Periodic and subharmonic solutions for a 2\textsuperscript{n}th-order nonlinear difference equation

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Abstract

By using the critical point method, some new criteria are obtained for the existence and multiplicity of periodic and subharmonic solutions to a 2\textsuperscript{n}th-order nonlinear difference equation. The proof is based on the Linking Theorem in combination with variational technique. Our results generalize and improve some known ones.

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1. Introduction

Existence of periodic solutions of higher-order differential equations has been the subject of many investigations [8,19-21,34,38,39]. By using various methods and techniques, such as fixed point theory, the Kaplan-Yorke method, critical point theory, coincidence degree theory, bifurcation theory and dynamical system theory etc., a series of existence results for periodic solutions have been obtained in the literature. Difference equations, the discrete analogs of differential equations, occur widely in numerous settings and forms, both in mathematics itself and in its applications to statistics, computing, electrical circuit analysis, dynamical systems, economics, biology and other fields. For the general background of difference equations, one can refer to monographs [1,3,4,31]. Since the last decade, there has been much progress on the qualitative properties of difference equations, which included results on stability and attractivity [22,31,33,48] and results on oscillation and other topics [1-4,7,11-15,17,18,28-30,32,44-47]. Only a few papers discuss the periodic solutions of higher-order difference equations. Therefore, it is worthwhile to explore this topic.

Let \( N, Z \) and \( R \) denote the sets of all natural numbers, integers and real numbers respectively. For \( a, b \in Z \), define \( Z(a) = \{a, a+1, \cdots \} \), \( Z(a,b) = \{a, a+1, \cdots, b \} \) when \( a \leq b \). * denotes the transpose of a vector.

In this paper, we consider the following forward and backward difference equation

\[
\Delta^n (r_k \Delta^n u_{k-n}) = (-1)^n f(k, u_{k+1}, u_k, u_{k-1}), \quad n \in Z(3), \quad k \in Z,
\]

where \( \Delta \) is the forward difference operator \( \Delta u_k = u_{k+1} - u_k \), \( \Delta^n u_k = \Delta(\Delta^{n-1} u_k) \), \( r_k \) is real valued for each \( k \in Z \), \( f \in C(Z \times R^3, R) \), \( r_k \) and \( f(k, v_1, v_2, v_3) \) are \( T \)-periodic in \( k \) for a given positive integer \( T \).

We may think of (1.1) as a discrete analogue of the following 2nth-order functional differential equation

\[
\frac{d^n}{dt^n} \left[ r(t) \frac{d^n u(t)}{dt^n} \right] = (-1)^n f(t, u(t+1), u(t), u(t-1)), \quad t \in R.
\]

Equations similar in structure to (1.2) arise in the study of the existence of solitary waves of lattice differential equations, see Smets and Willem [42].

The widely used tools for the existence of periodic solutions of difference equations are the various fixed point theorems in cones [1,3,4,27]. It is well known that critical point theory is a powerful tool that deals with the problems of differential equations [8,10,12,25,26,43]. Only since 2003, critical point theory has been employed to establish sufficient conditions on the existence of periodic solutions of difference equations. By using the critical point theory, Guo and Yu [28-30] and Shi et al. [41] established sufficient conditions on the existence of periodic solutions of second-order nonlinear difference equations. Compared to first-order or second-order difference equations, the study of higher-order equations has received considerably less attention (see, for example, [1,5,6,11-15,17,18,23,31,35,37] and the references contained therein). Ahlbrandt and Peterson [5] in 1994 studied the 2nth-order difference equation of the form

\[
\sum_{i=0}^{n} \Delta^i \left( r_i(k-i) \Delta^i u(k-i) \right) = 0
\]

in the context of the discrete calculus of variations, and Peil and Peterson [37] studied the asymptotic behavior of solutions of (1.3) with \( r_i(k) \equiv 0 \) for \( 1 \leq i \leq n-1 \). In 1998, Anderson [6] considered (1.3) for \( k \in Z(a) \), and obtained a formulation of generalized zeros and \((n,n)\)-disconjugacy for (1.3). Migda [35] in 2004 studied an \( m \)th-order linear difference equation. In 2007, Cai and Yu [9] have obtained some criteria for the existence
of periodic solutions of a 2nd-order difference equation

\[ \Delta^n (r_{k-n} \Delta^n u_{k-n}) + f(k, u_k) = 0, \quad n \in \mathbb{Z}(3), \ k \in \mathbb{Z}. \]

for the case where \( f \) grows superlinearly at both 0 and \( \infty \). However, to the best of our
knowledge, the results on periodic solutions of higher-order nonlinear difference equations
are very scarce in the literature. Furthermore, since (1.1) contains both advance and
retardation, there are very few manuscripts dealing with this subject. The main purpose
of this paper is to give some sufficient conditions for the existence and multiplicity of
periodic and subharmonic solutions to a 2nd-order nonlinear difference equation. The
main approach used in our paper is a variational technique and the Linking Theorem.
Particularly, our results not only generalize the results in the literature [9], but also
improve them. In fact, one can see the following Remarks 1.2 and 1.4 for details. The
motivation for the present work stems from the recent papers in [13,24].

Let

\[ \varrho = \min_{k \in \mathbb{Z}(1,T)} \{ r_k \}, \quad \bar{\varrho} = \max_{k \in \mathbb{Z}(1,T)} \{ r_k \}. \]

Our main results are as follows.

**Theorem 1.1.** Assume that the following hypotheses are satisfied:

(i) \( r_k > 0, \forall k \in \mathbb{Z} \),

(ii) there exists a functional \( F(k, v_1, v_2) \in C^1(\mathbb{Z} \times \mathbb{R}^2, \mathbb{R}) \) with \( F(k, v_1, v_2) \geq 0 \) and it satisfies

\[ F(k + T, v_1, v_2) = F(k, v_1, v_2), \]

\[ \partial F(k - 1, v_2, v_3) + \partial F(k, v_1, v_2) = f(k, v_1, v_2, v_3); \]

(iii) there exist constants \( \delta_1 > 0, \alpha \in \left( 0, \frac{1}{4} \zeta_{\lambda_{\text{max}}} \right) \) such that

\[ F(k, v_1, v_2) \geq \alpha (v_1^2 + v_2^2), \quad \forall k \in \mathbb{Z} \text{ and } v_1^2 + v_2^2 \leq \delta_1^2; \]

(iv) there exist constants \( \rho_1 > 0, \zeta > 0, \beta \in \left( \frac{1}{4} \zeta + \zeta_{\lambda_{\text{max}}}, +\infty \right) \) such that

\[ F(k, v_1, v_2) \geq \beta (v_1^2 + v_2^2) - \zeta, \quad \forall k \in \mathbb{Z} \text{ and } v_1^2 + v_2^2 \geq \rho_1^2; \]

Then for any given positive integer \( m > 0 \), (1.1) has at least three \( mT \)-periodic solutions.

**Remark 1.1.** By (iii) it is easy to see that there exists a constant \( \zeta' > 0 \) such that

\[ F(k, v_1, v_2) \geq \beta (v_1^2 + v_2^2) - \zeta', \quad \forall (k, v_1, v_2) \in \mathbb{Z} \times \mathbb{R}^2. \]

As a matter of fact, let \( \zeta_1 = \max \{ |F(k, v_1, v_2) - \beta (v_1^2 + v_2^2)|, k \in \mathbb{Z}, v_1^2 + v_2^2 \leq \rho_1^2 \} \),

\( \zeta' = \zeta + \zeta_1 \), we can easily get the desired result.

**Corollary 1.1.** Assume that (i) and (ii) - (iv) are satisfied. Then for any given positive
integer \( m > 0 \), (1.1) has at least two nontrivial \( mT \)-periodic solutions.

**Remark 1.2.** Corollary 1.1 reduces to Theorem 1.1 in [9].

**Theorem 1.2.** Assume that (i), (ii) and the following conditions are satisfied:

(iii) \( \lim_{\rho \to 0} \frac{F(k, v_1, v_2)}{\rho} = 0, \rho = \sqrt{v_1^2 + v_2^2}, \forall (k, v_1, v_2) \in \mathbb{Z} \times \mathbb{R}^2; \)

(iv) there exist constants \( R_1 > 0 \) and \( \theta > 2 \) such that for \( k \in \mathbb{Z} \) and \( v_1^2 + v_2^2 \geq R_1^2 \),

\[ 0 < \theta F(k, v_1, v_2) \leq \frac{\partial F(k, v_1, v_2)}{v_1} v_1 + \frac{\partial F(k, v_1, v_2)}{v_2} v_2. \]
Remark 1.3. Assumption (F_3) implies that there exist constants \( a_1 > 0 \) and \( a_2 > 0 \) such that
\[
(F_3') \quad F(k, v_1, v_2) \geq a_1 \left( \sqrt{v_1^2 + v_2^2} \right)^{\theta} - a_2, \quad \forall (k, v_1, v_2) \in \mathbb{Z} \times \mathbb{R}^2.
\]

Corollary 1.2. Assume that (r) and (F_1), (F_4), (F_5) are satisfied. Then for any given positive integer \( m > 0 \), (1.1) has at least two nontrivial \( mT \)-periodic solutions.

If \( f(k, u_{k+1}, u_k, u_{k-1}) = q_k g(u_k) \), (1.1) reduces to the following 2nth-order nonlinear equation,
\[
(1.5) \quad \Delta^n (r_{k-n} u_{k-n}) = (-1)^n q_k g(u_k), \quad k \in \mathbb{Z},
\]
where \( g \in C(\mathbb{R}, \mathbb{R}), q_{k+T} = q_k > 0 \), for all \( k \in \mathbb{Z} \). Then, we have the following results.

Theorem 1.3. Assume that (r) and the following hypotheses are satisfied:

(G_1) there exists a functional \( G(v) \in C^1(\mathbb{R}, \mathbb{R}) \) with \( G(v) \geq 0 \) and it satisfies
\[
G'(v) = g(v),
\]

(G_2) there exist constants \( \delta_2 > 0 \), \( \alpha \in (0, \frac{1}{2} \lambda_{\text{min}}^n) \) such that
\[
G(v) \leq \alpha |v|^2, \quad \text{for } |v| \leq \delta_2;
\]

(G_3) there exist constants \( \rho_2 > 0 \), \( \zeta > 0 \), \( \beta \in \left( \frac{1}{2} \lambda_{\text{max}}, +\infty \right) \) such that
\[
G(v) \geq \beta |v|^2 - \zeta, \quad \text{for } |v| \geq \rho_2,
\]
where \( \lambda_{\text{min}}, \lambda_{\text{max}} \) are constants which can be referred to (2.7).

Then for any given positive integer \( m > 0 \), (1.5) has at least three \( mT \)-periodic solutions.

Corollary 1.3. Assume that (r) and (G_1) - (G_3) are satisfied. Then for any given positive integer \( m > 0 \), (1.5) has at least two nontrivial \( mT \)-periodic solutions.

Remark 1.4. Corollary 1.3 reduces to Corollary 1.1 in [9].

The rest of the paper is organized as follows. First, in Section 2, we shall establish the variational framework associated with (1.1) and transfer the problem of the existence of periodic solutions of (1.1) into that of the existence of critical points of the corresponding functional. Some related fundamental results will also be recalled. Then, in Section 3, we shall complete the proof of the results by using the critical point method. Finally, in Section 4, we shall give an example to illustrate the main result.

For the basic knowledge of variational methods, the reader is referred to [27,34,36,40].

2. Variational structure and some lemmas

In order to apply the critical point theory, we shall establish the corresponding variational framework for (1.1) and give some lemmas which will be of fundamental importance in proving our main results. First, we state some basic notations.

Let \( S \) be the set of sequences \( u = (\cdots, u_{-k}, \cdots, u_{-1}, u_0, u_1, \cdots, u_k, \cdots) = \{u_k\}_{k=-\infty}^{+\infty} \), that is
\[
S = \{ \{u_k\}_{k \in \mathbb{Z}} \mid k \in \mathbb{Z} \}.
\]

For any \( u, v \in S \), \( a, b \in \mathbb{R} \), \( au + bv \) is defined by
\[
au + bv = \{au_k + bv_k\}_{k=-\infty}^{+\infty}.
\]
Then $S$ is a vector space.

For any given positive integers $m$ and $T$, $E_{mT}$ is defined as a subspace of $S$ by

$$E_{mT} = \{ u \in S | u_{k+mT} = u_k, \forall k \in \mathbb{Z} \}.$$  

Clearly, $E_{mT}$ is isomorphic to $\mathbb{R}^{mT}$. $E_{mT}$ can be equipped with the inner product

$$(u, v) = \sum_{j=1}^{mT} u_j v_j, \forall u, v \in E_{mT},$$

by which the norm $\| \cdot \|$ can be induced by

$$\|u\| = \left( \sum_{j=1}^{mT} |u_j|^2 \right)^{\frac{1}{2}}, \forall u \in E_{mT}.$$  

It is obvious that $E_{mT}$ with the inner product (2.1) is a finite dimensional Hilbert space

and linearly homeomorphic to $\mathbb{R}^{mT}$.

On the other hand, we define the norm $\| \cdot \|_s$ on $E_{mT}$ as follows:

$$\|u\|_s = \left( \sum_{j=1}^{mT} |u_j|^s \right)^{\frac{1}{s}}, \forall u \in E_{mT}.$$  

for all $u \in E_{mT}$ and $s > 1$.

Since $\|u\|_s$ and $\|u\|_2$ are equivalent, there exist constants $c_1, c_2$ such that $c_2 \geq c_1 > 0$, and

$$c_1 \|u\|_2 \leq \|u\|_s \leq c_2 \|u\|_2, \forall u \in E_{mT}.$$  

Clearly, $\|u\| = \|u\|_2$. For all $u \in E_{mT}$, define the functional $J$ on $E_{mT}$ as follows:

$$J(u) = \frac{1}{2} \sum_{k=1}^{mT} \Delta^2 u_{k-1}^2 - \sum_{k=1}^{mT} F(k, u_{k+1}, u_k),$$

where

$$\frac{\partial F(k-1, v_2, v_3)}{\partial v_2} + \frac{\partial F(k, v_1, v_2)}{\partial v_2} = f(k, v_1, v_2, v_3).$$  

Clearly, $J \in C^1(E_{mT}, \mathbb{R})$ and for any $u = \{ u_k \}_{k \in \mathbb{Z}} \in E_{mT}$, by using $u_0 = u_{mT}$, $u_1 = u_{mT+1}$, we can compute the partial derivative as

$$\frac{\partial J}{\partial u_k} = (-1)^n \Delta^3 u_{k-n} - f(k, u_{k+1}, u_k, u_{k-1}).$$  

Thus, $u$ is a critical point of $J$ on $E_{mT}$ if and only if

$$\Delta^3 (u_{k-n} - f(k, u_{k+1}, u_k, u_{k-1})) = (-1)^n f(k, u_{k+1}, u_k, u_{k-1}), \forall k \in \mathbb{Z}(1, mT).$$  

Due to the periodicity of $u = \{ u_k \}_{k \in \mathbb{Z}} \in E_{mT}$ and $f(k, v_1, v_2, v_3)$ in the first variable $k$, we reduce the existence of periodic solutions of (1.1) to the existence of critical points of $J$ on $E_{mT}$. That is, the functional $J$ is just the variational framework of (1.1).

Let $P$ be the $mT \times mT$ matrix defined by

$$P = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & -1 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ -1 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}.$$
By matrix theory, we see that the eigenvalues of $P$ are

$$\lambda_j = 2 \left(1 - \cos \frac{2j}{mT} \pi\right), j = 0, 1, 2, \ldots, mT - 1.$$  

Thus, $\lambda_0 = 0, \lambda_1 > 0, \lambda_2 > 0, \ldots, \lambda_{mT-1} > 0$. Therefore,

$$\lambda_{\min} = \min\{\lambda_1, \lambda_2, \ldots, \lambda_{mT-1}\} = 2 \left(1 - \cos \frac{2}{mT} \pi\right),$$

$$\lambda_{\max} = \max\{\lambda_1, \lambda_2, \ldots, \lambda_{mT-1}\} = \begin{cases} 4, & \text{when } mT \text{ is even,} \\ 2 \left(1 + \cos \frac{1}{mT} \pi\right), & \text{when } mT \text{ is odd.} \end{cases}$$

Let $W = \ker P = \{u \in E_{mT} | Pu = 0 \in \mathbb{R}^{mT}\}$. Then

$$W = \{u \in E_{mT} | u = \{c\}, c \in \mathbb{R}\}.$$

Let $V$ be the direct orthogonal complement of $E_{mT}$ to $W$, i.e., $E_{mT} = V \oplus W$. For convenience, we identify $u \in E_{mT}$ with $u = (u_1, u_2, \ldots, u_{mT})^\ast$.

Let $E$ be a real Banach space, $J \in C^1(E, \mathbb{R})$, i.e., $J$ is a continuously Fréchet-differentiable functional defined on $E$. $J$ is said to satisfy the Palais-Smale condition (P.S. condition for short) if any sequence $\{u^{(i)}\} \subset E$ for which $\{J(u^{(i)})\}$ is bounded and $J'(u^{(i)}) \to 0 (i \to \infty)$ possesses a convergent subsequence in $E$.

Let $B_\rho$ denote the open ball in $E$ about 0 of radius $\rho$ and let $\partial B_\rho$ denote its boundary.

**Lemma 2.1 (Linking Theorem [40])**. Let $E$ be a real Banach space, $E = E_1 \oplus E_2$, where $E_1$ is finite dimensional. Suppose that $J \in C^1(E, \mathbb{R})$ satisfies the P.S. condition and (J1) there exist constants $a > 0$ and $\rho > 0$ such that $J|_{\partial B_\rho \cap E_2} \geq a$;

(J2) there exists an $e \in \partial B_\rho \cap E_2$ and a constant $R_0 \geq \rho$ such that $J|_{\partial Q} \leq 0$, where $Q = (\partial B_{R_0} \cap E_1) \oplus \{se | 0 < s < R_0\}$.

Then $J$ possesses a critical value $c \geq a$, where

$$c = \inf_{h \in \Gamma} \sup_{u \in Q} J(h(u)),$$

and $\Gamma = \{h \in C(\bar{Q}, E) | h|_{\partial Q} = id\}$, where id denotes the identity operator.

**Lemma 2.2**. Assume that (r), (F1) and (F3) are satisfied. Then the functional $J$ is bounded from above in $E_{mT}$.

**Proof**. By (F3) and (2.4), for any $u \in E_{mT},$

$$J(u) = \frac{1}{2} \sum_{k=1}^{mT} r_{k-1} (\Delta^n u_{k-1}, \Delta^n u_{k-1}) - \sum_{k=1}^{mT} F(k, u_{k+1}, u_k)$$

$$= \frac{1}{2} \sum_{k=1}^{mT} r_k (\Delta^n u_k, \Delta^n u_k) - \sum_{k=1}^{mT} F(k, u_{k+1}, u_k)$$

$$\leq \frac{\bar{r}}{2} x^\ast Px - \sum_{k=1}^{mT} \left[\beta(u_{k+1}^2 + u_k^2) - \zeta'\right]$$

$$\leq \frac{\bar{r}}{2} \lambda_{\max} \|x\|^2 - 2\beta \|u\|^2 + mT\zeta',$$

where $x = (\Delta^{n-1} u_1, \Delta^{n-1} u_2, \ldots, \Delta^{n-1} u_{mT})^\ast$. Since

$$\|x\|^2 = \sum_{k=1}^{mT} (\Delta^{n-2} u_{k+1} - \Delta^{n-2} u_k)^2 \leq \lambda_{\max} \sum_{k=1}^{mT} (\Delta^{n-2} u_k)^2 \leq \lambda_{\max}^{-1} \|u\|^2,$$

$$\|x\|^2 = \sum_{k=1}^{mT} (\Delta^{n-2} u_{k+1} - \Delta^{n-2} u_k)^2 \leq \lambda_{\max} \sum_{k=1}^{mT} (\Delta^{n-2} u_k)^2 \leq \lambda_{\max}^{-1} \|u\|^2,$$
we have
\[ J(u) \leq \left( \frac{r}{2} \lambda_{\text{max}} - 2\beta \right) \|u\|^2 + mT\zeta' \leq mT\zeta'. \]

The proof of Lemma 2.2 is complete. \(\square\)

**Remark 2.1.** The case \(m = 1\) is trivial. For the case \(m = 2\), \(P\) has a different form, namely,
\[ P = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}. \]
However, in this special case, the argument need not to be changed and we omit it.

**Lemma 2.3.** Assume that \(\tau\), \((F_1)\) and \((F_3)\) are satisfied. Then the functional \(J\) satisfies the P.S. condition.

**Proof.** Let \(\{J(u^{(i)})\}\) be a bounded sequence from the lower bound, i.e., there exists a positive constant \(M_1\) such that
\[-M_1 \leq J(u^{(i)}) \leq M_1, \quad \forall i \in \mathbb{N}.\]
By the proof of Lemma 2.2, it is easy to see that
\[-M_1 \leq J(u^{(i)}) \leq \left( \frac{r}{2} \lambda_{\text{max}} - 2\beta \right) \|u^{(i)}\|^2 + mT\zeta', \quad \forall i \in \mathbb{N}.\]
Therefore,
\[ \left( 2\beta - \frac{r}{2} \lambda_{\text{max}} \right) \|u^{(i)}\|^2 \leq M_1 + mT\zeta'. \]
Since \(\beta > \frac{1}{4} r \lambda_{\text{max}},\) it is not difficult to know that \(\{u^{(i)}\}\) is a bounded sequence in \(E_{mT}.\)
As a consequence, \(\{u^{(i)}\}\) possesses a convergence subsequence in \(E_{mT}.\) Thus the P.S. condition is verified. \(\square\)

### 3. Proof of the main results

In this Section, we shall prove our main results by using the critical point theory.

#### 3.1. Proof of Theorem 1.1

Assumptions \((F_1)\) and \((F_2)\) imply that \(F(k, 0) = 0\) and \(f(k, 0) = 0\) for \(k \in \mathbb{Z}\). Then \(u = 0\) is a trivial \(mT\)-periodic solution of (1.1).

By Lemma 2.2, \(J\) is bounded from the upper on \(E_{mT}.\). We define \(c_0 = \sup_{u \in E_{mT}} J(u).\)

The proof of Lemma 2.2 implies \(\lim_{\|u\|_2 \to +\infty} J(u) = -\infty.\) This means that \(-J(u)\) is coercive.

By the continuity of \(J(u)\), there exists \(\bar{u} \in E_{mT}\) such that \(J(\bar{u}) = c_0.\) Clearly, \(\bar{u}\) is a critical point of \(J\).

We claim that \(c_0 > 0.\) Indeed, by \((F_2),\) for any \(u \in V, \|u\|_2 \leq \delta,\) we have
\[
J(u) = \frac{1}{2} \sum_{k=1}^{mT} r_{k-1} (\Delta^n u_{k-1}, \Delta^n u_{k-1}) - \sum_{k=1}^{mT} F(k, u_{k+1}, u_k) \\
= \frac{1}{2} \sum_{k=1}^{mT} r_k (\Delta^n u_k, \Delta^n u_k) - \sum_{k=1}^{mT} F(k, u_{k+1}, u_k) \\
\geq \frac{1}{2} \sum_{k=1}^{mT} r^* P x - \alpha \sum_{k=1}^{mT} (u_{k+1}^2 + u_k^2). 
\]
Take where \(x = (\Delta^{n-1}u_1, \Delta^{n-1}u_2, \cdots, \Delta^{n-1}u_{mT})^T\). Since
\[
\|x\|^2 = \sum_{k=1}^{mT} (\Delta^{n-2}u_{k+1} - \Delta^{n-2}u_k)^2 \geq \lambda_{\min} \sum_{k=1}^{mT} (\Delta^{n-2}u_k)^2 \geq \lambda_{\min}^n \|u\|^2_2,
\]
we have
\[
J(u) = \left(\frac{1}{2} \lambda_{\min}^n - 2\alpha\right) \|u\|^2_2.
\]
Take \(\sigma = \left(\frac{1}{2} \lambda_{\min}^n - 2\alpha\right) \delta_{\min}^2\). Then
\[
J(u) \geq \sigma, \quad \forall u \in V \cap \partial B_{\delta_{\min}}.
\]
Therefore, \(c_0 = \sup_{u \in E_{mT}} J(u) \geq \sigma > 0\). At the same time, we have also proved that there exist constants \(\sigma > 0\) and \(\delta_{1} > 0\) such that \(J|_{\partial B_{\delta_{1}} \cap V} \geq \sigma\). That is to say, \(J\) satisfies the condition \((J_1)\) of the Linking Theorem.

Noting that \(\sum_{k=1}^{mT} r_{k-1} (\Delta^n u_{k-1})^2 = 0\), for all \(u \in W\), we have
\[
J(u) = \frac{1}{2} \sum_{k=1}^{mT} r_{k-1} (\Delta^n u_k) - \sum_{k=1}^{mT} F(k, u_{k+1}, u_k) = -\sum_{k=1}^{mT} F(k, u_{k+1}, u_k) \leq 0.
\]
Thus, the critical point \(\bar{u}\) of \(J\) corresponding to the critical value \(c_0\) is a nontrivial \(mT\)-periodic solution of (1.1).

In order to obtain another nontrivial \(mT\)-periodic solution of (1.1) different from \(\bar{u}\), we need to use the conclusion of Lemma 2.1. We have known that \(J\) satisfies the P.S. condition on \(E_{mT}\). In the following, we shall verify the condition \((J_2)\).

Take \(e \in \partial B_{\delta_{\min}} \cap V\), for any \(z \in W\) and \(s \in \mathbb{R}\), let \(u = se + z\). Then
\[
J(u) = \frac{1}{2} \sum_{k=1}^{mT} r_{k-1} (\Delta^n u_k) - \sum_{k=1}^{mT} F(k, u_{k+1}, u_k)
\]
\[
\leq \frac{\bar{r}}{2} s^2 \sum_{k=1}^{mT} (\Delta^n e_k, \Delta^n e_k) - \sum_{k=1}^{mT} F(k, s e_{k+1} + z_{k+1}, s e_k + z_k)
\]
\[
\leq \frac{\bar{r}}{2} s^2 \|y\|_2^2 - \sum_{k=1}^{mT} \left\{ \beta \left[ (s e_{k+1} + z_{k+1})^2 + (s e_k + z_k)^2 \right] - \zeta' \right\}
\]
\[
\leq \frac{\bar{r}}{2} s^2 ||\|y\|_2^2 - 2\beta \sum_{k=1}^{mT} (s e_k + z_k)^2 + mT\zeta'
\]
\[
= \frac{\bar{r}}{2} s^2 \lambda_{\max} ||y||_2^2 - 2\beta s^2 - 2\beta \|z\|^2 + mT\zeta',
\]
where \(y = (\Delta^{n-1}e_1, \Delta^{n-1}e_2, \cdots, \Delta^{n-1}e_{mT})^T\). Since
\[
||y||_2^2 = \sum_{k=1}^{mT} (\Delta^{n-2}e_{k+1} - \Delta^{n-2}e_k)^2 \leq \lambda_{\max} \sum_{k=1}^{mT} (\Delta^{n-2}e_k)^2 \leq \lambda_{\max}^n,
\]
we have
\[
J(u) \leq \left(\frac{\bar{r}}{2} \lambda_{\max}^n - 2\beta\right) s^2 - 2\beta \|z\|^2 + mT\zeta' \leq -2\beta \|z\|^2 + mT\zeta'.
\]
Thus, there exists a positive constant $R_2 > \delta_1$ such that for any $u \in \partial Q$, $J(u) \leq 0$, where $Q = (B_{R_2} \cap W) \oplus \{ s \epsilon [0 < s < R_3] \}$. By the Linking Theorem, $J$ possesses a critical value $c \geq \sigma > 0$, where
\[
c = \inf_{h \in \Gamma} \sup_{u \in Q} J(h(u)),
\]
and $\Gamma = \{ h \in C(\bar{Q}, E_{m, T}) \mid h|_{\partial Q} = id \}$.

Let $\tilde{u} \in E_{m, T}$ be a critical point associated to the critical value $c$ of $J$, i.e., $J(\tilde{u}) = c$. If $\tilde{u} \neq \bar{u}$, then the conclusion of Theorem 1.1 holds. Otherwise, $\tilde{u} = \bar{u}$. Then $c_0 = J(\bar{u}) = J(\tilde{u}) = c$, that is $\sup_{u \in E_{m, T}} J(u) = \inf_{h \in \Gamma} \sup_{u \in Q} J(h(u))$. Choosing $h = id$, we have $\sup_{u \in Q} J(u) = c_0$. Since the choice of $\varepsilon \in \partial B_1 \cap V$ is arbitrary, we can take $-\varepsilon \in \partial B_1 \cap V$. Similarly, there exists a positive number $R_3 > \delta_1$, for any $u \in \partial Q_1$, $J(u) \leq 0$, where $Q_1 = (\tilde{B}_{R_3} \cap W) \oplus \{ \varepsilon \in [0 < \varepsilon < R_3] \}$.

Again, by the Linking Theorem, $J$ possesses a critical value $c' \geq \sigma > 0$, where
\[
c' = \inf_{h \in \Gamma_1} \sup_{u \in Q_1} J(h(u)),
\]
and $\Gamma_1 = \{ h \in C(\bar{Q}_1, E_{m, T}) \mid h|_{\partial Q_1} = id \}$.

If $c' \neq c_0$, then the proof is finished. If $c' = c_0$, then $\sup_{u \in Q_1} J(u) = c_0$. Due to the fact $J|_{\partial Q_1} \leq 0$ and $J|_{\partial Q_1} \leq 0$, $J$ attains its maximum at some points in the interior of sets $Q$ and $Q_1$. However, $Q \cap Q_1 \subset W$ and $J(u) \leq 0$ for any $u \in W$. Therefore, there must be a point $u' \in E_{m, T}$, $u' \neq \tilde{u}$ and $J(u') = c' = c_0$. The proof of Theorem 1.1 is complete. □

**Remark 3.1.** Similarly to above argument, we can also prove Theorems 1.2 and 1.3. For simplicity, we omit their proofs.

**Remark 3.2.** Due to Theorems 1.1, 1.2 and 1.3, the conclusion of Corollaries 1.1, 1.2 and 1.3 is obviously true.

### 4. Example

As an application of Theorem 1.1, we give an example to illustrate our main result.

**Example 4.1.** For all $n \in \mathbb{Z}(3)$, $k \in \mathbb{Z}$, assume that
\[
\Delta^n (r_{k-n} \Delta^n u_{k-n}) = (4.1)
\]
\[
(-1)^n \mu u_k \left[ (8 + \sin^2 \left( \frac{\pi k}{T} \right) ) \left( u_{k+1}^2 + u_k^2 \right) \frac{2}{\pi} - 1 + \left( 8 + \sin^2 \left( \frac{\pi (k-1)}{T} \right) \right) \left( u_k^2 + u_{k-1}^2 \right) \frac{2}{\pi} - 1 \right],
\]
where $r_k$ is real valued for each $k \in \mathbb{Z}$ and $r_{k+T} = r_k > 0$, $\mu > 2$, $T$ is a given positive integer.

We have
\[
f(k, v_1, v_2, v_3) = \mu v_2 \left[ (8 + \sin^2 \left( \frac{\pi k}{T} \right) ) \left( v_1^2 + v_2^2 \right) \frac{2}{\pi} - 1 + \left( 8 + \sin^2 \left( \frac{\pi (k-1)}{T} \right) \right) \left( v_2^2 + v_3^2 \right) \frac{2}{\pi} - 1 \right]
\]
and
\[
F(k, v_1, v_2) = \left[ 8 + \sin^2 \left( \frac{\pi k}{T} \right) \right] \left( v_1^2 + v_2^2 \right) \frac{2}{\pi}.
\]
Then
\[
\frac{\partial F(k-1,v_2,v_3)}{\partial v_2} + \frac{\partial F(k,v_1,v_2)}{\partial v_2} = \mu v_2 \left[ \left( 8 + \sin^2 \left( \frac{\pi k}{T} \right) \right) \left( v_2^2 + v_3^2 \right)^{\frac{5}{2}} + \left( 8 + \sin^2 \left( \frac{\pi (k-1)}{T} \right) \right) \left( v_2^2 + v_3^2 \right)^{\frac{5}{2}-1} \right].
\]

It is easy to verify all the assumptions of Theorem 1.1 are satisfied. Consequently, for any given positive integer \( m > 0 \), (4.1) has at least three \( mT \)-periodic solutions.

References


