B \theta,_i\theta,_i dV ds$$

where $t \in [0, t_0]$.
In this inequality we have used the notations

$$N_1^2 = \max\left(B_{ij\alpha}\ B_{ij\alpha}\right)(x_s),\ N_2^2 = \max\left(D_{ij\alpha}\ D_{ij\alpha}\right)(x_s),$$
$$N_3^2 = \max\left(F_{ijk\alpha}\ F_{ijk\alpha}\right)(x_s),\ N_4^2 = \max\left(G_{mnij}\ G_{mnij}\right)(x_s), \tag{3.20}$$
$$N_5^2 = \max\left(F_{mnrij}\ F_{mnrij}\right)(x_s),\ N_6^2 = \max\left(D_{ijmnr}\ D_{ijmnr}\right)(x_s),$$

where $(x_s) \in \overline{B}$.

On the other hand, by using the initial conditions (3.4) and the consitutive relation (2.7), we arrive to the conclusion that:

$$\begin{aligned}\int_B \omega_\alpha\omega_\alpha dV &= \int_0^t \frac{d}{ds}\left(\int_B \omega_\alpha\omega_\alpha dV\right)ds = 2\int_0^t\left(\int_B \omega_\alpha\dot{\omega}_\alpha dV\right)ds = \\ &= 2\int_0^t\int_B \left(g_{ij\alpha}\varepsilon_{ij}\omega_\alpha + h_{ij\alpha}\gamma_{ij}\omega_\alpha + l_{ijs\alpha}\kappa_{ijs}\omega_\alpha + \right. \\ &\quad \left. + p_\alpha\theta\omega_\alpha + q_{\alpha\beta}\omega_\alpha\omega_\beta + r_i\omega_\alpha\theta_{,\,i}\right) dVds\end{aligned} \tag{3.21}$$

Now, by using, again, the Schwarz's inequality and the arithmetic - geometric mean inequality (3.14) to the right side of the relation (3.21). So, we deduce that for an arbitrary positive constant π_9 the following inequality hold:

$$\begin{aligned}\int_B \omega_\alpha\ \omega_\alpha\ dV &\le \pi_9^2\int_0^t\int_B \theta_{,\,i}\ \theta_{,\,i}\ dVds + \\ &+ \left(\frac{Q_1^2}{\pi_9^2} + Q_5^2 + Q_6^2 + 3\right)\int_0^t\int_B \omega_\alpha\ \omega_\alpha\ dVds + \\ &+ Q_2^2\int_0^t\int_B \varepsilon_{ij}\ \varepsilon_{ij}\ dVds + Q_3^2\int_0^t\int_B \gamma_{ij}\ \gamma_{ij}\ dVds + \\ &+ Q_4^2\int_0^t\int_B \kappa_{ijs}\ \kappa_{ijs}\ dVds + \int_0^t\int_B \theta^2 dVds\end{aligned} \tag{3.22}$$

where $t \in [0, t_0]$.
In this inequality we have used the notations

$$Q_1^2 = \max\left(r_{i\alpha}\ r_{i\alpha}\right)(x_s),\ Q_2^2 = \max\left(g_{ij\alpha}\ g_{ij\alpha}\right)(x_s),$$
$$Q_3^2 = \max\left(h_{ij\alpha}\ h_{ij\alpha}\right)(x_s),\ Q_4^2 = \max\left(l_{ijk\alpha}\ l_{ijk\alpha}\right)(x_s), \tag{3.23}$$
$$Q_5^2 = \max\left(p_\alpha\ p_\alpha\right)(x_s),\ Q_6^2 = \max\left[\left(q_{i\alpha}\ q_{i\alpha}\right)(x_s)\right]^{1/2},$$

where $(x_s) \in \overline{B}$.
If we denote by m_4 the quantity

$$m_4 = \max\left\{\frac{Q_1^2}{\pi_9^2} + Q_5^2 + Q_6^2 + 3,\ Q_2^2,\ Q_3^2,\ Q_4^2,\ 1\right\},$$

then, from (3.21) we obtain the following inequality

$$\begin{aligned}\int_B \omega_\alpha\ \omega_\alpha\ dV &\le \pi_9^2\pi_{10}^2\int_0^t\int_B \theta_{,\,i}\ \theta_{,\,i}\ dVds + \\ &+ m_4\pi_{10}^2\int_0^t\int_B \left(\varepsilon_{ij}\varepsilon_{ij} + \gamma_{ij}\gamma_{ij} + \kappa_{ijs}\kappa_{ijs} + \theta^2 + \omega_\alpha\ \omega_\alpha\right) dVds\end{aligned} \tag{3.24}$$

which is satisfied for an arbitrary positive constant π_{10}.
From (3.19) and (3.24) we obtain

$$\begin{aligned}
&m_0\int_B\left(\dot{u}_i\dot{u}_i+\dot{\varphi}_{ij}\dot{\varphi}_{ij}+\theta^2\right)dV+\left[m_0-\left(\pi_6^2+N_4^2+N_5^2\right)\right]\int_B\varepsilon_{ij}\;\varepsilon_{ij}\;dV+\\
&+\left(m_0-\pi_7^2-N_6^2-2\right)\int_B\gamma_{ij}\;\gamma_{ij}\;dV+\left(m_0-\pi_8^2-3\right)\int_B\kappa_{ijs}\;\kappa_{ijs}\;dV+\\
&+\left(\pi_{10}^2-\frac{N_1^2}{\pi_6^2}-\frac{N_2^2}{\pi_7^2}-\frac{N_3^2}{\pi_8^2}\right)\int_B\omega_\alpha\omega_\alpha dV\le\left(m_1-\pi_9^2-\pi_{10}^2\right)\int_0^t\int_B\theta,_i\theta,_i dV ds+\\
&+\left(m_2+m_4\pi_{10}^2\right)\int_0^t\int_B\left(\varepsilon_{ij}\varepsilon_{ij}+\gamma_{ij}\gamma_{ij}+\kappa_{ijs}\kappa_{ijs}+\theta^2+\omega_\alpha\;\omega_\alpha\right)dV ds
\end{aligned}\tag{3.25}$$

We choose the arbitrary constants π_6, π_7, π_8, π_9 and π_{10} so that

$$m_5\equiv m_0-\pi_6^2-N_4^2-N_5^2>0,\;m_6\equiv m_0-\pi_7^2-N_6^2-2>0,$$
$$m_7\equiv m_0-\pi_8^2-3>0,\;m_8\equiv\pi_{10}^2-\frac{N_1^2}{\pi_6^2}-\frac{N_2^2}{\pi_7^2}-\frac{N_3^2}{\pi_8^2}>0,$$
$$m_9\equiv m_1-\pi_9^2\;\pi_{10}^2>0,$$

and thus we are lead to

$$\begin{aligned}
&\left(m_2+m_4\pi_{10}^2\right)\int_0^t\int_B\left(\varepsilon_{ij}\varepsilon_{ij}+\gamma_{ij}\gamma_{ij}+\kappa_{ijs}\kappa_{ijs}+\theta^2+\omega_\alpha\;\omega_\alpha\right)dV ds\ge\\
&\ge m_0\int_B\left(\dot{u}_i\dot{u}_i+\dot{\varphi}_{ij}\dot{\varphi}_{ij}+\theta^2\right)dV+m_5\int_B\varepsilon_{ij}\varepsilon_{ij}dV+m_6\int_B\gamma_{ij}\gamma_{ij}dV+\\
&+m_7\int_B\kappa_{ijs}\kappa_{ijs}dV+m_8\int_B\omega_\alpha\;\omega_\alpha dV+m_9\int_B\theta,_i\theta,_i dV dV\ge\\
&\ge m_{10}\int_B\left(\dot{u}_i\dot{u}_i+\dot{\varphi}_{ij}\dot{\varphi}_{ij}+\varepsilon_{ij}\varepsilon_{ij}+\gamma_{ij}\gamma_{ij}+\kappa_{ijs}\kappa_{ijs}+\theta^2+\omega_\alpha\;\omega_\alpha\right)dV,
\end{aligned}\tag{3.26}$$

where the signification of the constant m_{10} is

$$m_{10}=\min\left\{m_0,\;m_5,\;m_6,\;m_7,\;m_8\right\}.$$

It is easy to observe that

$$\begin{aligned}
&\int_0^t\int_B\left(\dot{u}_i\dot{u}_i+\dot{\varphi}_{ij}\dot{\varphi}_{ij}+\varepsilon_{ij}\;\varepsilon_{ij}+\gamma_{ij}\;\gamma_{ij}+\kappa_{ijs}\;\kappa_{ijs}+\theta^2+\omega_\alpha\;\omega_\alpha\right)dV ds\ge\\
&\ge\int_0^t\int_B\left(\varepsilon_{ij}\;\varepsilon_{ij}+\gamma_{ij}\;\gamma_{ij}+\kappa_{ijs}\;\kappa_{ijs}+\theta^2+\omega_\alpha\;\omega_\alpha\right)dV ds
\end{aligned}\tag{3.27}$$

Finally, if we choose

$$m_3=\frac{\left(m_2+m_4\pi_{10}^2\right)}{m_{10}}$$

then from (3.26) and (3.27) we arrive at the desired result (3.17) and Theorem 3.3 is proved. □

Theorem 3.1, Theorem 3.2 and Theorem 3.3 form the basis of the main result of this study: the uniqueness of mixed initial-boundary value problem for thermoelastic dipolar body with internal state variables.

3.4. Theorem. *Assume that the hypotheses (i) - (v) hold. Then there exists at most one solution of the problem defined by the equations (2.1), (2.2) and (2.6) with the initial conditions (2.9) and the boundary conditions (2.10).*

Proof. Suppose that the mixed problem has two solutions. Then the difference of these solutions is solution for the above mentioned problem P_0. For our aim it is suffice to show that the function $y(t)$ defined by

$$y(t) = \int_B \left(\dot{u}_i \dot{u}_i + \dot{\varphi}_{ij} \dot{\varphi}_{ij} + \varepsilon_{ij}\varepsilon_{ij} + \gamma_{ij}\gamma_{ij} + \kappa_{ijr}\kappa_{ijr} + \theta^2 + \omega_\alpha \omega_\alpha \right) dV$$

vanishes on $[0, t_0]$.

If we assume the contrary, i.e. $y(t) \neq 0$, this is absurdum because the inequality (3.17) and Gronwall's inequality imply that $y(t) \equiv 0$ on $[0, t_0]$ and Theorem 3.4 is concluded. □

Conclusion. The existence of internal state variables do not affect the uniqueness of solution of the mixed problem for dipolar thermoelastic materials.

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References

[1] Anand, L. and Gurtin, M. E. *A theory of amorphous solids undergoing large deformations*, Int. J. Solids Struct. **40**, 1465–1487, 2003

[2] Bouvard, J. L., Ward, D. K., Hossain, D., Marin, E. B., Bammann, D. J. and Horstemeyer M. F., *A general inelastic internal state variable model for amorphous glassy polymers*, Acta Mechanica, **213** 1–2, 71-96, 2010

[3] Chirita, S. *On the linear theory of thermo-viscoelastic materials with internal state variables*, Arch. Mech., **33**, 455–464, 1982

[4] Marin, M. *An evolutionary equation in thermoelasticity of dipolar bodies*, Journal of Mathematical Physics, **40** 3, 1391–1399, 1999

[5] Marin, M. *A partition of energy in thermoelsticity of microstretch bodies*, Nonlinear Analysis: RWA, **11** 4, 2436–2447, 2010

[6] Marin, M. *Some estimates on vibrations in thermoelasticity of dipolar bodies*, Journal of Vibration and Control, **16** 1, 33–47, 2010

[7] Marin, M. *Lagrange identity method for microstretch thermoelastic materials* J. Mathematical Analysis and Applications, **363** 1, 275–286, 2010

[8] Marin, M., Agarwal, R. P. and Mahmoud, S. R. *Modeling a microstretch thermoelastic body with two temperature*, Abstract and Applied Analysis, **2013**, 1–7, 2013

[9] Nachlinger, R. R. and Nunziato, J. W. *Wave propagation and uniqueness theorem for elastic materials with ISV*, Int. J. Engng. Sci., **14**, 31-38, 1976

[10] Pop, N., *An algorithm for solving nonsmooth variational inequalities arising in frictional quasistatic contact problems*, Carpathian Journal of Mathematics, **24** 1, 110–119, 2008

[11] Pop, N., Cioban, H. and Horvat-Marc, A., *Finite element method used in contact problems with dry friction*, Computational Materials Science, **50** 4, 1283–1285, 2011

[12] Sherburn, J. A., Horstemeyer, M. F., Bammann, D. J. and Baumgardner, R. R. *Application of the Bammann inelasticity internal state variable constitutive model to geological materials*, Geophysical J. Int., **184** 3, 1023–1036, 2011

[13] Solanki, K. N. and Bammann, D. J. *A thermodynamic framework for a gradient theory of continuum damage*, (American Acad.Mech.Conf., New Orleans, 2008).

[14] Wei, C. and Dewoolkar, M. M. *Formulation of capillary hysteresis with internal state variables*, Water Resources Research, **42**, 16 pp., 2006