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# ON THE DERIVATION OF EXPLICIT FORMULAE FOR SOLUTIONS OF THE WAVE EQUATION IN HYPERBOLIC SPACE

Gusein Sh. Guseinov*

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**Abstract**

We offer a new approach to solving the initial value problem for the wave equation in hyperbolic space in arbitrary dimensions. Our approach is based on the spectral analysis of the Laplace-Beltrami operator in hyperbolic space and some structural formulae for rapidly decreasing functions of this operator.

**Keywords:** Hyperbolic space, Laplace-Beltrami operator, wave equation, spectral projection.

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## 1. Introduction

The $n$-dimentional hyperbolic space $H^n$ can be realized as the set

(1.1) $$H^n = \{z = (x_1, \dots, x_{n-1}, y) : -\infty < x_j < \infty \ (1 \le j \le n-1),\ 0 < y < \infty\}.$$

The $H^n$ is a homogeneous space of the group

(1.2) $$G = SO^+(1,n) = \left\{g \in GL(n+1,\mathbb{R}) : g^T J g = J,\ \det g = 1,\ g_{00} > 0\right\},$$

where $GL(n+1,\mathbb{R})$ is the group of all nonsingular real $(n+1)\times(n+1)$ matrices $g = [g_{jk}]_{j,k=0}^n$, $J$ is the $(n+1)\times(n+1)$ diagonal matrix whose the first diagonal element equals $-1$ and the remaining diagonal elements are all equal to 1; the symbol $T$ stands for the matrix transposition.

The group $G = SO^+(1,n)$ acts in $H^n$ as follows: If $g \in G$, $g = [g_{jk}]_{j,k=0}^n$ and $z = (x_1, \dots, x_{n-1}, y)$, then the point

$$gz = z' = (x'_1, \dots, x'_{n-1}, y')$$

has the coordinates

(1.3) $$x'_j = \frac{(g_{j0}+g_{jn})\,|z|^2 + 2\sum_{k=1}^{n-1} g_{jk}x_k + g_{j0} - g_{jn}}{c_g\,|z|^2 + 2\sum_{k=1}^{n-1}(g_{0k}-g_{nk})x_k + d_g} \quad (1 \le j \le n-1),$$

*Department of Mathematics, Atilim University, 06836 Incek, Ankara, Turkey. E-Mail: guseinov@atilim.edu.tr

$$y' = \frac{2y}{c_g |z|^2 + 2\sum_{k=1}^{n-1}(g_{0k} - g_{nk})x_k + d_g}, \tag{1.4}$$

where

$$|z|^2 = x_1^2 + \ldots + x_{n-1}^2 + y^2,$$

$$c_g = g_{00} + g_{0n} - g_{n0} - g_{nn}, \quad d_g = g_{00} + g_{nn} - g_{0n} - g_{n0}.$$

The invariant (under the action of $G$) Riemannian metric $ds^2$ and the invariant volume element $dv(z)$ associated with it have the form

$$ds^2 = \frac{dx_1^2 + \ldots + dx_{n-1}^2 + dy^2}{y^2}, \quad dv(z) = \frac{dx_1 \cdots dx_{n-1} dy}{y^n}. \tag{1.5}$$

Denote by $L$ the invariant differential operator (Laplace-Beltrami operator)

$$L = y^2 \left( \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_{n-1}^2} \right) + y^n \frac{\partial}{\partial y} \left( \frac{1}{y^{n-2}} \frac{\partial}{\partial y} \right). \tag{1.6}$$

An invariant of a pair of points, $u(z, z')$, we choose in the form

$$u(z, z') = \frac{|z - z'|^2}{yy'} = \frac{(x_1 - x_1')^2 + \ldots + (x_{n-1} - x_{n-1}')^2 + (y - y')^2}{yy'} \tag{1.7}$$

so that $u(gz, gz') = u(z, z')$ for all $g \in G$ and $z, z' \in H^n$. The non-Euclidean (hyperbolic) distance $\rho(z, z')$ on $H^n$, generated by the metric $ds^2$, has the form

$$\rho(z, z') = \ln \frac{\left|z - \overline{z'}\right| + |z - z'|}{\left|z - \overline{z'}\right| - |z - z'|}, \tag{1.8}$$

where we put $\overline{z} = (x_1, \ldots, x_{n-1}, -y)$ for $z = (x_1, \ldots, x_{n-1}, y)$. It follows from (1.7) and (1.8) that

$$u = 2\cosh\rho - 2 = 4\sinh^2 \frac{\rho}{2}. \tag{1.9}$$

The wave equation in the hyperbolic space $H^n$ for a function

$$w(x_1, \ldots, x_{n-1}, y, t) = w(z, t)$$

of $n$ space variables $x_1, \ldots, x_{n-1}, y$ and the time $t$ is given by

$$\frac{\partial^2 w}{\partial t^2} = L_1 w, \tag{1.10}$$

where

$$L_1 = L + \left(\frac{n-1}{2}\right)^2 = y^2 \left( \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_{n-1}^2} \right) + y^n \frac{\partial}{\partial y} \left( \frac{1}{y^{n-2}} \frac{\partial}{\partial y} \right) + \frac{(n-1)^2}{4}. \tag{1.11}$$

Note that, as is indicated in [3, 4], the right form of the wave equation in hyperbolic space should be Eq. (1.10) with the term $(n-1)^2/4$ because this equation without the term $(n-1)^2/4$ does not satisfy Huygens' principle. However, Eq. (1.10) with the term $(n-1)^2/4$ satisfies Huygens' principle, as the usual wave equation in Euclidean space, when $n$ is odd. Note also that the spectrum of the operator $L$ defined by (1.6) fills the interval $(-\infty, -(n-1)^2/4]$ whereas the spectrum of the operator $L_1$ defined by (1.11) fills the interval $(-\infty, 0]$.

In the Cauchy problem (initial value problem) one asks for a solution $w(z, t)$ of (1.10) defined for $z \in H^n$, $t \geq 0$ that satisfies equation (1.10) for $z \in H^n$, $t > 0$ and the initial conditions

$$w(z, 0) = \varphi(z), \quad \frac{\partial w(z, 0)}{\partial t} = \psi(z) \quad (z \in H^n). \tag{1.12}$$

Let us denote by $w(z,t) = N_\varphi(z,t)$ the solution of the problem

$$(1.13)\qquad \frac{\partial^2 w}{\partial t^2} = L_1 w, \quad z \in H^n,\ t > 0,$$

$$(1.14)\qquad w(z,0) = \varphi(z), \quad \frac{\partial w(z,0)}{\partial t} = 0, \quad z \in H^n.$$

It is easy to see that then the function

$$(1.15)\qquad \widetilde{w}(z,t) = \int_0^t w(z,\tau)d\tau$$

is the solution of the problem

$$(1.16)\qquad \frac{\partial^2 \widetilde{w}}{\partial t^2} = L_1 \widetilde{w}, \quad z \in H^n,\ t > 0,$$

$$(1.17)\qquad \widetilde{w}(z,0) = 0, \quad \frac{\partial \widetilde{w}(z,0)}{\partial t} = \varphi(z), \quad z \in H^n.$$

Indeed, integrating (1.13) we get

$$\int_0^t \frac{\partial^2 w(z,\tau)}{\partial \tau^2} d\tau = \int_0^t L_1 w(z,\tau)d\tau = L_1 \int_0^t w(z,\tau)d\tau = L_1 \widetilde{w}(z,t).$$

Hence

$$(1.18)\qquad \frac{\partial w(z,t)}{\partial t} - \frac{\partial w(z,0)}{\partial t} = L_1 \widetilde{w}(z,t) \quad \text{or} \quad \frac{\partial w(z,t)}{\partial t} = L_1 \widetilde{w}(z,t),$$

by the second condition in (1.14). On the other hand, from (1.15),

$$(1.19)\qquad \frac{\partial \widetilde{w}(z,t)}{\partial t} = w(z,t), \quad \frac{\partial^2 \widetilde{w}(z,t)}{\partial t^2} = \frac{\partial w(z,t)}{\partial t}.$$

Comparing (1.18) and (1.19) we get equation (1.16). Besides,

$$\widetilde{w}(z,0) = 0, \quad \frac{\partial \widetilde{w}(z,0)}{\partial t} = w(z,0) = \varphi(z)$$

so that initial conditions in (1.17) are also satisfied.

Consequently, the solution $w(z,t)$ of problem (1.10), (1.12) is represented in the form

$$w(z,t) = N_\varphi(z,t) + \int_0^t N_\psi(z,\tau)d\tau.$$

It follows that it is enough to know an explicit form of the solution $N_\varphi(z,t)$ of problem (1.13), (1.14). It is known [3, 4] that

$$(1.20)\qquad N_\varphi(z,t) = \frac{1}{2^{m+1}\pi^m}\left(\frac{\partial}{\partial t}\frac{1}{\sinh t}\right)^m \int_{\rho(z,z')=t} \varphi(z')dS_{z'} \quad \text{if} \quad n = 2m+1,$$

$$(1.21)\qquad N_\varphi(z,t) = \frac{1}{2^m\pi^m}\left(\frac{\partial}{\partial t}\frac{1}{\sinh t}\right)^{m-1} \frac{\partial}{\partial t}\int_{\rho(z,z')<t} \frac{\varphi(z')dv(z')}{\sqrt{2(\cosh t - \cosh \rho)}} \quad \text{if} \quad n = 2m,$$

where $z = (x_1, \ldots, x_{n-1}, y)$, $z' = (x'_1, \ldots, x'_{n-1}, y')$, $\rho = \rho(z,z')$ is the hyperbolic distance of $z$ from $z'$ defined by (1.8), $dS_{z'}$ is the surface element of the sphere $\{z' \in H^n : \rho(z,z') = t\}$, and $dv(z')$ is the volume element as defined in (1.5).

In the present paper, we give a new proof of formulae (1.20), (1.21) for the solution of problem (1.13), (1.14). Our method of the proof is based on the spectral theory of the Laplace-Beltrami operator. A similar method was recently applied by the author to the wave equation in Euclidean space in [2].

Note that in the case $n = 1$ problem (1.10), (1.12) becomes

$$\frac{\partial^2 w}{\partial t^2} = y\frac{\partial}{\partial y}\left(y\frac{\partial w}{dy}\right), \tag{1.22}$$

$$w(y,0) = \varphi(y), \quad \frac{\partial w(y,0)}{\partial t} = \psi(y), \tag{1.23}$$

where $w = w(y,t)$, $t \geq 0$, and $y \in H^1 = (0,\infty)$. Making the change of variables

$$\xi = ye^t,\ \eta = ye^{-t}, \quad \text{i.e.} \quad y = \sqrt{\xi\eta},\ t = \ln\sqrt{\frac{\xi}{\eta}},$$

we transform Eq. (1.22) into

$$\frac{\partial^2 \widetilde{w}(\xi,\eta)}{\partial\xi\partial\eta} = 0, \quad \text{where} \quad \widetilde{w}(\xi,\eta) = w\left(\sqrt{\xi\eta}, \ln\sqrt{\frac{\xi}{\eta}}\right).$$

Hence $\widetilde{w}(\xi,\eta) = \theta_1(\xi) + \theta_2(\eta)$, where $\theta_1, \theta_2$ are arbitrary differentiable functions. Returning to the old variables we get that the general solution of Eq. (1.22) has the form

$$w(y,t) = \theta_1(ye^t) + \theta_2(ye^{-t}).$$

Using this we find easily that the solution of problem (1.22), (1.23) is

$$w(y,t) = \frac{\varphi(ye^t) + \varphi(ye^{-t})}{2} + \frac{1}{2}\int_{ye^{-t}}^{ye^t} \psi(y')dy'.$$

The paper is organized as follows. In Section 2, we describe the structure of arbitrary rapidly decreasing function of the Laplace-Beltrami operator, showing that it is an integral operator and giving an explicit formula for its kernel. Next we use these results in Section 3 to derive the explicit representation formulae (1.20) and (1.21) for the classical solution to the initial value problem (1.13), (1.14). The final section is an appendix and contains some additional material about the hyperbolic space.

## 2. Structure of arbitrary function of the Laplace-Beltrami operator

We denote by $L^2(H^n, dv)$ the Hilbert space of all complex-valued measurable functions $f(z)$ defined on $H^n$ such that

$$\int_{H^n} |f(z)|^2\, dv(z) < \infty,$$

with the inner product

$$(f_1, f_2) = \int_{H^n} f_1(z)\overline{f_2(z)}dv(z),$$

where $dv(z)$ is the invariant volume element defined in (1.5). Let $A$ be the selfadjoint positive operator obtained as the closure of the symmetric operator $A'$ determined in the Hilbert space $L^2(H^n, dv)$ by the differential expression

$$-L_1 = -y^2\left(\frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_{n-1}^2}\right) - y^n\frac{\partial}{\partial y}\left(\frac{1}{y^{n-2}}\frac{\partial}{\partial y}\right) - \left(\frac{n-1}{2}\right)^2 \tag{2.1}$$

on the domain of definition $D(A') = C_0^\infty(H^n)$ that is the set of all infinitely differentiable functions on $H^n$ with compact (with respect to the hyperbolic distance) support. Let $E_\mu$ denote the resolution of the identity (the spectral projection) for $A$ :

$$Af = \int_0^\infty \mu dE_\mu f, \quad f \in D(A).$$

Next, let $h(t)$ be any infinitely differentiable even function on the axis $-\infty < t < \infty$ with compact support and

$$\widetilde{h}(\lambda) = \int_{-\infty}^{\infty} h(t) e^{i\lambda t} dt \tag{2.2}$$

its Fourier transform. Note that the function $\widetilde{h}(\lambda)$ tends to zero as $|\lambda| \to \infty$ ($\lambda \in \mathbb{R}$) faster than any negative power of $|\lambda|$. Consider the operator $\widetilde{h}(\sqrt{A})$ defined according to the general theory of selfadjoint operators (see [1]):

$$\widetilde{h}(\sqrt{A}) f = \int_0^{\infty} \widetilde{h}(\sqrt{\mu}) dE_\mu f, \quad f \in L^2(H^n, dv). \tag{2.3}$$

The following theorem describes the structure of the operator $\widetilde{h}(\sqrt{A})$ showing that it is an integral operator and giving an explicit fomula for its kernel in terms of the function $h(t)$.

**2.1. Theorem.** *The operator $\widetilde{h}(\sqrt{A})$ is an integral operator*

$$\widetilde{h}(\sqrt{A}) f(z) = \int_{H^n} \mathcal{K}(z, z') f(z') dv(z'), \quad f \in L^2(H^n, dv). \tag{2.4}$$

*Further, there is a smooth function $k(t)$ defined on the interval $0 \le t < \infty$ such that*

$$\mathcal{K}(z, z') = k(u(z, z')), \tag{2.5}$$

*where $u(z, z')$ is the invariant of a pair of points given in (1.7). The function $k(t)$ depends on the function $h(t)$ as follows. If we set*

$$Q(t) = h\left(\cosh^{-1}\left(1 + \frac{t}{2}\right)\right), \quad i.e. \quad Q(e^t + e^{-t} - 2) = h(t), \quad 0 \le t < \infty, \tag{2.6}$$

*then*

$$k(t) = \begin{cases} \frac{(-1)^m}{\pi^m} Q^{(m)}(t) & \text{if} \quad n = 2m + 1, \\ \frac{(-1)^m}{\pi^m} \int_t^{\infty} \frac{Q^{(m)}(\eta)}{\sqrt{\eta - t}} d\eta & \text{if} \quad n = 2m, \end{cases} \tag{2.7}$$

*where $Q^{(m)}(t)$ denotes the $m$-th order derivative of $Q(t)$. If $\operatorname{supp} h(t) \subset (-a, a)$, then $\operatorname{supp} k(t) \subset [0, 4\sinh^2 \frac{a}{2})$. For any solution $\psi(z, \lambda)$ of the equation*

$$-L_1 \psi(z, \lambda) = \lambda^2 \psi(z, \lambda) \tag{2.8}$$

*the equality*

$$\int_{H^n} k(u(z, z')) \psi(z', \lambda) dv(z') = \widetilde{h}(\lambda) \psi(z, \lambda) \tag{2.9}$$

*holds, where $-L_1$ has the form (2.1).*

*Proof.* First we consider the case $n = 1$. In this case, the statements of the theorem take the following form: the operator $\widetilde{h}(\sqrt{A})$ is an integral operator of the form

$$\widetilde{h}(\sqrt{A}) f(y) = \int_0^{\infty} h\left(\ln \frac{y}{y'}\right) f(y') \frac{dy'}{y'}, \tag{2.10}$$

and for any solution $\psi(y, \lambda)$ of the equation

$$-y \frac{d}{dy}\left(y \frac{d}{dy} \psi(y, \lambda)\right) = \lambda^2 \psi(y, \lambda) \tag{2.11}$$

the equality

$$\int_0^{\infty} h\left(\ln \frac{y}{y'}\right) \psi(y', \lambda) \frac{dy'}{y'} = \widetilde{h}(\lambda) \psi(y, \lambda) \tag{2.12}$$

holds.

To prove these statements note that, in the case $n = 1$, the operator $A$ is generated in the Hilbert space $L^2[(0,\infty), dy/y]$ by the operation $-(yd/dy)^2$ and the operator $A^{1/2}$ by the operation $iyd/dy$. The resolvent $R_\mu = (A - \mu I)^{-1}$ of the operator $A$ has the form

$$R_\mu f(y) = \frac{i}{2\sqrt{\mu}} \int_0^\infty e^{i|\ln y - \ln y'|\sqrt{\mu}} f(y') \frac{dy'}{y'},$$

while the spectral projection $E_\mu$ of the operator $A$ has the form

$$E_\mu f(y) = \int_0^\infty \frac{\sin\sqrt{\mu}(\ln y - \ln y')}{\pi(\ln y - \ln y')} f(y') \frac{dy'}{y'}, \quad 0 \le \mu < \infty,$$

$$E_\mu = 0 \quad \text{for} \quad \mu < 0.$$

Therefore,

$$\begin{aligned}\widetilde{h}(\sqrt{A}) f(y) &= \int_0^\infty \widetilde{h}(\sqrt{\mu}) dE_\mu f(y) \\ &= \int_0^\infty \widetilde{h}(\sqrt{\mu}) \left\{ \int_0^\infty \frac{\cos\sqrt{\mu}(\ln y - \ln y')}{2\pi\sqrt{\mu}} f(y') \frac{dy'}{y'} \right\} d\mu \\ &= \int_0^\infty \left\{ \frac{1}{\pi} \int_0^\infty \widetilde{h}(\lambda) \cos\lambda(\ln y - \ln y') d\lambda \right\} f(y') \frac{dy'}{y'} = \int_0^\infty h\left(\ln\frac{y}{y'}\right) f(y') \frac{dy'}{y'},\end{aligned}$$

where we have used the inversion formula for the Fourier cosine transform. Therefore, (2.10) is proved. To prove (2.12) note that the general solution of Eq. (2.11) is

$$\psi(y,\lambda) = \begin{cases} c_1 \cos(\lambda \ln y) + c_2 \sin(\lambda \ln y) & \text{if} \quad \lambda \ne 0, \\ c_1 + c_2 \ln y & \text{if} \quad \lambda = 0, \end{cases}$$

where $c_1$ and $c_2$ are arbitrary constants. Then, we have, for $\lambda \ne 0$,

$$\begin{aligned}&\int_0^\infty h\left(\ln\frac{y}{y'}\right) \psi(y',\lambda) \frac{dy'}{y'} \\ &= c_1 \int_0^\infty h\left(\ln\frac{y}{y'}\right) \cos(\lambda \ln y') \frac{dy'}{y'} + c_2 \int_0^\infty h\left(\ln\frac{y}{y'}\right) \sin(\lambda \ln y') \frac{dy'}{y'} \\ &= c_1 \int_{-\infty}^\infty h(t) \cos\lambda(\ln y - t) dt + c_2 \int_{-\infty}^\infty h(t) \sin\lambda(\ln y - t) dt \\ &= c_1 \int_{-\infty}^\infty h(t)[\cos(\lambda \ln y)\cos\lambda t + \sin(\lambda \ln y)\sin\lambda t] dt \\ &+ c_2 \int_{-\infty}^\infty h(t)[\sin(\lambda \ln y)\cos\lambda t - \sin\lambda t \cos(\lambda \ln y)] dt \\ &= c_1 \cos(\lambda \ln y) \int_{-\infty}^\infty h(t)\cos\lambda t dt + c_2 \sin(\lambda \ln y) \int_{-\infty}^\infty h(t) \cos\lambda t dt \\ &= [c_1 \cos(\lambda \ln y) + c_2 \sin(\lambda \ln y)] \int_{-\infty}^\infty h(t)\cos\lambda t dt = \psi(y,\lambda)\widetilde{h}(\lambda),\end{aligned}$$

where we have used the fact that the function $h(t)$ is even and therefore

$$\int_{-\infty}^\infty h(t) \sin\lambda t dt = 0.$$

The same reslut can be obtained similarly for $\lambda = 0$. Thus, (2.12) is also proved.

Now we consider the case $n \ge 2$. We shall use the known [5] integral representation

$$R_\mu f(z) = \int_{H^n} r(z, z'; \mu) f(z') dv(z'),$$

of the resolvent $R_\mu = (A - \mu I)^{-1}$ of the operator $A$, where the kernel $r(z, z'; \mu)$ is given by

$$(2.13) \quad r(z, z'; \mu) = \begin{cases} \omega\left(u(z, z'); \frac{n-1}{2} + i\sqrt{\mu}\right), & Im\mu < 0, \\ \omega\left(u(z, z'); \frac{n-1}{2} - i\sqrt{\mu}\right), & Im\mu > 0, \end{cases}$$

in which $u(z, z')$ is the invariant of a pair of points defined by (1.7) and the function $\omega(u; s)$ is given by the classical integral

$$\omega(u; s) = \frac{\Gamma(s)}{2^n \pi^{\frac{n}{2}} \Gamma\left(s - \frac{n}{2} + 1\right)} \int_0^1 [t(1-t)]^{s-\frac{n}{2}} \left(t + \frac{u}{4}\right)^{-s} dt,$$

which converges absolutely for $u > 0$ and complex $s = \sigma + i\tau$ with $\sigma > \frac{n-2}{2}$; $\Gamma(s)$ is the gamma-function. Next, according to the general spectral ıheory of selfadjoint operators [1, p. 150, Formula (11)], we have

$$dE_\mu f(z) = \frac{1}{2\pi i}(R_{\mu+i0} - R_{\mu-i0}) f(z) d\mu.$$

Therefore from (2.3) it follows that the representation (2.4) holds with

$$(2.14) \quad \mathcal{K}(z, z') = \frac{1}{2\pi i} \int_0^\infty \widetilde{h}(\sqrt{\mu})[r(z, z'; \mu + i0) - r(z, z'; \mu - i0)] d\mu.$$

Now the representation (2.5), which expresses that $\mathcal{K}(z, z')$ is a function of $u(z, z')$, follows from (2.14) by (2.13).

To prove (2.9) we use (2.14) by virtue of which we have

$$(2.15) \quad \int_{H^n} k(u(z, z'))\psi(z', \lambda) dv(z') = \int_{H^n} \mathcal{K}(z, z')\psi(z', \lambda) dv(z')$$

$$= \lim_{\varepsilon \to +0} \int_{H^n} \left\{ \frac{1}{2\pi i} \int_0^\infty \widetilde{h}(\sqrt{\mu})[r(z, z'; \mu + i\varepsilon) - r(z, z'; \mu - i\varepsilon)] d\mu \right\} \psi(z', \lambda) dv(z')$$

$$= \psi(z, \lambda) \lim_{\varepsilon \to +0} \frac{\varepsilon}{\pi} \int_0^\infty \frac{\widetilde{h}(\sqrt{\mu})}{(\mu - \lambda^2)^2 + \varepsilon^2} d\mu = \psi(z, \lambda) \widetilde{h}(\lambda).$$

Here we have used the fact, as it follows from (2.8), that

$$(-L_1 - \zeta)\psi(z, \lambda) = (\lambda^2 - \zeta)\psi(z, \lambda),$$

that is,

$$\psi(z, \lambda) = (\lambda^2 - \zeta)(-L_1 - \zeta)^{-1}\psi(z, \lambda),$$

and therefore

$$\int_{H^n} r(z, z'; \zeta)\psi(z', \lambda) dv(z') = \frac{1}{\lambda^2 - \zeta}\psi(z, \lambda).$$

Finally, to deduce the explicit formulae (2.6), (2.7), we take

$$\psi(z, \lambda) = y^{\frac{n-1}{2} + i\lambda}$$

in (2.9):

$$\int_{H^n} k\left(\frac{|z - z'|^2}{yy'}\right) y'^{\frac{n-1}{2} + i\lambda} dv(z') = \widetilde{h}(\lambda) y^{\frac{n-1}{2} + i\lambda}.$$

Hence, putting $x = (x_1, \ldots, x_{n-1})$ and $x' = (x'_1, \ldots, x'_{n-1})$, we can write

$$(2.16) \quad \int_{H^n} k\left(\frac{(y - y')^2}{yy'} + \frac{|x - x'|^2}{yy'}\right) y'^{\frac{n-1}{2} + i\lambda} \frac{dy'}{y'^n} dx' = \widetilde{h}(\lambda) y^{\frac{n-1}{2} + i\lambda}.$$

If we set

$$\frac{y}{y'}+\frac{y'}{y}-2=\eta, \tag{2.17}$$

then the left-hand side of (2.16) equals

$$\int_0^\infty \left\{\int_{\mathbb{R}^{n-1}} k\left(\eta+\frac{|x-x'|^2}{yy'}\right)dx'\right\} {y'}^{i\lambda-\frac{n+1}{2}}dy'.$$

On the other hand,

$$\int_{\mathbb{R}^{n-1}} k\left(\eta+\frac{|x-x'|^2}{yy'}\right)dx' = \int_0^\infty \left\{\int_{|x-x'|=r} k\left(\eta+\frac{|x-x'|^2}{yy'}\right)dS\right\}dr$$

$$=\int_0^\infty k\left(\eta+\frac{r^2}{yy'}\right)\left\{\int_{|x-x'|=r} dS\right\}dr = \sigma_{n-1}\int_0^\infty r^{n-2}k\left(\eta+\frac{r^2}{yy'}\right)dr$$

$$=\frac{1}{2}\sigma_{n-1}(yy')^{\frac{n-1}{2}}\int_\eta^\infty (t-\eta)^{\frac{n-3}{2}}k(t)dt,$$

where

$$\sigma_n=\frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}$$

is the surface area of the $(n-1)$-dimensional unit sphere in $\mathbb{R}^n$ ($\Gamma$ is the gamma function) and $dS$ denotes the surface element of the sphere $\{x'\in\mathbb{R}^{n-1}: |x-x'|=r\}$. Therefore setting

$$Q(\eta)=\frac{1}{2}\sigma_{n-1}\int_\eta^\infty (t-\eta)^{\frac{n-3}{2}}k(t)dt, \tag{2.18}$$

we get that (2.16) takes the form

$$\int_0^\infty Q(\eta){y'}^{i\lambda-1}dy'=\widetilde{h}(\lambda)y^{i\lambda}.$$

Substituting here the expression of $\eta$ given in (2.17) and making then the change of variables $y'=ye^t$, we obtain

$$\int_{-\infty}^\infty Q(e^{-t}+e^t-2)e^{i\lambda t}dt=\widetilde{h}(\lambda)=\int_{-\infty}^\infty h(t)e^{i\lambda t}dt.$$

Hence (2.6) follows. Further, it is not diffucult to check that the formula (2.18) for $n\geq 2$ is equivalent to (2.7), see [2, Appendix].

Since $h(t)$ is smooth and has a compact support, it follows from (2.6), (2.7) that the function $k(t)$ also is smooth and has a compact support; more precisely, if $suppg(t)\subset(-a,a)$, then $suppk(t)\subset[0,4\sinh^2\frac{a}{2})$. This implies, in particular, convergence of the integral in (2.9) for each fixed $z$. The theorem is proved. □

## 3. Derivation of formulae (1.20), (1.21)

Consider the Cauchy problem (1.13), (1.14):

$$\frac{\partial^2 w}{\partial t^2}=L_1 w, \quad z\in H^n,\ t>0, \tag{3.1}$$

$$w(z,0)=\varphi(z), \quad \frac{\partial w(z,0)}{\partial t}=0, \quad z\in H^n, \tag{3.2}$$

where $w = w(z,t)$, $t \geq 0$, $z = (x_1, \ldots, x_{n-1}, y) \in H^n$, $\varphi(z) \in C_0^\infty(H^n)$. The solution of problem (3.1), (3.2) can be written in the form

$$w(z,t) = \cos\left(t\sqrt{A}\right)\varphi(z) = \int_0^\infty \cos\left(t\sqrt{\mu}\right) dE_\mu \varphi(z).$$

Multiply the last equation by $2h(t)$ and integrate with respect to $t$ from 0 to $\infty$ to get

$$2\int_0^\infty h(t)w(z,t)dt = \int_0^\infty \widetilde{h}(\sqrt{\mu})dE_\mu\varphi(z) = \widetilde{h}(\sqrt{A})\varphi(z).$$

Thus, we have obtained the identity

$$\widetilde{h}(\sqrt{A})\varphi(z) = 2\int_0^\infty h(t)w(z,t)dt, \tag{3.3}$$

where $h(t)$ is any infinitely differentiable even function on the axis $-\infty < t < \infty$ with compact support, $\widetilde{h}(\lambda)$ is its Fourier transform (2.2), $\varphi(z) \in C_0^\infty(H^n)$, and $w(z,t)$ is the solution of the Cauchy problem (3.1), (3.2). Further, by Theorem 2.1 we can rewrite (3.3) in the form

$$\int_{H^n} k(u(z,z'))\varphi(z')dv(z') = 2\int_0^\infty h(t)w(z,t)dt, \tag{3.4}$$

where the function $k(t)$, $0 \leq t < \infty$, is determined from the function $h(t)$ by means of formulae (2.6), (2.7).

Next we will transform the left-hand side of (3.4) using formulae (2.6), (2.7).

First we consider the case $n = 1$. In this case, (3.4) takes the form

$$\int_0^\infty k\left(\frac{(y-y')^2}{yy'}\right)\varphi(y')\frac{dy'}{y'} = 2\int_0^\infty h(t)w(y,t)dt \tag{3.5}$$

and from (2.6), (2.7) we have

$$k(e^t + e^{-t} - 2) = Q(e^t + e^{-t} - 2) = h(t).$$

Therefore, making the change of variables $y' = ye^t$ and taking into account the evenness of the function $h(t)$, we can write

$$\int_0^\infty k\left(\frac{(y-y')^2}{yy'}\right)\varphi(y')\frac{dy'}{y'} = \int_{-\infty}^\infty k(e^t + e^{-t} - 2)\varphi(ye^t)dt$$

$$= \int_{-\infty}^\infty h(t)\varphi(ye^t)dt = \int_0^\infty h(t)[\varphi(ye^t) + \varphi(ye^{-t})]dt.$$

Substituting this in the left-hand side of (3.5), we obtain

$$\int_0^\infty h(t)[\varphi(ye^t) + \varphi(ye^{-t})]dt = 2\int_0^\infty h(t)w(y,t)dt.$$

Hence, by the arbitrariness of the smooth even funciton $h(t)$ with compact support, we get

$$w(y,t) = \frac{\varphi(ye^t) + \varphi(ye^{-t})}{2}.$$

Further assume that $n \geq 2$. Passing to the geodesic polar coordinates with center at $z$ and setting $\rho(z,z') = t$, where $\rho(z,z')$ is the hyperbolic distance between the points $z$ and $z'$ defined by (1.8), and taking into account (1.9), we have

$$\int_{H^n} k(u(z,z'))\varphi(z')dv(z') = \int_0^\infty k(2\cosh t - 2)\left\{\int_{\rho(z,z')=t}\varphi(z')dS_{z'}\right\}dt,$$

where $dS_{z'}$ is the surface element of the sphere $\{z' \in H^n : \rho(z,z') = t\}$. Let us set

$$(3.6)\qquad P_\varphi(z,t) = \int_{\rho(z,z')=t} \varphi(z')dS_{z'} = (\sinh^{n-1} t)\int_{|\omega|=1} \varphi(z,t,\omega)dS_\omega,$$

where to write the second equality we have passed to the spherical (geodesic) coordinates $(t,\omega)$ with origin $z$, $\omega \in \mathbb{R}^n$, $|\omega| = 1$; and used $dS_{z'} = \sinh^{n-1} t dS_\omega$ in which $dS_\omega$ is the surface element on the unit sphere $\{\omega \in \mathbb{R}^n : |\omega| = 1\}$. Then we can rewrite the last formula in the form

$$\int_{H^n} k(u(z,z'))\varphi(z')dv(z') = \int_0^\infty k(2\cosh t - 2)P_\varphi(z,t)dt.$$

Therefore (3.4) becomes

$$(3.7)\qquad \int_0^\infty k(2\cosh t - 2)P_\varphi(z,t)dt = 2\int_0^\infty h(t)w(z,t)dt.$$

Consider the cases of odd and even $n$ separately.

Let $n = 2m+1$ ($m \in \mathbb{N}$). Then, by (2.7) we have

$$k(2\cosh t - 2) = \frac{(-1)^m}{\pi^m} Q^{(m)}(2\cosh t - 2)$$

and it follows from (2.6) (by successive differentiation) that

$$(3.8)\qquad Q^{(m)}(2\cosh t - 2) = \left(\frac{1}{2\sinh t}\frac{\partial}{\partial t}\right)^m h(t).$$

Therefore

$$k(2\cosh t - 2) = \frac{(-1)^m}{2^m\pi^m}\left(\frac{1}{\sinh t}\frac{\partial}{\partial t}\right)^m h(t)$$

and (3.7) takes the form

$$(3.9)\qquad \frac{(-1)^m}{2^m\pi^m}\int_0^\infty \left\{\left(\frac{1}{\sinh t}\frac{\partial}{\partial t}\right)^m h(t)\right\} P_\varphi(z,t)dt = 2\int_0^\infty h(t)w(z,t)dt.$$

Further, integrating $m$ times by parts, we get

$$\int_0^\infty \left\{\left(\frac{1}{\sinh t}\frac{\partial}{\partial t}\right)^m h(t)\right\} P_\varphi(z,t)dt$$

$$= R(z,t)\,\Big|_{t=0}^{t=\infty} + (-1)^m \int_0^\infty h(t)\left(\frac{\partial}{\partial t}\frac{1}{\sinh t}\right)^m P_\varphi(z,t)dt,$$

where

$$(3.10)\qquad R(z,t) = \sum_{k=1}^m \frac{(-1)^{k-1}}{\sinh t}\left\{\left(\frac{1}{\sinh t}\frac{\partial}{\partial t}\right)^{m-k} h(t)\right\}\left(\frac{\partial}{\partial t}\frac{1}{\sinh t}\right)^{k-1} P_\varphi(z,t)$$

$$= \sum_{k=1}^m \frac{(-1)^{k-1}}{\sinh t}\left\{\left(\frac{1}{\sinh t}\frac{\partial}{\partial t}\right)^{m-k} h(t)\right\}$$

$$\times \left(\frac{\partial}{\partial t}\frac{1}{\sinh t}\right)^{k-1} \sinh^{2m} t \int_{|\omega|=1} \varphi(z,t,\omega)dS_\omega.$$

Since $h(t)$ is identically zero for large values of $t$, we have from (3.10) that $R(z,\infty) = 0$. Also, it follows directly from (3.10) that $R(z,0) = 0$. Therefore, (3.9) becomes

$$(3.11)\qquad \frac{1}{2^m\pi^m}\int_0^\infty h(t)\left(\frac{\partial}{\partial t}\frac{1}{\sinh t}\right)^m P_\varphi(z,t)dt = 2\int_0^\infty h(t)w(z,t)dt.$$

Since in (3.11) $h(t)$ is an arbitrary smooth even function with compact support, we obtain that

$$w(z,t) = \frac{1}{2^{m+1}\pi^m}\left(\frac{\partial}{\partial t}\frac{1}{\sinh t}\right)^m P_\varphi(z,t).$$

This coincides with (1.20) by (3.6).

Now let us consider the case $n = 2m$ ($m \in \mathbb{N}$). In this case, by (2.7) we have

$$\begin{aligned} k(2\cosh r - 2) &= \frac{(-1)^m}{\pi^m}\int_{2\cosh r-2}^\infty \frac{Q^{(m)}(\eta)}{\sqrt{\eta-(2\cosh r-2)}}d\eta \\ &= \frac{(-1)^m}{\pi^m}\int_r^\infty \frac{Q^{(m)}(2\cosh t-2)2\sinh t}{\sqrt{2(\cosh t-\cosh r)}}dt \end{aligned}$$

and therefore

$$\begin{aligned} &\int_0^\infty k(2\cosh r-2)P_\varphi(z,r)dr \\ &= \frac{(-1)^m}{\pi^m}\int_0^\infty \left\{\int_r^\infty \frac{Q^{(m)}(2\cosh t-2)2\sinh t}{\sqrt{2(\cosh t-\cosh r)}}dt\right\}P_\varphi(z,r)dr \\ &= \frac{(-1)^m}{\pi^m}\int_0^\infty \left\{\int_r^\infty \frac{Q^{(m)}(2\cosh t-2)2\sinh t}{\sqrt{2(\cosh t-\cosh r)}}dt\right\}\left\{\int_{\rho(z,z')=r}\varphi(z')dS_{z'}\right\}dr. \end{aligned}$$

Next, we have

$$\begin{aligned} &\left\{\int_r^\infty \frac{Q^{(m)}(2\cosh t-2)2\sinh t}{\sqrt{2(\cosh t-\cosh r)}}dt\right\}\left\{\int_{\rho(z,z')=r}\varphi(z')dS_{z'}\right\} \\ &= \int_r^\infty Q^{(m)}(2\cosh t-2)2\sinh t\left[\int_{\rho(z,z')=r}\frac{\varphi(z')dS_{z'}}{\sqrt{2[\cosh t-\cosh\rho(z,z')]}}\right]dt. \end{aligned}$$

Therefore

$$\begin{aligned} &\int_0^\infty k(2\cosh r-2)P_\varphi(z,r)dr \\ &= \frac{(-1)^m}{\pi^m}\int_0^\infty Q^{(m)}(2\cosh t-2)2\sinh t \\ &\times\left\{\int_0^t\left[\int_{\rho(z,z')=r}\frac{\varphi(z')dS_{z'}}{\sqrt{2[\cosh t-\cosh\rho(z,z')]}}\right]dr\right\}dt. \end{aligned}$$

Hence, setting

$$\begin{aligned} \text{(3.12)}\qquad H_\varphi(z,t) &:= \int_0^t\left[\int_{\rho(z,z')=r}\frac{\varphi(z')dS_{z'}}{\sqrt{2[\cosh t-\cosh\rho(z,z')]}}\right]dr \\ &= \int_{\rho(z,z')<t}\frac{\varphi(z')dv(z')}{\sqrt{2[\cosh t-\cosh\rho(z,z')]}}, \end{aligned}$$

we get

$$\int_0^\infty k(2\cosh r-2)P_\varphi(z,r)dr = \frac{(-1)^m}{\pi^m}\int_0^\infty Q^{(m)}(2\cosh t-2)2(\sinh t)H_\varphi(z,t)dt.$$

Substituting this in the left-hand side of (3.7) (beforehand replacing $t$ by $r$ in the left-hand side of (3.7)), we obtain

$$\frac{(-1)^m}{\pi^m}\int_0^\infty Q^{(m)}(2\cosh t-2)2(\sinh t)H_\varphi(z,t)dt = 2\int_0^\infty h(t)w(z,t)dt$$

or, using (3.8),

$$\frac{(-1)^m}{2^{m-1}\pi^m}\int_0^\infty \left\{\left(\frac{1}{\sinh t}\frac{\partial}{\partial t}\right)^m h(t)\right\}(\sinh t)H_\varphi(z,t)dt = 2\int_0^\infty h(t)w(z,t)dt. \tag{3.13}$$

Further, integrating $m$ times by parts, we get

$$\begin{aligned}&\int_0^\infty \left\{\left(\frac{1}{\sinh t}\frac{\partial}{\partial t}\right)^m h(t)\right\}(\sinh t)H_\varphi(z,t)dt\\ &= L(z,t)\,\Big|_{t=0}^{t=\infty} + (-1)^m\int_0^\infty h(t)\left(\frac{\partial}{\partial t}\frac{1}{\sinh t}\right)^m(\sinh t)H_\varphi(z,t)dt,\end{aligned}$$

where

$$L(z,t) = \sum_{k=1}^m (-1)^{k-1}\left\{\left(\frac{1}{\sinh t}\frac{\partial}{\partial t}\right)^{m-k} h(t)\right\}\left(\frac{\partial}{\partial t}\frac{1}{\sinh t}\right)^{k-1}(\sinh t)H_\varphi(z,t). \tag{3.14}$$

Since $h(t)$ is identically zero for large values of $t$, we have from (3.14) that $L(z,\infty)=0$. Also, using the expression of $H_\varphi(z,t)$,

$$\begin{aligned}H_\varphi(z,t) &= \int_0^t \left[\int_{\rho(z,z')=r}\frac{\varphi(z')dS_{z'}}{\sqrt{2[\cosh t-\cosh\rho(z,z')]}}\right]dr\\ &= \int_0^t \sinh^{2m-1} r\left[\int_{|\omega|=1}\frac{\varphi(z,r,\omega)dS_\omega}{\sqrt{2(\cosh t-\cosh r)}}\right]dr\\ &= \int_0^t \frac{\sinh^{2m-1} r}{\sqrt{2[\cosh t-\cosh r)]}}\left[\int_{|\omega|=1}\varphi(z,r,\omega)dS_\omega\right]dr\\ &= \int_0^t [2(\cosh t-\cosh\xi)]^{2m-2}\left[\int_{|\omega|=1}\varphi(z,\sqrt{2(\cosh t-\cosh\xi)},\omega)dS_\omega\right]d\xi,\end{aligned}$$

we can check directly from (3.14) that $L(z,0)=0$. Therefore, (3.13) becomes

$$\frac{1}{2^{m-1}\pi^m}\int_0^\infty h(t)\left(\frac{\partial}{\partial t}\frac{1}{\sinh t}\right)^m(\sinh t)H_\varphi(z,t)dt = 2\int_0^\infty h(t)w(z,t)dt. \tag{3.15}$$

Since in (3.15) $h(t)$ is an arbitrary smooth even function with compact support, we obtain that

$$\begin{aligned}w(z,t) &= \frac{1}{2^m\pi^m}\left(\frac{\partial}{\partial t}\frac{1}{\sinh t}\right)^m(\sinh t)H_\varphi(z,t)\\ &= \frac{1}{2^m\pi^m}\left(\frac{\partial}{\partial t}\frac{1}{\sinh t}\right)^{m-1}\frac{\partial}{\partial t}H_\varphi(z,t).\end{aligned}$$

This coincides with (1.21) by (3.12).

## 4. Appendix

Note that the group

$$O(1,n) = \left\{g\in GL(n+1,\mathbb{R}) : g^t Jg = J\right\}$$

consists of four connected components while the group

$$SO(1,n) = \left\{g\in GL(n+1,\mathbb{R}) : g^t Jg = J,\ \det g = 1\right\}$$

consists of two. By the condition $g_{00}>0$ in (1.2) there is selected the connected component $SO^+(1,n)$ containing the identity matrix.

With a view to an explanation of the formulae in (1.3), (1.4) note that the set $H^n$ defined by (1.1) is only one of possible models of the $n$-dimensional hyberbolic space. Another its model is the set

$$L^n = \{\xi = (\xi_0, \xi_1, \dots, \xi_n) \in \mathbb{R}^{n+1} : (\xi, \xi)_1 = -1,\ \xi_0 \geq 1\},$$

where $(\xi, \eta)_1 = -\xi_0\eta_0 + \xi_1\eta_1 + \dots + \xi_n\eta_n = (J\xi, \eta)$. Thus, $L^n$ is a quadratic surface (hemisphere of imaginary radius $i$) in the $(n+1)$-dimensional Minkowski space $\mathbb{R}_1^{n+1}$ with the Riemannian metric

$$ds^2 = -d\xi_0^2 + d\xi_1^2 + \dots + d\xi_n^2$$

(this metric is positive definite on $L^n$). In other words, $L^n$ presents one of the two sheets of the hyperboloid $\xi_0^2 = 1 + \xi_1^2 + \dots + \xi_n^2$ in $\mathbb{R}^{n+1}$, namely, the one for which $\xi_0 \geq 1$. Each element $g \in SO^+(1, n)$ acts on $L^n$ in a natural way: If $\xi \in L^n$ then the image $g\xi$ of the point $\xi$ under the transformation $g$ is computed as the product of the matrix $g$ with the column vector $\xi$ (i.e., $g$ acts as a linear transformation).

The models $L^n$ and $H^n$ are isometric to each other. The isometry is realized by the mapping

$$\chi : L^n \to H^n, \quad \xi = (\xi_0, \xi_1, \dots, \xi_n) \longmapsto z = (x_1, \dots, x_{n-1}, y),$$

where

$$x_j = \frac{\xi_j}{\xi_0 - \xi_n}\ (1 \leq j \leq n-1), \quad y = \frac{1}{\xi_0 - \xi_n}.$$

The inverse mapping $\chi^{-1}$ is given by the formulae

$$\xi_0 = \frac{|z|^2 + 1}{2y}, \quad \xi_j = \frac{x_j}{y}\ (1 \leq j \leq n-1), \quad \xi_n = \frac{|z|^2 - 1}{2y}.$$

Now it is natural to define the action of the group $SO^+(1, n)$ on $H^n$ by the formula: If $g = [g_{jk}]_{j,k=0}^n \in SO^+(1, n)$ and $z \in H^n$, then

$$z' = gz = (\chi g \chi^{-1})(z).$$

Hence formulae (1.3), (1.4) follow.

Let us bring the well-known specializations of the cases $n = 1$, $n = 2$, and $n = 3$.

(i) For $n = 1$, the (one-dimensional) hyperbolic space $H^1$ consists of the positive semi-axis

$$H^1 = \{y : 0 < y < \infty).$$

The group of motions $SO^+(1, 1)$ consists of the matrices

$$g = \begin{bmatrix} \cosh\psi & \sinh\psi \\ \sinh\psi & \cosh\psi \end{bmatrix}, \quad -\infty < \psi < \infty.$$

The action formula takes the form

$$g : y \longmapsto y', \quad y' = e^{\psi} y.$$

The group $SO^+(1, 1)$ is isomorphic to the (multiplicative) group of positive real numbers $\mathbb{R}_+$. The isomorphism is given by the mapping

$$\Phi : SO^+(1, 1) \to \mathbb{R}_+, \quad g = \begin{bmatrix} \cosh\psi & \sinh\psi \\ \sinh\psi & \cosh\psi \end{bmatrix} \longmapsto \alpha = e^{\psi}.$$

An element (number) $\alpha \in \mathbb{R}_+$ acts on $H^1$ by the formula: $\alpha : y \longmapsto y'$, $y' = \alpha y$. The invariant Riemannian metric, volume element, Laplace-Beltrami operator, invariant of a pair of points, and hyperbolic distance on $H^1$ are

$$ds^2 = \frac{dy^2}{y^2}, \quad dv(y) = \frac{dy}{y}, \quad L = y\frac{d}{dy}\left(y\frac{d}{dy}\right) = y^2\frac{d^2}{dy^2} + y\frac{d}{dy},$$

$$u(y,y')=\frac{|y-y'|^2}{yy'},\quad \rho(y,y')=\ln\frac{y+y'+|y-y'|}{y+y'-|y-y'|}=\left|\ln y-\ln y'\right|,$$

respectively.

(ii) For $n=2$, the hyperbolic plane $H^2$ is realized as the upper half-plane

$$H^2=\{z=x+iy\in\mathbb{C}:-\infty<x<\infty,\ 0<y<\infty\}$$

in the complex plane $\mathbb{C}$. The group $SO^+(1,2)$ is isomorphic to the factor group $PSL(2,\mathbb{R})=SL(2,\mathbb{R})/\{\pm I_2\}$, where $SL(2,\mathbb{R})$ is the group of real $2\times 2$ matrices with determinat 1. One does not usually distinguish between an element $g\in PSL(2,\mathbb{R})$ and its preimages $\pm g$ in the group $SL(2,\mathbb{R})$, since this does not lead to any misunderstanding. The element

$$g=\begin{bmatrix} a & b\\ c & d\end{bmatrix}\in SL(2,\mathbb{R})$$

acts on $H^2$ by a linear-fractional transformation

$$g:z\longmapsto z',\quad z'=gz=\frac{az+b}{cz+d}.$$

The matrices $g$ and $-g$ are identified, since they define the same transformation of $H^2$. The main invariant objects associated with the space $H^2$ take the form

$$ds^2=\frac{dx^2+dy^2}{y^2},\quad dv(z)=\frac{dxdy}{y^2},\quad L=y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right),$$

$$u(z,z')=\frac{|z-z'|^2}{yy'},\quad \rho(z,z')=\ln\frac{\left|z-\overline{z'}\right|+|z-z'|}{\left|z-\overline{z'}\right|-|z-z'|},$$

where the bar over a complex number denotes the complex conjugate.

(iii) For $n=3$, three-dimensional hyperbolic space $H^3$ is realized as the upper half-space in three-dimensional Euclidean space,

$$H^3=\{z=(w,y):w=x_1+ix_2\in\mathbb{C},\ 0<y<\infty\}.$$

The group $SO^+(1,3)$ is isomorphic to the factor group $PSL(2,\mathbb{C})=SL(2,\mathbb{C})/\{\pm I_2\}$, where $SL(2,\mathbb{C})$ is the group of complex $2\times 2$ matrices with determinat one. The group $PSL(2,\mathbb{C})$ acts on $H^3$ as follows: If

$$g=\begin{bmatrix} a & b\\ c & d\end{bmatrix}\in SL(2,\mathbb{C})$$

and $z=(w,y)$, then the point $gz=z'=(w',y')$ has coordianates

$$w'=\frac{(aw+b)(\overline{cw+d})+a\bar{c}y^2}{|cw+d|^2+|c|^2y^2},\quad y'=\frac{y}{|cw+d|^2+|c|^2y^2}.$$

The main invariant objects associated with the space $H^3$ take the form

$$ds^2=\frac{dx_1^2+dx_2^2+dy^2}{y^2},\quad dv(z)=\frac{dx_1dx_2dy}{y^3},\quad u(z,z')=\frac{|z-z'|^2}{yy'},$$

$$L=y^2\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial y^2}\right)-y\frac{\partial}{\partial y},\quad \rho(z,z')=\ln\frac{\left|z-\overline{z'}\right|+|z-z'|}{\left|z-\overline{z'}\right|-|z-z'|},$$

where $|z-z'|^2=(x_1-x_1')^2+(x_2-x_2')^2+(y-y')^2$ and $\overline{z}=(w,-y)$ for $z=(w,y)$, $z'=(w',y')$.

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