A NOTE ON CERTAIN CENTRAL DIFFERENTIAL IDENTITIES WITH GENERALIZED DERIVATIONS

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Abstract

Let $R$ be a noncommutative prime ring of characteristic different from 2 with right Utumi quotient ring $U$ and extended centroid $C$, $I$ a nonzero right ideal of $R$. Let $f(x_1,\ldots,x_n)$ be a non-central multilinear polynomial over $C$, $m \geq 1$ a fixed integer, $a$ a fixed element of $R$, $G$ a non-zero generalized derivation of $R$. If $aG(f(r_1,\ldots,r_n))^m \in Z(R)$ for all $r_1,\ldots,r_n \in I$, then one of the following holds:

1. $aI = aG(I) = (0)$;
2. $G(x) = qx$, for some $q \in U$ and $aqI = 0$;
3. $[f(x_1,\ldots,x_n),x_1]x_{n+2}$ is an identity for $I$;
4. $G(x) = cx + [q,x]$ for all $x \in R$, where $c,q \in U$ such that $cI = 0$ and $[g,I]I = 0$;
5. dim$_C(RC) \leq 4$;
6. $G(x) = \alpha x$, for some $\alpha \in C$; moreover $a \in C$ and $f(x_1,\ldots,x_n)^m$ is central valued on $R$.

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1. Introduction and Preliminaries

Throughout this paper unless specially stated, \( R \) always denotes a prime ring with center \( Z(R) \), \( U \) its right Utumi quotient ring and \( C \) its extended centroid (which is the center of \( U \)). The definitions, the axiomatic formulations and the properties of this quotient ring \( U \) can be found in [1]. In any case, when \( R \) is a prime ring, all that we need about \( U \) is that:

1. \( R \subset U \);
2. \( U \) is a prime ring with identity;
3. The center of \( U \), denoted by \( C \), is a field which is called the extended centroid of \( R \).

By a derivation of \( R \) we mean that an additive map \( d \) from \( R \) into itself satisfies the rule \( d(xy) = d(x)y + xd(y) \) for all \( x, y \in R \). For any \( x, y \in R \), the symbol \([x, y]\) stands for the commutator \( xy - yx \). For \( b \in U \), we use \( \text{ad}(b) \) to denote the inner derivation induced by \( b \); that is, \( \text{ad}(b)(x) = [b, x] \) for \( x \in R \). An additive mapping \( g : R \rightarrow R \) is called a generalized derivation of \( R \) if there exists a derivation \( d \) of \( R \) such that \( g(xy) = g(x)y + xd(y) \) for all \( x, y \in R \) [8]. Obviously any derivation is a generalized derivation. Moreover, other basic examples of generalized derivations are the following: (i) \( g(x) = ax + xb \), for \( a, b \in R \); (ii) \( g(x) = ax \), for some \( a \in R \). Many authors have studied generalized derivations in the context of prime and semiprime rings (see [8, 11, 16]).

In [2] M. Bresar proved that if \( R \) is a semiprime ring, \( d \) a nonzero derivation of \( R \) and \( a \in R \) such that \( \text{ad}(x)^m = 0 \), for all \( x \in R \), where \( m \) is a fixed integer, then \( \text{ad}(R) = 0 \) when \( R \) is \((m-1)!\)-torsion free. In [15] T.K. Lee and J.S. Lin proved Bresar’s result without the \((m-1)!\)-torsion free assumption on \( R \). They studied the Lie ideal case and, for the prime case, they showed that if \( R \) is a prime ring with a derivation \( d \neq 0 \), \( L \) is an ideal of \( R \), \( a \in R \) such that \( \text{ad}(u)^m = 0 \), for all \( u \in L \), where \( m \) is fixed, then \( \text{ad}(L) = 0 \) unless the case when \( \text{char}(R) = 2 \) and \( \dim_C RC = 4 \). In addition, if \([L, L] \neq 0 \), then \( \text{ad}(R) = 0 \).

In [4] C.M. Chang and T.K. Lee established a unified version of the previous results for prime rings. More precisely they proved the following theorem: let \( R \) be a prime ring, \( \varphi \) a nonzero right ideal of \( R \), \( d \) a nonzero derivation of \( R \), \( a \in R \) such that \( \text{ad}(\varphi)^m = 0 \), \( \varphi \in Z(R) \) \((d(\varphi))^{m-1}a \in Z(R)\). If \([\varphi, \varphi]_\varphi \neq 0 \) and \( \dim_C RC > 4 \), then either \( \text{ad}(\varphi) = 0 \) \((a = 0 \) resp.) or \( d \) is the inner derivation induced by some \( q \in U \) such that \( qq = 0 \).

Recently in the first part of [3], C.M. Chang generalized above results by proving that if \( R \) is a prime ring with extended centroid \( C \), \( I \) a non-zero right ideal of \( R \), \( d \) a non-zero derivation of \( R \), \( f(x_1, \ldots, x_n) \) a multilinear polynomial over \( C \), \( a \in R \) and \( m \geq 1 \) a fixed integer such that \( \text{ad}(f(r_1, \ldots, r_n))^m = 0 \) for all \( r_1, \ldots, r_n \in I \), then either \( aI = d(I)I = 0 \) or \([f(x_1, \ldots, x_n), x_{n+1}]x_{n+2} \) is an identity for \( I \).

In [7] the second author obtained some results under the assumption that \( I \) is a nonzero right ideal of a noncommutative prime ring \( R \), \( G \) is a generalized derivation of \( R \), \( m \) is a fixed positive integer, \( f(x_1, \ldots, x_n) \) is a non-central multilinear polynomial over \( C \) such that \( aG(f(r_1, \ldots, r_n))^m = 0 \) for all \( r_1, \ldots, r_n \in I \). In this case one of the following holds:

1. \( aI = aG(I) = 0 \);
2. \( G(x) = qx \), for some \( q \in U \) and \( aqI = 0 \);
3. \([f(x_1, \ldots, x_n), x_{n+1}]x_{n+2} \) is an identity for \( I \);
4. \( G(x) = cx + [q, x] \) for all \( x \in R \), where \( c, q \in U \) such that \( cI = 0 \) and \([q, I]I = 0 \).

Motivated by the above results we will prove:

**1.1. Theorem.** Let \( R \) be a noncommutative prime ring of characteristic different from 2 with right Utumi quotient ring \( U \) and extended centroid \( C \), \( I \) a nonzero right ideal of \( R \).
Let \( f(x_1, \ldots, x_n) \) be a non-central multilinear polynomial over \( C \), \( m \geq 1 \) a fixed integer, \( a \) a fixed element of \( R \), \( G \) a non-zero generalized derivation of \( R \). If \( aG(f(r_1, \ldots, r_n))^m \in Z(R) \) for all \( r_1, \ldots, r_n \in I \), then one of the following holds:

1. \( aI = aG(I) = \{0\} \);
2. \( G(x) = qx \) for some \( q \in U \) and \( aqI = 0 \);
3. \([f(x_1, \ldots, x_n), x_{n+1}]_{x_{n+2}}\) is an identity for \( I \);
4. \( G(x) = cx + [q, x] \) for all \( x \in R \), where \( c, q \in U \) such that \( cI = 0 \) and \([q, I]I = 0\);
5. \( \dim_C(\mathcal{R}C) \leq 4 \);
6. \( G(x) = \alpha x \), for some \( \alpha \in C \); moreover \( a \in C \) and \( f(x_1, \ldots, x_n)^m \) is central valued on \( R \).

In order to prove our Theorem we will use frequently the theory of generalized polynomial identities and differential identities (see [1, 9, 13, 17]). In particular we need to recall the following:

**1.2. Remark.** In [11], T.K. Lee proved that every generalized derivation \( G \) of \( R \) can be uniquely extended to a generalized derivation of \( U \). In particular, there exists \( a \in U \) and a derivation \( d \) of \( U \) such that \( G(x) = ax + d(x) \) for all \( x \in U \) [11, Theorem 3].

**1.3. Remark.** We need to recall the following notation:

\[
f(x_1, \ldots, x_n) = x_1x_2 \cdot \ldots \cdot x_n + \sum_{\sigma \in S_n, \sigma \neq 1} \alpha_{\sigma}x_{\sigma(1)} \ldots x_{\sigma(n)}
\]

for some \( \alpha_{\sigma} \in C \) and we denote by \( f^d(x_1, \ldots, x_n) \) the polynomial obtained from \( f(x_1, \ldots, x_n) \) by replacing each coefficient \( \alpha_{\sigma} \) with \( d(\alpha_{\sigma} \cdot 1) \). Thus, for \( d \) a usual derivation, we write \( d(f(r_1, \ldots, r_n)) = f^d(r_1, \ldots, r_n) + \sum_i f(r_1, \ldots, d(r_i), \ldots, r_n) \), for all \( r_1, \ldots, r_n \in R \).

Finally we also recall the following:

**1.4. Definition.** By a differential polynomial \( f(d_i(x_i)) \) over \( U \) we mean a generalized polynomial with coefficients in \( U \) and with variables acted on by derivation words, that is, \( f(z_{ij}) \) is a generalized polynomial in variables \( z_{ij} \) and with coefficients in \( U \), and each \( d_i \) is either a derivation word or the identity map of \( R \).

In particular in this note we consider the differential polynomial

\[
f(x_1, \ldots, x_n, d(x_1), \ldots, d(x_n)),
\]

that is, we will consider the case when a derivation \( d \) and the identity map act on the variables.

We say that the differential polynomial \( f(d_i(x_i)) \) is a **central differential identity** (central DI) for a right ideal \( \mathfrak{g} \) of \( R \) if \( f(z_{ij}) \) has no constant term and \( f(d_i(r_i)) \in \mathfrak{g} \) for all \( r_1, \ldots, r_n \in \mathfrak{g} \), but there exist \( s_1, \ldots, s_n \in \mathfrak{g} \) such that \( f(d_i(s_i)) \neq 0 \) (for more details we refer the reader to [4]).

**Proof.** Firstly we prove Theorem 1.1. We consider \( G(x) = cx + d(x) \), for some \( c \in U \) and a derivation \( d \) on \( U \). If \( aG(f(r_1, \ldots, r_n))^m = 0 \) for all \( r_1, \ldots, r_n \in I \), the result follows from [7]. Hence we suppose there exist \( s_1, \ldots, s_n \in I \) such that \( aG(f(s_1, \ldots, s_n))^m \neq 0 \). Therefore \( aG(f(x_1, \ldots, x_n))^m \in Z(R) \) is a central DI for \( I \), then by [4, Theorem 1], \( R \) is a PI-ring. Thus by Posner’s Theorem (see for example [18, Theorem 1.7.9]), \( RC \) is a finite-dimensional central simple algebra over \( C \) and \( RC \cong M_k(F) \), the ring of \( k \times k \) matrices over \( F \), for some integer \( k \) and some finite-dimensional central division algebra \( F \) over \( C \). We note that in this case \( a \) is invertible, therefore \( aG(f(x_1, \ldots, x_n))^m \in Z(R) \) if and only if \( G(f(x_1, \ldots, x_n))^m a \in Z(R) \). By [13, Theorem 2], \( G(f(r_1, \ldots, r_n))^m a \in C \).
for all \( r_1, \ldots, r_n \in IC \). In order to prove our result we may replace \( R \) with \( RC \) and \( I \) with \( IC \), so that we assume without loss of generality that \( R \cong M_k(F) \). Since \( I \) satisfies
\[
\left( cf(x_1, \ldots, x_n) + f^d(x_1, \ldots, x_n) + \sum_{i=1}^{n} f(x_1, \ldots, d(x_i), \ldots, x_n) \right)^m a \in C,
\]
then for all \( y \in R, I \) also satisfies
\[
\left( cf(x_1y, \ldots, x_n) + f^d(x_1y, \ldots, x_n) + f(d(x_1)y + x_1d(y), x_2, \ldots, x_n) + \sum_{i=2}^{n} f(x_1y, \ldots, d(x_i), \ldots, x_n) \right)^m a \in C.
\]

In the light of Kharchenko’s theory [9], we divide the proof into two cases:

If the derivation \( d \) is not inner, \( I \) satisfies
\[
\left( cf(x_1y, \ldots, x_n) + f^d(x_1y, \ldots, x_n) + f(d(x_1)y + x_1z, x_2, \ldots, x_n) + \sum_{i=2}^{n} f(x_1y, \ldots, d(x_i), \ldots, x_n) \right)^m a \in C,
\]
where the variable \( z \) falls in \( R \). In particular, for \( y = 0 \), \( I \) satisfies \( f(x_1z, \ldots, x_n)^m a \in C \) for all \( z \in R \), that is, \( I \) satisfies \( f(x_1, \ldots, x_n)^m a \in C \).

In case there are \( x_1, \ldots, x_n \in I \) such that \( f(x_1, \ldots, x_n)^m a \neq 0 \), then by [14, Theorem 1], \( f(x_1, \ldots, x_n)^m \) is central valued on \( R \) and also \( a \in C \). Thus \( I = R \) and \( G(f(x_1, \ldots, x_n))^m \) is central valued on \( R \). Hence, by [19], either \( f(x_1, \ldots, x_n) \) is central valued on \( R \), or \( R \) satisfies \( s_4 \) the standard identity of degree 4, or there exists \( a \in C \) such that \( G(x) = ax \).

In any case we are done.

On the other hand, if \( I \) satisfies \( f(x_1, \ldots, x_n)^m a \). Then, by [6] we get the conclusion that either \( a = 0 \) or \( f(x_1, \ldots, x_n)x_{n+1} \) is an identity for \( I \).

Let now \( d \) be the inner derivation induced by \( q \in U \), namely \( d(x) = [q, x] \), then we have \( G(x) = (c + q)x - xq \). In this case \( I \) satisfies
\[
\left( (c + q)f(x_1, \ldots, x_n) + f(x_1, \ldots, x_n)(-q) \right)^m a \in C.
\]

Denote by \( K \) the algebraic closure of \( F \) if \( F \) is infinite, otherwise let \( K = F \). Then \( M_k(F) \otimes_C K \cong M_l(K) \) for some \( l \geq 2 \). By [12, Lemma 2] and [10, Proposition], it follows that \( ((c + q)f(r_1, \ldots, r_n) + f(r_1, \ldots, r_n)(-q))^m a \in Z(M_l(K)) \) for all \( r_1, \ldots, r_n \in IC \otimes_C K \). Also in this case we assume, without loss of generality, that \( R = M_l(K) \) and \( I = \sum_{i=1}^{l} e_i R \), where \( t \leq l \).

If \( l = 2 \) we are done, thus we suppose that \( l \geq 3 \). By [3, Lemma 3], if \( f(x_1, \ldots, x_n), x_{n+1}x_{n+2} \) is not an identity for \( I \), then for all \( \alpha \in F, i \leq l \) and \( j \neq i \) there exist \( r_1, \ldots, r_n \in R \) such that \( f(r_1, \ldots, r_n) = \alpha e_{ij} \). Without loss of generality we may consider \( f(r_1, \ldots, r_n) = e_{ij} \). Therefore \(((c + q)e_{ij} + e_{ij}(-q))^m a \in Z(M_l(K))\). Since \(((c + q)e_{ij} + e_{ij}(-q))^m a \) has rank \( \leq 2 \), then it is zero in \( M_l(K) \), hence \(((c + q)e_{ij} + e_{ij}(-q))^m = 0 \), since \( a \) is invertible. This means both \( e_{ij}((c + q)e_{ij} + e_{ij}(-q))^m = 0 \) and \(((c + q)e_{ij} + e_{ij}(-q))^m = 0 \). Therefore the \((j, i)\)-entries of the matrices \( c \) and \( a \) are zero, so that \( qI \subseteq I \) and \( cI \subseteq I \). This means that \( G(I) \subseteq I \) and so \( G(f(r_1, \ldots, r_n))^m a \in I \cap K \), for all \( r_1, \ldots, r_n \in I \), implies \( I = R = M_l(K) \). Therefore \( R \) satisfies (1).

In the light of this, we may repeat the previous argument, for any \( i \neq j \) and with no assumption on \( i \) and \( j \). There are \( r_1, \ldots, r_n \in R \) such that \( f(r_1, \ldots, r_n) = e_{ij} \) and \(((c + q)e_{ij} + e_{ij}(-q))^m a \in Z(M_l(K))\). As above we have that \(((c + q)e_{ij} + e_{ij}(-q))^m = 0 \). Since it holds for all \( i \neq j \), it follows that both \( c \) and \( q \) are diagonal matrices in \( R \) and a standard argument shows that both \( c \) and \( q \) are central matrices in \( R \). Thus \( G(x) = cx \).
for \( c \in C \), and \((c^m)af(x_1, \ldots, x_n)_m \in C \) is satisfied by \( R \). Consider the following subset of \( R \):

\[
A = \{ x \in R : x f(r_1, \ldots, r_n)_m \in C, \ \forall r_1, \ldots, r_n \in R \}.
\]

Of course \( A \) is a subgroup of \( R \) which is invariant under the action of all the inner \( K \)-automorphisms. By \([5]\) either \( A \subseteq Z(R) \) or \( [R, R] \subseteq A \). In the first case \( a \in Z(R) \) and \( f(x_1, \ldots, x_n)_m \) is central valued on \( R \). In the second one, for all \( i \neq j \), \( e_{ij} f(x_1, \ldots, x_n)_m \in Z(R) \). By commuting this last with \( e_{ij} \) we get

\[
0 = [e_{ij} f(r_1, \ldots, r_n)_m, e_{ij}] = e_{ij} f(r_1, \ldots, r_n)_m e_{ij},
\]

for all \( r_1, \ldots, r_n \in R \). This means that \( f(r_1, \ldots, r_n)_m \) is a diagonal matrix on \( R \), and as above we obtain that \( f(r_1, \ldots, r_n)_m \) is a central matrix, for all \( r_1, \ldots, r_n \in R \). As a consequence, once again \( a \in Z(R) \).

As a consequence of the previous theorem we also have the following:

**1.5. Corollary.** Let \( R \) be a noncommutative prime ring of characteristic different from 2 with right Utumi quotient ring \( U \) and extended centroid \( C \), \( I \) a nonzero right ideal of \( R \). Let \( m \geq 1 \) be a fixed integer, \( a \) a fixed element of \( R \), \( G \) a generalized derivation of \( R \). If \( a G(r)_m \in Z(R) \) for all \( r \in I \), then one of the following holds:

1. \( a I = a G(I) = (0) \);
2. \( G(x) = qx \), for some \( q \in U \) and \( a q I = 0 \);
3. \( [x_1, x_2]_{x_3} \) is an identity for \( I \);
4. \( G(x) = cx + [q, x] \) for all \( x \in R \), where \( c, q \in U \) such that \( c I = 0 \) and \( [q, I] I = 0 \);
5. \( \dim C(RC) \leq 4 \).

We would like to conclude this note with the following results, which are easy reductions of the previous ones:

**1.6. Corollary.** Let \( R \) be a noncommutative prime ring of characteristic different from 2 with right Utumi quotient ring \( U \) and extended centroid \( C \), \( I \) a non-zero two-sided ideal of \( R \). Let \( f(x_1, \ldots, x_n) \) be a non-central multilinear polynomial over \( C \), \( m \geq 1 \) a fixed integer, \( a \) a non-zero fixed element of \( R \), \( G \) a non-zero generalized derivation of \( R \). If \( a G(f(r_1, \ldots, r_n))_m \in Z(R) \) for all \( r_1, \ldots, r_n \in I \), then one of the following holds:

1. \( G(x) = qx \), for some \( q \in U \) and \( a q I = 0 \);
2. \( R \) satisfies \( s_4 \), the standard identity of degree 4;
3. \( G(x) = cx + [q, x] \) for some \( c \in C \); moreover \( a \in C \) and \( f(x_1, \ldots, x_n)_m \) is central valued on \( R \).

**1.7. Corollary.** Let \( R \) be a noncommutative prime ring of characteristic different from 2 with right Utumi quotient ring \( U \) and extended centroid \( C \), \( I \) a non-zero two-sided ideal of \( R \). Let \( m \geq 1 \) be a fixed integer, \( a \) a non-zero fixed element of \( R \), \( G \) a non-zero generalized derivation of \( R \). If \( a G(r)_m \in Z(R) \) for all \( r \in I \), then one of the following holds:

1. \( G(x) = qx \), for some \( q \in U \) and \( a q I = 0 \);
2. \( R \) satisfies \( s_4 \), the standard identity of degree 4.

**References**


