A note on the endomorphism ring of finitely presented modules of the projective dimension $\leq 1$

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Abstract

In this paper, we study the behavior of endomorphism rings of a cyclic, finitely presented module of projective dimension $\leq 1$. This class of modules extends to arbitrary rings the class of couniformly presented modules over local rings.

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1. Introduction

Throughout this paper, all rings will be associative with identity and modules will be unital right modules. For any ring $R$, the Jacobson radical of $R$ will be denoted by $J(R)$.

Recall that $M_R$ is couniform if it has dual Goldie dimension one (if and only if it is non-zero and the sum of any two proper submodules of $M_R$ is a proper submodule of $M_R$). It is well known that a projective right module $P_R$ is couniform if and only if $\text{End}(P_R)$ is a local ring, if and only if there exists an idempotent $e \in R$ with $P_R \cong eR$ and $eRe$ a local ring, if and only if is a finitely generated module with a unique maximal submodule.

In [7], Facchini and Girardi introduced and studied the notion of couniformly presented modules. A module $M_R$ is called couniformly presented if it is non-zero and there exists an exact sequence

$$0 \rightarrow C_R \xrightarrow{i} P_R \rightarrow M_R \rightarrow 0$$

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with $P_R$ projective and both $C_R$ and $P_R$ couniform modules. In this case, every endomorphism $f$ of $M_R$ lifts to an endomorphism $f_0$ of its projective cover $P_R$, and we will denote by $f_1$ the restriction to $C_R$ of $f_0$. Hence we have a commutative diagram

$$
\begin{array}{cccc}
0 & \rightarrow & C_R & \xrightarrow{\iota} & P_R & \rightarrow & M_R & \rightarrow & 0 \\
\downarrow f_1 & & \downarrow f_0 & & \downarrow f & \\
0 & \rightarrow & C_R & \xrightarrow{\iota} & P_R & \rightarrow & M_R & \rightarrow & 0.
\end{array}
$$

In [7, Theorem 2.5], Facchini and Girardi proved that:

- Let $0 \rightarrow C_R \rightarrow P_R \rightarrow M_R \rightarrow 0$ be a couniform presentation of a couniformly presented module $M_R$. Set $K := \{ f \in \text{End}(M_R) \mid f \text{ is not surjective} \}$ and $I := \{ f \in \text{End}(M_R) \mid f_1: C_R \rightarrow C_R \text{ is not surjective} \}$. Then $K$ and $I$ are completely prime two-sided ideals of $\text{End}(M_R)$, and the union $K \cup I$ is the set of all non-invertible elements of $\text{End}(M_R)$. Moreover, one of the following two conditions holds:

(a) Either $\text{End}(M_R)$ is a local ring, or

(b) $K$ and $I$ are the two maximal right, maximal left ideals of $\text{End}(M_R)$.

If $M_R$ and $M_R'$ are two couniformly presented modules with couniform presentations $0 \rightarrow C_R \rightarrow P_R \rightarrow M_R \rightarrow 0$ and $0 \rightarrow C_R' \rightarrow P_R' \rightarrow M_R' \rightarrow 0$, we say that $M_R$ and $M_R'$ have the same lower part, and we write $[M_R]_l = [M_R']_l$, if there are two homomorphisms $f_0: P_R \rightarrow P_R'$ and $f_0': P_R' \rightarrow P_R$ such that $f_0(C_R) = C_R'$ and $f_0'(C_R') = C_R$.

Recall that a ring $R$ is semilocal if $R/J(R)$ is semisimple artinian, that is, isomorphic to a finite direct product of rings $M_n(D_i)$ of $n \times n$ matrices over division rings $D_i$. A ring $R$ is homogeneous semilocal if $R/J(R)$ is simple artinian, that is, isomorphic to the ring $M_n(D)$ of all $n \times n$ matrices for some positive integer $n$ and some division ring $D$ [2, 4]. Examples of such rings include all local rings and all simple Artinian rings. If $R$ is a homogeneous semilocal ring, then so are the rings $eRe$ and $M_n(R)$, where $e$ is a nonzero idempotent element of $R$ and $M_n(R)$ is the matrix ring over $R$. Also, homogeneous semilocal rings appear in a natural way when one localizes a right Noetherian ring with respect to a right localizable prime ideal.

In [4], Corisello and Facchini showed that:

- a homogeneous semilocal ring has a unique maximal proper two-sided ideal and a unique simple module up to isomorphism. Similarly, as in the case of local rings, a homogeneous semilocal ring has only one indecomposable projective module $P_R$ up to isomorphism, and all projective modules are direct sums of copies of this $P_R$.

- for a module $M$ over any ring $R$, the Krull-Schmidt theorem holds for $M$ provided $\text{End}_R(M)$ is homogeneous semilocal—that is, the direct sum decomposition of $M$ into indecomposable summands is unique up to isomorphism.

In [2], Barioli-Facchini-Raggi proved that:

- The later result fails to extend to modules $M_R$ with finite direct sum decompositions whose indecomposable summands have homogeneous semilocal endomorphism rings.

- If a module $M$ over a ring $R$ has two decompositions $M = M_1 \oplus \cdots \oplus M_t = N_1 \oplus \cdots \oplus N_s$ where all the summands are indecomposable with homogeneous semilocal endomorphism rings, then these two decompositions are isomorphic.
2. The endomorphism ring

The following results describe the endomorphism ring of a cyclic, finitely presented module of projective dimension \( \leq 1 \) over a local ring. Throughout this paper, we will assume that \( M_R \neq 0 \).

2.1. Theorem. Let \( R \) be a local ring and let \( M_R := R_R/I \) be a cyclic, finitely presented module of projective dimension \( \leq 1 \). Suppose \( \text{Ext}^1_R(M_R, R_R) = 0 \).

Assume \( 0 \neq I \neq R \) and let \( E \) be the idealizer of the right ideal \( I \) of \( R \), that is, the set of all \( r \in R \) with \( rI \subseteq I \), so that \( \text{End}(M_R) \cong E/I \). Set \( L := \{ r \in R \mid rI \subseteq IJ(R) \} \) and \( K := E \cap J(R) \). Let \( \psi: E \to \text{End}_R(I/IJ(R)) \) be the ring morphism defined by

\[ \psi(e)(x + IJ(R)) = ex + IJ(R), \]

for every \( e \in E \) and \( x \in I \). Let \( n \) be the dimension of the right vector space \( I/IJ(R) \) over the division ring \( R/J(R) \). Then:

1. \( L \) and \( K \) are prime two-sided ideals of \( E \) containing \( I \) and \( K \) is a completely prime ideal of \( E \);
2. For every \( e \in E \), the element \( e + I \) of \( E/I \) is invertible in \( E/I \) if and only if \( e + J(R) \) is invertible in \( R/J(R) \) and \( \psi(e) \) is invertible in \( \text{End}_R(I/IJ(R)) \);
3. The quotient ring \( E/L \) is isomorphic to the ring \( M_n(R/J(R)) \) of all \( n \times n \) matrices over the division ring \( R/J(R) \);
4. Exactly one of the following two conditions holds:
   a. Either \( K \subseteq L \), in which case \( E/I \) is a homogeneous semilocal ring with Jacobson radical \( L/I \), or
   b. \( L \) and \( K \) are not comparable.

Proof. (1) and (3). Notice that \( L \) is contained in \( E \) and is the kernel of \( \psi \), so that \( L \) is a two-sided ideal of \( E \). Trivially, \( I \) is contained in \( L \). Let us prove that \( \psi \) is onto. Let \( f: I/IJ(R) \to I/IJ(R) \) be a morphism. Since \( M_R := R_R/I \) is of projective dimension \( \leq 1 \), the ideal \( I_R \) is projective, so that \( f \) lifts to a morphism \( f': I_R \to I_R \). Apply the functor \( \text{Hom}(\_ , R_R) \) to the exact sequence \( 0 \to I_R \to R_R \to M_R \to 0 \), getting a short exact sequence

\[ 0 \to \text{Hom}(M_R, R_R) \to \text{Hom}(R_R, R_R) \to \text{Hom}(I_R, R_R) \to 0 \]

because \( \text{Ext}^1_R(M_R, R_R) = 0 \). Hence \( f' \) can be extended to a morphism \( f'': R_R \to R_R \), which is necessarily left multiplication by an element \( r \in R \). Since \( f'' \) restricts to the endomorphism \( f' \) of \( I_R \), we get that \( r \in E \), and \( \psi(e) = f \). This proves that \( \psi \) is an onto ring morphism, so that

\[ E/L \cong E/\ker \psi \cong \text{End}_R(I/IJ(R)) \cong M_n(R/J(R)). \]

This proves (3).

As \( \text{End}_R(I/IJ(R)) \cong M_n(R/J(R)) \) is a simple ring, it follows that \( L \) is a prime ideal and a maximal two-sided ideal. Similarly, \( K \) is the kernel of the composite morphism \( \varphi: E \to R/J(R) \) of the embedding \( E \to R \) and the canonical projection \( R \to R/J(R) \). Since \( R/J(R) \) is a division ring, we get that \( K \) is a completely prime, two-sided ideal of \( E \) containing \( I \). This concludes the proof of (1).
(2). \(\Rightarrow\) Since \(\varphi(I) = 0\) and \(\psi(I) = 0\), the morphisms \(\varphi\) and \(\psi\) induce morphisms 
\(\widetilde{\varphi}: E/I \to R/J(R)\) and \(\widetilde{\psi}: E/I \to \text{End}(I/IJ(R))\), respectively. Hence \(e + I\) invertible implies \(\varphi(e) = e + J(R)\) invertible in \(R/J(R)\) and \(\psi(e)\) is invertible in \(\text{End}_R(I/IJ(R))\).

\(\Leftarrow\) Assume that \(e \in E\) and that \(\varphi(e)\) and \(\psi(e)\) are invertible in \(R/J(R)\) and \(\text{End}_R(I/IJ(R))\), respectively. Then we have a commutative diagram with exact rows

\[
\begin{array}{ccccccc}
0 & \rightarrow & I & \rightarrow & R_R & \rightarrow & R_R/I & \rightarrow & 0 \\
\downarrow{\epsilon} && \downarrow{\epsilon} && \downarrow{\epsilon} && \downarrow{\epsilon} \\
0 & \rightarrow & I & \rightarrow & R_R & \rightarrow & R_R/I & \rightarrow & 0.
\end{array}
\]

Now \(\varphi(e) = e + J(R)\) invertible implies that \(e \in E \setminus J(R)\), and so \(e\) is invertible in \(R\). Hence the middle vertical arrow is an isomorphism. Since \(\psi(e)\) is invertible, it is an automorphism of \(I/IJ(R)\), and so \(e(I/IJ(R)) = I/IJ(R)\), that is, \(eI + IJ(R) = I\). By Nakayama’s Lemma, \(eI = I\). Hence the left vertical arrow is an epimorphism. By the Snake Lemma, the right vertical arrow is a monomorphism, hence an isomorphism. That is, \(e + I\) is invertible in \(E/I\).

(4) We have the three cases (a) \(L \subseteq K\), (b) \(K \subseteq L\), and (c) \(L \nsubsetneq K\) and \(K \nsubseteq L\).

Assume \(L \subseteq K\). In this case, \(L \subseteq K \subseteq E\) implies that \(0 \subseteq K/L \subseteq E/L\), so that \(E/L \cong M_n(R/J(R))\) has a proper non-zero two-sided ideal. This is impossible, because \(M_n(R/J)\) is a simple ring. Hence this case cannot occur.

Assume \(K \subseteq L\). From (2), it follows that an element \(e + I\) of \(E/I\) is invertible in \(E/I\) if and only if \(e + J(R)\) is invertible in \(R/J(R)\) and \(e + L\) is invertible in \(E/L\). Hence, in order to prove (4) in this case \(K \subseteq L\), it suffices to prove that \(J(E/I) = L/I\).

\(\subseteq\) If \(e + I \in J(E/I)\), then \(1 - xey + I\) is invertible in \(E/I\) for every \(x, y \in E\). Thus \(1 - xey + L\) is invertible in \(E/L\) for all \(x, y \in E\), so that \(e + L \in J(E/L)\). But \(E/L \cong M_n(R/J(R))\) has Jacobson radical zero so that \(e \in L\).

\(\supseteq\) Take \(l + I \in L/I\) with \(l \in L\). Then \(1 - xly + L = 1 + L\) in \(E/L\) for every \(x, y \in E\). Hence \(1 - xly + L\) is invertible in \(E/L\). In particular, \(1 - xly \notin L\). Thus \(1 - xly \notin K\), so that \(1 - xly \notin J(R)\). As \(R/J(R)\) is a division ring, it follows that \(1 - xly + J(R)\) is invertible in \(R/J(R)\). Thus \(1 - xly + I\) is invertible in \(E/I\), and \(l \in J(E/I)\).

It is known that a finitely presented module over a semilocal ring always has a semilocal endomorphism ring. We have the following natural question.

2.2. Question. Characterize \(J(E/I)\). This was done in [1] for cyclically presented modules.

As far as Question 2.2 is concerned, notice that, in the proof of Theorem 2.1(2), we have seen that the mapping

\[
\tilde{\varphi} \times \tilde{\psi}: E/J \to R/J(R) \times \text{End}(I/IJ(R))
\]

is a local morphism, so that its kernel \(K/I \cap L/I\) is contained in \(J(E/I)\). In particular, when \(K \subseteq L\), we have that \(L/I = J(E/I)\) as we have seen in Theorem 2.1(4)(a). We are not able to describe \(J(E/I)\) when \(K\) and \(K\) are not comparable.

2.3. Remark. Let \(R\) be a local right self-injective ring. Let \(M_R\) be a cyclic and finitely presented module of projective dimension \(\leq 1\). Since \(R_R\) is injective, we have that \(\text{Ext}_R^1(M_R, R_R) = 0\). Thus, Theorem 2.1 can be applied.

Let \(A\) and \(B\) be two modules. We say that:

- \(A\) and \(B\) have the same monogeny class, and write \([A]_m = [B]_m\), if there exist a monomorphism \(A \to B\) and a monomorphism \(B \to A\) [5];
• $A$ and $B$ have the same epigeny class, and write $[A]_{ek} = [B]_{ek}$, if there exist an
epimorphism $A \to B$ and an epimorphism $B \to A$;

It is clear that a module $A$ has the same monogeny (epigeny) class as the zero module
if and only if $A = 0$.

• Two cyclically presented modules $R/aR$ and $R/bR$ over a local ring $R$ are said to
have the same lower part, denoted $[R/aR]_{l} = [R/bR]_{l}$, if there exist $r, s \in R$ such that
$raR = bR$ and $sbR = aR$ [1].

• If $M_{R}$ and $M'_{R}$ are two couniformly presented modules with couniform presenta-
tions

$$0 \to C_{R} \to P_{R} \to M_{R} \to 0$$

and

$$0 \to C'_{R} \to P'_{R} \to M'_{R} \to 0,$$

we say that $M_{R}$ and $M'_{R}$ have the same lower part, and we write $[M_{R}]_{l} = [M'_{R}]_{l}$, if there are two homomorphisms $f_{0} : P_{R} \to P'_{R}$ and $f'_{0} : P'_{R} \to P_{R}$ such that $f_{0}(C_{R}) = C'_{R}$ and$f'_{0}(C'_{R}) = C_{R}$ [7].

2.4. Theorem. Let $R$ be a semiperfect ring and let $R_{R}/L$ be a cyclic uniform right
$R$-module with $L \neq 0$. Let $E$ be the idealizer of the right ideal $L$ of $R$, that is, the set of
all $r \in R$ with $rL \subseteq L$, so that

$$\text{End}(R_{R}/L) \cong E/L.$$  

Similarly, let $E'$ be the idealizer of the right ideal $L + J(R)$ of $R$, so that

$$\text{End}(R_{R}/(L + J(R))) \cong E'/((L + J(R)).$$

Set $I := \{e \in E \mid \text{left multiplication by } e + I \text{ is a non-injective endomorphism of } R_{R}/L\}$
and $K := E \cap L + J(R)$. Then:

1. $I$ and $K$ are two two-sided ideals of $E$ containing $L$, and $I$ is completely prime
in $E$.
2. For every $e \in E$, the element $e + L$ of $E/L$ is invertible in $E/L$ if and only if
$e + L + J(R)$ is invertible in $E'/L + J(R)$ and $e \notin I$.
3. Moreover:
   (a) If $I \subset K$, then every epimorphism $R_{R}/L \to R_{R}/L$ is an automorphism of
$R_{R}/L$.
   (b) $K \subset I$ if and only if $[R_{R}/L]_{m} = [L + J(R)/L]_{m}$.

Proof. (1) We know that $\text{End}(R_{R}/L) \cong E/L$. Every endomorphism $e + L$ of $R_{R}/L$ ex-
tends to an endomorphism $e_{1}$ of the injective envelope $E(R_{R}/L)$. Define a ring morphism

$$\varphi : E \to \text{End}(E(R_{R}/L))/J(\text{End}(E(R_{R}/L)))$$

by $\varphi(e) = e_{1} + J(\text{End}(E(R_{R}/L)))$ for every $e \in E$. Since $R_{R}/L$ is uniform, the injective
envelope $E(R_{R}/L)$ is indecomposable, the endomorphism ring $\text{End}(E(R_{R}/L))$ is a local
ring, and the Jacobson radical $J(\text{End}(E(R_{R}/L)))$ consists of all non-injective endomor-
phisms of $E(R_{R}/L)$. It follows that $I$, which is equal to the kernel of the ring morphism
$\varphi$, whose range is the division ring

$$\text{End}(E(R_{R}/L))/J(\text{End}(E(R_{R}/L))),$$

must be a completely prime two-sided ideal of $E$. The remaining part of statement (1)
is easily checked.

(2) We have already seen that there is a ring morphism

$$\varphi : E \to \text{End}(E(R_{R}/L))/J(\text{End}(E(R_{R}/L)))$$
whose kernel is $I$. Hence if $e \in E$ and $e + L$ is invertible in $E/L$, then $\varphi(e)$ must be invertible in the division ring $\text{End}(E(R_R/L))/J(\text{End}(E(R_R/L)))$. Thus $\varphi(e) \neq 0$, that is, $e \notin \ker \varphi = I$. Similarly, we can consider the ring morphism

$$
\psi: E \to \text{End}(R_R/L + J(R))
$$

defined by $\psi(e)(r + L + J(R)) = er + L + J(R)$ for every $e \in E$ and every $r \in R$. Its kernel is $K$, which contains $L$. Hence $e + L$ invertible in $E/L$ implies $\psi(e)$ invertible in $\text{End}(R_R/L + J(R))$. But

$$
\text{End}(R_R/(L + J(R))) \cong E'/(L + J(R)),
$$

so that $e + L + J(R)$ must be invertible in $E'/L + J(R)$.

Conversely, assume $e \in E$, $e + L + J(R)$ invertible in $E'/L + J(R)$ and $e \notin I$. We want to show that $e + L$ is invertible in $E/L$. Since $E/L \cong \text{End}(R_R/L)$, this is equivalent to showing that left multiplication $\mu_e: R_R/L \to R_R/L$ by $e$ is an automorphism of $R_R/L$. Now $e \notin I$ is equivalent to $\mu_e$ is injective by definition of $I$. In order to show that $\mu_e$ is onto as well, it suffices to prove that $\mu_e$ induces an onto endomorphism

$$(R_R/L)/(R_R/L)J(R) \to (R_R/L)/(R_R/L)J(R)$$

by Nakayama’s Lemma. But $(R_R/L)J(R) = L + J(R)/L$, so that

$$(R_R/L)/(R_R/L)J(R) \cong R_R/L + J(R).$$

Hence $e + L + J(R)$ invertible in $E'/L + J(R)$ means that $\text{End}(E(R_R/L)J(R))$ means that the endomorphism $\psi(e)$ of $R_R/L + J(R)$ induced by $\mu_e$ is onto, as desired.

(3) (a) Assume $I \subseteq K$. Let $e + L: R_R/L \to R_R/L$ be an epimorphism with $e \in E$. Then the induced morphism $\psi(e): R_R/L + J(R) \to R_R/L + J(R)$ is also an epimorphism, so that it is an automorphism because $R_R/L + J(R)$ is a semisimple module of finite Goldie dimension. In the isomorphism

$$
\text{End}(R_R/(L + J(R))) \cong E'/(L + J(R)),
$$

we obtain that $e + L + J(R)$ is invertible in the ring $E'/L + J(R))$. Thus $e \notin K$. Hence $e \notin I$. It follows from (2) that $e + L$ is invertible, that is, it is an automorphism of $R_R/L$.

(b) Assume $K \not\subseteq I$. Then there is an element $f \in K$, $f \notin I$. Thus $f \in E$ induces an endomorphism $f$ of $R_R/L$. Now $f \notin I$ means that $f$ is injective, and $f \in K$ means that the image of $f$ is contained in $L + J(R)/L$. Hence $[R_R/L]_m = [L + J(R)/L]_m$. Conversely, if $[R_R/L]_m = [L + J(R)/L]_m$, then there is a monomorphism $f: R_R/L \to L + J(R)/L$. If we compose it with the inclusion $L + J(R)/L \to R_R/L$ we get an endomorphism of $R_R/L$ which is in $K$ but not in $I$. Hence $K \not\subseteq I$. \hfill \Box

We finish this study with the following result.

2.5. Theorem. Let $R$ be a semiperfect ring, let $R/L, R/L'$ be two cyclic uniform modules with $L \neq 0$ and $L' \neq 0$ proper right ideals of $R$. Assume that either

1. every monomorphism $R_R/L \to R_R/L$ is an automorphism of $R_R/L$, or
2. every epimorphism $R_R/L \to R_R/L$ is an automorphism of $R_R/L$, or
3. $[R_R/L]_m = [L + J(R)/L]_m$.

Then the followings are equivalent.

(a) $R_R/L \cong R_R/L'$
(b) $[R_R/L]_m = [R_R/L']_m$ and $[R_R/L]_c = [R_R/L']_c$.

Proof. Assume $[R_R/L]_m = [R_R/L']_m$ and $[R_R/L]_c = [R_R/L']_c$. Then there are monomorphisms $\alpha: R_R/L \to R_R/L'$ and $\beta: R_R/L' \to R_R/L$ and epimorphisms $\alpha: R_R/L \to R_R/L'$ and $\beta: R_R/L' \to R_R/L$. Then $\beta\alpha$ is a monomorphism $R_R/L \to R_R/L$ and $\beta\alpha'$ is an epimorphism $R_R/L \to R_R/L$. If hypothesis (a) holds, then $\beta\alpha$ is an automorphism
of $R_R/L$ that factors through $R_R/L'$, so that $R_R/L$ is isomorphic to a direct summand of $R_R/L'$. But $R_R/L \neq 0$ and $R_R/L'$ is uniform, so that $R_R/L \cong R_R/L'$. This proves our theorem under hypothesis (a). Dually one proves that the theorem holds when hypothesis (b) holds.

Assume now that hypothesis (c) holds, i.e., $[R_R/L]_m = [L + J(R)/L]_m$. Equivalently, there exists a monomorphism $\gamma: R_R/L \to R_R/L$ whose image is contained in $L + J(R)/L$. Now if either $\alpha$ or $\alpha'$ are isomorphisms, then the existence of $\alpha$ or $\alpha'$ shows that $R_R/L \cong R_R/L'$. This allows us to conclude. Thus we can assume that $\alpha$ is not an epimorphism and $\alpha'$ is not a monomorphism. Then $\alpha' + \alpha\gamma: R_R/L \to R_R/L'$ is an isomorphism, because:

1. It is injective, because it is the sum of the injective morphism $\alpha\gamma: R_R/L \to R_R/L'$ and the non-injective morphism $\alpha': R_R/L \to R_R/L'$, and $R_R/L$ is uniform.

2. The ideal $J(R)$ is superfluous in $R_R$ by Nakayama’s Lemma. Considering the canonical projection $R_R \to R_R/L$, it follows that $L + J(R)/L$ is superfluous in $R_R/L$. Applying the morphism $\alpha: R/L \to R/L'$, we get that the image of $\alpha\gamma$ is contained in $\alpha(L + J(R)/L)$, hence is a superfluous submodule of $R/L'$. Thus the sum of $\alpha\gamma$ and the surjective morphism $\alpha': R/L \to R/L'$ is a surjective morphism $\alpha' + \alpha\gamma: R_R/L \to R_R/L'$.

Thus $\alpha + \alpha\gamma$ is an isomorphism of $R_R/L$ onto $R_R/L'$.

2.6. Remark. By Theorem 2.4, the only case in which we cannot apply Theorem 2.5 is when $I$ is properly contained in $I$. Namely, if $K \not\subseteq I$, then $[R_R/L]_m = [L + J(R)/L]_m$ and we can apply Theorem 2.5(a); if $K \subseteq I$, then either $K$ is properly contained in $I$, which is the case still unknown, or $K = I$, but in the latter case every epimorphism $R_R/L \to R_R/L$ is an automorphism of $R_R/L$ by Theorem 2.4(1).

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