On star-K-Menger spaces

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Abstract
A space \(X\) is star-K-Menger if for each sequence \((U_n : n \in \mathbb{N})\) of open covers of \(X\) there exists a sequence \((K_n : n \in \mathbb{N})\) of compact subsets of \(X\) such that \(\{\text{St}(K_n, U_n) : n \in \mathbb{N}\}\) is an open cover of \(X\). In this paper, we investigate the relationship between star-K-Menger spaces and related spaces, and study topological properties of star-K-Menger spaces.

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1. Introduction

By a space, we mean a topological space. We give definitions of terms which are used in this paper. Let \(\mathbb{N}\) denote the set of positive integers. Let \(X\) be a space and \(U\) a collection of subsets of \(X\). For \(A \subseteq X\), let \(\text{St}(A, U) = \bigcup\{U \in U : U \cap A \neq \emptyset\}\). As usual, we write \(\text{St}(x, U)\) instead of \(\text{St}(\{x\}, U)\).

Let \(A\) and \(B\) be collections of open covers of a space \(X\). Then the symbol \(S_1(A, B)\) denotes the selection hypothesis that for each sequence \((U_n : n \in \mathbb{N})\) of elements of \(A\) there exists a sequence \((V_n : n \in \mathbb{N})\) such that for each \(n \in \mathbb{N}\), \(V_n\) is a finite subset of \(U_n\) and \(\bigcup_{n \in \mathbb{N}} V_n\) is an element of \(B\) (see [3,8]).

Kočinac [4,5] introduced star selection hypothesis similar to the previous ones. Let \(A\) and \(B\) be collections of open covers of a space \(X\). Then:

(A) The symbol \(S_{*1}(A, B)\) denotes the selection hypothesis that for each sequence \((U_n : n \in \mathbb{N})\) of elements of \(A\) there exists a sequence \((V_n : n \in \mathbb{N})\) such that for each \(n \in \mathbb{N}\), \(V_n\) is a finite subset of \(U_n\) and \(\bigcup_{n \in \mathbb{N}} \{\text{St}(V, U_n) : V \in \mathcal{V}_n\}\) is an element of \(B\).
(B) The symbol $SS^*_\text{comp}(A, B)$ (or $SS^*_\text{fin}(A, B)$) denotes the selection hypothesis that for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $A$ there exists a sequence $(K_n : n \in \mathbb{N})$ of compact (resp., finite) subsets of $X$ such that $\{St(K_n, \mathcal{U}_n) : n \in \mathbb{N}\} \in B$.

Let $\mathcal{O}$ denote the collection of all open covers of $X$.

1. Definition. ([4,5]) A space $X$ is said to be star-Menger if it satisfies the selection hypothesis $SS^*_\text{fin}(\mathcal{O}, \mathcal{O})$.

2. Definition. ([4,5]) A space $X$ is said to be star-K-Menger (strongly star-Menger) if it satisfies the selection hypothesis $SS^*_\text{comp}(\mathcal{O}, \mathcal{O})$ (resp., $SS^*_\text{fin}(\mathcal{O}, \mathcal{O})$).

3. Definition. ([1,7]) A space $X$ is said to be starcompact (star-Lindelöf) if for every open cover $\mathcal{U}$ of $X$ there exists a finite (resp., countable, respectively) $\mathcal{V} \subseteq \mathcal{U}$ such that $St(\cup \mathcal{V}, \mathcal{U}) = X$.

4. Definition. ([1,6,9]) A space $X$ is said to be $K$-starcompact (strongly starcompact, strongly star-Lindelöf, star-L-Lindelöf) if for every open cover $\mathcal{U}$ of $X$ there exists a compact (resp., finite, countable, Lindelöf) subset $F$ of $X$ such that $St(F, \mathcal{U}) = X$.

From the definitions, it is clear that every $K$-starcompact space is star-K-Menger, every strongly star-Menger space is star-K-Menger and every star-K-Menger space is star-Menger. Since every $\sigma$-compact subset is Lindelöf, thus every star-K-Menger space is star-L-Lindelöf. But the converses do not hold (see Examples 2.1, 2.2, 2.3 and 2.4 below).

Kočinac [4,5] studied the star-Menger and related spaces. In this paper, our purpose is to investigate the relationship between star-K-Menger spaces and related spaces, and study topological properties of star-K-Menger spaces.

Throughout this paper, let $\omega$ denote the first infinite cardinal, $\omega_1$ the first uncountable cardinal, $c$ the cardinality of the set of all real numbers. For a cardinal $\kappa$, let $\kappa^+$ be the smallest cardinal greater than $\kappa$. For each pair of ordinals $\alpha, \beta$ with $\alpha < \beta$, we write $[\alpha, \beta) = \{ \gamma : \alpha \leq \gamma < \beta \}$, $(\alpha, \beta) = \{ \gamma : \alpha < \gamma \leq \beta \}$, $[\alpha, \beta] = \{ \gamma : \alpha \leq \gamma \leq \beta \}$ and $[\alpha, \beta) = \{ \gamma : \alpha \leq \gamma < \beta \}$. As usual, a cardinal is an initial ordinal and an ordinal is the set of smaller ordinals. A cardinal is often viewed as a space with the usual order topology. Other terms and symbols that we do not define follow [2].

2. Star-K-Menger spaces and related spaces

In this section, we give some examples showing that the relationship between star-K-Menger spaces and other related spaces.

2.1. Example. There exists a Tychonoff star-K-Menger space $X$ which is not $K$-starcompact.

Proof. Let $X = \omega$ be the countably infinite discrete space. Clearly, $X$ is not $K$-starcompact. Since $X$ is countable, the singleton sets can serve as the compact sets which witness that $X$ is star-K-Menger, which completes the proof.

2.2. Example. There exists a Tychonoff star-K-Menger space which is not strongly star-Menger.

Proof. Let $D = \{d_\alpha : \alpha < c\}$ be a discrete space of cardinality $c$ and let $aD = D \cup \{d^+\}$ be one-point comactification of $D$. Let

$$X = (aD \times [0, c^+)) \cup (D \times \{c^+\})$$

be the subspace of the product space $aD \times [0, c^+]$. Clearly, $X$ is a Tychonoff space.
First we show that \( X \) is star-K-Menger; we only show that \( X \) is K-starcompact, since every K-starcompact space is star-K-Menger. To this end, let \( \mathcal{U} \) be an open cover of \( X \). For each \( \alpha < \eta \), there exists \( U_\alpha \in \mathcal{U} \) such that \( \langle d_\alpha, c^+ \rangle \subseteq U_\alpha \). For each \( \alpha < c \), we can find \( \beta_\alpha < c^+ \) such that \( \{d_\alpha \} \times \{\beta_\alpha, c^+ \} \subseteq U_\alpha \). Let \( \beta = \sup\{\beta_\alpha : \alpha < c\} \). Then \( \beta < c^+ \). Let \( K_1 = aD \times \{\beta\} \). Then \( K_1 \) is compact and \( U_\alpha \cap K_1 \neq \emptyset \) for each \( \alpha < c \). Hence \[
D \times \{c^+\} \subseteq St(K_1, \mathcal{U}).
\]
On the other hand, since \( aD \times [0, c^+] \) is countably compact and consequently \( aD \times [0, c^+] \) is strongly starcompact (see [1,6]), hence there exists a finite subset \( K_2 \) of \( aD \times [0, c^+] \) such that \( aD \times [0, c^+] \subseteq St(K_2, \mathcal{U}) \).

If we put \( K = K_1 \cup K_2 \). Then \( K \) is a compact subset of \( X \) such that \( X = St(K, \mathcal{U}) \), which shows that \( X \) is K-starcompact.

Next we show that \( X \) is not strongly star-Menger. We only show that \( X \) is not strongly star-Lindelöf, since every strongly star-Menger space is strongly star-Lindelöf. Let us consider the open cover \( \mathcal{U} = \{\{d_\alpha \} \times [0, c^+] : \alpha < c\} \cup \{aD \times [0, c^+]\} \)

of \( X \). It remains to show that \( St(F, \mathcal{U}) \neq X \) for any countable subset \( F \) of \( X \). To show this, let \( F \) any countable subset of \( X \). Then there exists \( \alpha_0 < c \) such that \( F \cap \{(d_{\alpha_0}) \times [0, c^+]\} = \emptyset \). Hence \( \langle d_{\alpha_0}, c^+ \rangle \notin St(F, \mathcal{U}) \), since \( \{d_{\alpha_0}\} \times [0, c^+] \) is the only element of \( \mathcal{U} \) containing the point \( \langle d_{\alpha_0}, c^+ \rangle \), which shows that \( X \) is not strongly star-Lindelöf.

\[ \square \]

2.3. Example. There exists a Tychonoff star-L-Lindelöf space which is not star-K-Menger.

Proof. Let \( D = \{d_\alpha : \alpha < \eta\} \) be a discrete space of cardinality \( \eta \) and let \( bD = D \cup \{d^*\} \), where \( d^* \notin D \). We topologize \( bD \) as follows: for each \( \alpha < \eta \), \( \{d_\alpha\} \) is isolated and a set \( U \) containing \( d^* \) is open if and only if \( bD \setminus U \) is countable. Then \( bD \) is Lindelöf and every compact subset of \( bD \) is finite. Let \( X = (bD \times [0, \omega]) \setminus \{(d^*, \omega)\} \) be the subspace of the product space \( bD \times [0, \omega] \). Then \( X \) is star-L-Lindelöf, since \( bD \times \omega \) is a Lindelöf dense subset of \( X \).

Next we show that \( X \) is not star-K-Menger. For each \( \alpha < \eta \), let \( U_\alpha = \{d_\alpha\} \times [0, \omega] \). For each \( n \in \omega \), let \( V_n = bD \times \{n-1\} \). For each \( n \in \mathbb{N} \), let \( U_n = \{U_\alpha : \alpha < \eta\} \cup \{V_n : n \in \mathbb{N}\} \).

Then \( U_n \) is an open cover of \( X \). Let us consider the sequence \( \langle U_n : n \in \mathbb{N}\rangle \) of open covers of \( X \). It suffices to show that \( \bigcup_{n \in \mathbb{N}} St(K_n, U_n) \neq X \) for any sequence \( \langle K_n : n \in \mathbb{N}\rangle \) of compact subsets of \( X \). Let \( \langle K_n : n \in \mathbb{N}\rangle \) be any sequence of compact subsets of \( X \). For each \( n \in \mathbb{N} \), since \( K_n \) is compact and \( \{d_\alpha, \omega : \alpha < \eta\} \) is a discrete closed subset of \( X \), the set \( K_n \cap \{d_\alpha, \omega : \alpha < \eta\} \) is finite. Then there exists \( \alpha_n < \eta \) such that \( K_n \cap \{d_\alpha, \omega : \alpha > \alpha_n\} = \emptyset \).

Let \( \alpha' = \sup\{\alpha_n : n \in \mathbb{N}\} \). Then \( \alpha' < \eta \) and \[
\bigcup_{n \in \mathbb{N}} K_n \cap \{d_\alpha, \omega : \alpha > \alpha'\} = \emptyset.
\]

For each \( n \in \mathbb{N} \), since \( K_n \cap V_m \) is finite for each \( m \in \mathbb{N} \), there exists \( \alpha_{nm} < \eta \) such that \( K_n \cap \{d_\alpha, n : \alpha > \alpha_{nm}\} = \emptyset \).
Let \( \alpha' = \sup\{\alpha_n : n \in \mathbb{N}\} \). Then \( \alpha' < \omega \) and
\[
K_n \cap \{(d_\alpha, m) : \alpha > \alpha', m \in \mathbb{N}\} = \emptyset.
\]
Let \( \alpha'' = \sup\{\alpha_n' : n \in \mathbb{N}\} \). Then \( \alpha'' < \omega \) and
\[
(\bigcup_{n \in \mathbb{N}} K_n) \cap \{(d_\alpha, m) : \alpha > \alpha'', m \in \mathbb{N}\} = \emptyset.
\]
If we pick \( \beta > \max\{\alpha', \alpha''\} \). Then \( U_\beta \cap K_n = \emptyset \) for each \( n \in \mathbb{N} \). Hence \( \{d_\beta, \omega\} \notin St(K_n, U_\alpha) \) for each \( n \in \mathbb{N} \), since \( U_\beta \) is the only element of \( U_\alpha \) containing the point \( \{d_\beta, \omega\} \) for each \( n \in \mathbb{N} \), which shows that \( X \) is not star-K-Menger.

2.4. Example. There exists a \( T_1 \) star-Menger space which is not star-K-Menger.

Proof. Let \( X = [0, \omega_1) \cup D \), where \( D = \{d_\alpha : \alpha < \omega_1\} \) is a set of cardinality \( \omega_1 \). We topologize \( X \) as follows: \([0, \omega_1)\) has the usual order topology and is an open subspace of \( X \); a basic neighborhood of a point \( d_\alpha \in D \) takes the form
\[
O_\alpha(d_\alpha) = \{d_\alpha\} \cup (\beta, \omega_1), \text{ where } \beta < \omega_1.
\]
Then \( X \) is a \( T_1 \) space.

First we show that \( X \) is star-Menger. We only show that \( X \) is starcompact, since every starcompact space is star-Menger. To this end, let \( U \) be an open cover of \( X \). Without loss of generality, we can assume that \( U \) consists of basic open subsets of \( X \). Thus it is sufficient to show that there exists a finite subset \( V \) of \( U \) such that \( St(\bigcup V, U) = X \). Since \([0, \omega_1)\) is countably compact, it is strongly starcompact (see [1,6]), then we can find a finite subset \( V \) of \( U \) such that \([0, \omega_1) \subseteq St(\bigcup V, U) \). On the other hand, if we pick \( \alpha_0 < \omega_1 \), then there exists \( U_{\alpha_0} \in U \) such that \( d_{\alpha_0} \in U_{\alpha_0} \). For each \( \alpha < \omega_1 \), there is \( U_\alpha \in U \) such that \( d_\alpha \in U_\alpha \). Hence we have \( U_{\alpha_0} \cap U_\alpha \neq \emptyset \) by the construction of the topology of \( X \). Therefore \( D \subseteq St(U_{\alpha_0}, U) \). If we put \( V = V_1 \cup \{U_{\alpha_0}\} \), then \( V \) is a finite subset of \( U \) and \( X = St(\bigcup V, U) \), which shows that \( X \) is starcompact.

Next we show that \( X \) is not star-K-Menger. For each \( n \in \mathbb{N} \), let
\[
U_n = \{O_\alpha(d_\alpha) : \alpha < \omega_1\} \cup \{(0, \omega_1)\}
\]
Then \( U_n \) is an open cover of \( X \). Let us consider the sequence \( \{U_n : n \in \mathbb{N}\} \) of open covers of \( X \). It suffices to show that \( \bigcup_{n \in \mathbb{N}} St(K_n, U_n) \neq X \) for any sequence \( \{K_n : n \in \mathbb{N}\} \) of compact subsets of \( X \). Let \( \{K_n : n \in \mathbb{N}\} \) be any sequence of compact subsets of \( X \). For each \( n \in \mathbb{N} \), the set \( K_n \cap \{d_\alpha : \alpha < \omega_1\} \) is finite, since \( K_n \) is compact and \( \{d_\alpha : \alpha < \omega_1\} \) is a discrete closed subset of \( X \). Then there exists \( \alpha_n < \omega_1 \) such that
\[
K_n \cap \{d_\alpha : \alpha > \alpha_n\} = \emptyset.
\]
Let \( \alpha' = \sup\{\alpha_n : n \in \mathbb{N}\} \). Then \( \alpha' < \omega_1 \) and
\[
\bigcup_{n \in \mathbb{N}} K_n \cap \{d_\alpha : \alpha > \alpha'\} = \emptyset.
\]
For each \( n \in \mathbb{N} \), the set \( K_n \) is compact and \([0, \omega_1)\) is countably compact. Hence \( K_n \cap [0, \omega_1) \) is bounded in \([0, \omega_1)\). Thus there exists \( \alpha'_n < \omega_1 \) such that
\[
K_n \cap (\alpha'_n, \omega_1) = \emptyset.
\]
Let \( \alpha'' = \sup\{\alpha'_n : n \in \mathbb{N}\} \). Then \( \alpha'' < \omega_1 \) and
\[
\bigcup_{n \in \mathbb{N}} K_n \cap (\alpha'', \omega_1) = \emptyset.
\]
If we pick $\beta > \max\{\alpha', \alpha''\}$. Then $O_{\beta}(d_{\beta}) \cap K_n = \emptyset$ for each $n \in \mathbb{N}$. Hence $d_{\beta} \notin \text{St}(K_n, \mathcal{U}_n)$ for each $n \in \mathbb{N}$, since $O_{\beta}(d_{\beta})$ is the only element of $\mathcal{U}_n$ containing the point $d_{\beta}$ for each $n \in \mathbb{N}$, which shows that $X$ is not star-K-Menger. \hfill $\square$

2.5. Remark. The author does not know if there exists a Hausdorff (or Tychonoff) star-Menger space which is not star-K-Menger.

3. Properties of star-K-Menger spaces

In this section, we study topological properties of star-K-Menger spaces. The space $X$ of the proof of Example 2.2 shows that a closed subset of a Tychonoff star-K-Menger space $X$ need not be star-K-Menger, since $D \times \{c^+\}$ is a discrete closed subset of cardinality $\mathfrak{c}$. Now we give an example showing that a regular-closed subset of a Tychonoff star-K-Menger space $X$ need not be star-K-Menger. Here a subset $A$ of a space $X$ is said to be regular-closed in $X$ if $\text{cl}_{X \cap X'} A = A$.

3.1. Example. There exists a Tychonoff star-K-Menger space having a regular-closed subspace which is not star-K-Menger.

Proof. Let $D = \{d_{\alpha} : \alpha < \mathfrak{c}\}$ be a discrete space of cardinality $\mathfrak{c}$ and let $aD = D \cup \{d^*\}$ be one-point compactification of $D$.

Let $S_1$ be the same space $X$ in the proof of Example 2.2. Then $S_1$ is a Tychonoff star-K-Menger space.

Let
\[ S_2 = (aD \times [0, \mathfrak{c})) \cup (D \times \{\mathfrak{c}\}) \]
be the subspace of the product space $aD \times [0, \mathfrak{c}]$. To show that $S_2$ is not star-K-Menger. For each $\alpha < \mathfrak{c}$, let
\[ U_\alpha = \{d_{\alpha}\} \times (\mathfrak{c}, \mathfrak{c}] \] and $V_\alpha = aD \times [0, \alpha)$.

For each $n \in \mathbb{N}$, let
\[ \mathcal{U}_n = \{U_{\alpha} : \alpha < \mathfrak{c}\} \cup \{V_{\alpha} : \alpha < \mathfrak{c}\}. \]

Then $\mathcal{U}_n$ is an open cover of $S_2$. Let us consider the sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of $S_2$. It suffices to show that $\bigcup_{n \in \mathbb{N}} \text{St}(K_n, \mathcal{U}_n) \neq X$ for any sequence $(K_n : n \in \mathbb{N})$ of compact subsets of $X$. Let $(K_n : n \in \mathbb{N})$ be any sequence of compact subsets of $X$. For each $n \in \mathbb{N}$, since $K_n$ is compact and $\{\{d_{\alpha}, c\} : \alpha < \mathfrak{c}\}$ is a discrete closed subset of $S_2$, the set $A_n = \{\alpha : \langle d_{\alpha}, c\rangle \in K_n\}$ is finite. Let
\[ K'_n = K_n \setminus \bigcup \{U_{\alpha} : \alpha \in A_n\}. \]

If $K'_n = \emptyset$. Then there exists $\alpha_n < \mathfrak{c}$ such that
\[ K_n \cap U_{\alpha_n} = \emptyset \text{ for each } \alpha > \alpha'_n. \]

If $K'_n \neq \emptyset$. Since $K'_n$ is closed in $K_n$, $K'_n$ is compact and $K'_n \subseteq aD \times [0, \mathfrak{c}]$. Then $\pi(K'_n)$ is a compact subset of a countable compact space $[0, \mathfrak{c}]$, where $\pi : aD \times [0, \mathfrak{c}] \to [0, \mathfrak{c}]$ is the projection. Hence $\pi(K'_n)$ is bounded in $[0, \mathfrak{c}]$. Thus there exists $\beta_n < \mathfrak{c}$ such that $\pi(K'_n) \cap (\beta_n, \mathfrak{c}) = \emptyset$. Choose $\alpha''_n > \max\{\alpha : \alpha \in A_n\} \cup \{\beta_n\}$. Then
\[ U_{\alpha'} \cap K_n = \emptyset \text{ for each } \alpha > \alpha''_n. \]

Hence, for each $n \in \mathbb{N}$ either $K'_n = \emptyset$ or $K'_n \neq \emptyset$, there exists $\alpha_n < \mathfrak{c}$ such that
\[ U_{\alpha} \cap K_n = \emptyset \text{ for each } \alpha > \alpha_n. \]

Let $\beta_0 = \sup\{\alpha_n : n \in \mathbb{N}\}$. Then $\beta_0 < \mathfrak{c}$ and
\[ U_{\alpha} \cap K_n = \emptyset \text{ for each } \alpha > \beta_0 \text{ and each } n \in \mathbb{N}. \]
If we pick \( \alpha' > \beta_0 \). Then
\[
U_{\alpha'} \cap K_n = \emptyset \text{ for each } n \in \mathbb{N}.
\]
Hence
\[
\langle d_{\alpha'}, c \rangle \notin \text{St}(K_n, \mathcal{U}_n) \text{ for each } n \in \mathbb{N},
\]
since \( U_{\alpha'} \) is the only element of \( \mathcal{U}_n \) containing the point \( \langle d_{\alpha'}, c \rangle \) for each \( n \in \mathbb{N} \), which shows that \( S_2 \) is not star-K-Menger.

We assume \( S_1 \cap S_2 = \emptyset \). Let \( \pi : D \times \{ c^+ \} \rightarrow D \times \{ c \} \) be a bijection and let \( X \) be the quotient image of the disjoint sum \( S_1 \oplus S_2 \) by identifying \( \langle d_\alpha, c^+ \rangle \) of \( S_1 \) with \( \pi(\langle d_\alpha, c^+ \rangle) \) of \( S_2 \) for every \( \alpha < c \). Let \( \varphi : S_1 \oplus S_2 \rightarrow X \) be the quotient map. It is clear that \( \varphi(S_2) \) is a regular-closed subspace of \( X \) which is not star-K-Menger, since it is homeomorphic to \( S_2 \).

Finally we show that \( X \) is star-K-Menger; we only show that \( X \) is K-starcompact, since every K-starcompact space is star-K-Menger. To this end, let \( \mathcal{U} \) be an open cover of \( X \). Since \( \varphi(S_1) \) is homeomorphic to \( S_1 \) and consequently \( \varphi(S_1) \) is K-starcompact. Thus there exists a compact subset \( K_1 \) of \( \varphi(S_1) \) such that
\[
\varphi(S_1) \subseteq \text{St}(K_1, \mathcal{U}).
\]
Since \( \varphi(aD \times [0, c)) \) is homeomorphic to \( aD \times [0, c) \), the set \( \varphi(aD \times [0, c)) \) is countably compact, hence it is strongly starcompact (see [1,6]). Thus we can find a finite subset \( K_2 \) of \( \varphi(aD \times [0, c)) \) such that
\[
\varphi(aD \times [0, c)) \subseteq \text{St}(K_2, \mathcal{U}).
\]
If we put \( K = K_1 \cup K_2 \). Then \( K \) is a compact subset of \( X \) such that \( X = \text{St}(K, \mathcal{U}) \), which shows that \( X \) is K-starcompact.

Since a continuous image of a K-starcompact space is K-starcompact, it is not difficult to show the following result.


Next we turn to consider preimages. To show that the preimage of a star-K-Menger space under a closed 2-to-1 continuous map need not be star-K-Menger, we use the the Alexandroff duplicate \( A(X) \) of a space \( X \). The underlying set \( A(X) \) is \( X \times \{0, 1\} \); each point of \( X \times \{1\} \) is isolated and a basic neighborhood of \( \langle x, 0 \rangle \in X \times \{0\} \) is a set of the form \( (U \times \{0\}) \cup ((U \times \{1\}) \setminus \{\langle x, 0 \rangle\}) \), where \( U \) is a neighborhood of \( x \) in \( X \).

3.3. Example. There exists a closed 2-to-1 continuous map \( f : X \rightarrow Y \) such that \( Y \) is a star-K-Menger space, but \( X \) is not star-K-Menger.

Proof. Let \( Y \) be the same space \( X \) in the proof of Example 2.2. As we proved in Example 2.2 above, \( Y \) is star-K-Menger. Let \( X \) be the Alexandroff duplicate \( A(Y) \). Then \( X \) is not star-K-Menger. In fact, let \( A = \{ \langle d_\alpha, c^+ \rangle, 1 : \alpha < c \} \). Then \( A \) is an open and closed subset of \( X \) with \( |A| = c \), and each point \( \langle d_\alpha, c^+ \rangle, 1 \) is isolated. Hence \( A(X) \) is not star-K-Menger, since every open and closed subset of a star-K-Menger space is star-K-Menger and \( A \) is not star-K-Menger. Let \( f : X \rightarrow Y \) be the projection. Then \( f \) is a closed 2-to-1 continuous map, which completes the proof.

Now, we give a positive result:

3.4. Theorem. Let \( f \) be an open perfect map from a space \( X \) to a star-K-Menger space \( Y \). Then \( X \) is star-K-Menger.
Proof. Since \( f(X) \) is open and closed in \( Y \), we may assume that \( f(X) = Y \). Let \( \{U_n : n \in \mathbb{N}\} \) be a sequence of open covers of \( X \) and let \( y \in Y \). For each \( n \in \mathbb{N} \), since \( f^{-1}(y) \) is compact, there exists a finite subcollection \( \cup_{n \in \mathbb{N}} U_n \) of \( U_n \) such that \( f^{-1}(y) \subseteq \bigcup U_{n \subseteq \mathbb{N}} \) and \( U \cap f^{-1}(y) \neq \emptyset \) for each \( U \in U_n \). Pick an open neighborhood \( V_n \) of \( y \) in \( Y \) such that \( f^{-1}(V_n) \subseteq \bigcup \{U : U \in U_n\} \), then we can assume that

\[(3.1) \quad V_n \subseteq \bigcap \{f(U) : U \in U_n\}, \]

because \( f \) is open. For each \( n \in \mathbb{N} \), taking such open set \( V_n \) for each \( y \in Y \), we have an open cover \( V = \{V_n : y \in Y\} \) of \( Y \). Thus \( \{V_n : n \in \mathbb{N}\} \) is a sequence of open covers of \( Y \), there exists a sequence \( \{K_n : n \in \mathbb{N}\} \) of compact subsets of \( Y \) such that \( (St(K_n, V_n) : n \in \mathbb{N}) \) is an open cover of \( Y \), since \( Y \) is star-K-Menger. Since \( f \) is perfect, the sequence \( \{f^{-1}(K_n) : n \in \mathbb{N}\} \) is the sequence of compact subsets of \( X \). To show that \( \{St(f^{-1}(K_n), U_n) : n \in \mathbb{N}\} \) is an open cover of \( X \). Let \( x \in X \). Then there exists a \( n \in \mathbb{N} \) and \( y \in Y \) such that \( f(x) \in V_n \) and \( V_n \cap K_n \neq \emptyset \). Since

\[x \in f^{-1}(V_n) \subseteq \bigcup \{U : U \in U_n\} \]

we can choose \( U \in U_n \) with \( x \in U \). Then \( V_n \subseteq f(U) \) by \( (3.1) \), and hence \( U \cap f^{-1}(K_n) \neq \emptyset \). Therefore \( x \in St(f^{-1}(K_n), U_n) \). Consequently, we have \( \{St(f^{-1}(K_n), U_n) : n \in \mathbb{N}\} \) is an open cover of \( X \), which shows that \( X \) is star-K-Menger. \( \square \)

By Theorem 3.4 we have the following corollary.

**3.5. Corollary.** Let \( X \) be a star-K-Menger space and \( Y \) a compact space. Then \( X \times Y \) is star-K-Menger.

However, the product of two star-K-Menger spaces need not be star-K-Menger. In fact, the following well-known example showing that the product of two countably compact (and hence star-K-Menger) spaces need not be star-K-Menger. Here we give a rough proof for the sake of completeness. For a Tychonoff space \( X \), let \( \beta X \) denote the Čech-Stone compactification of \( X \).

**3.6. Example.** There exists two countably compact spaces \( X \) and \( Y \) such that \( X \times Y \) is not star-K-Menger.

**Proof.** Let \( D \) be a discrete space of cardinality \( c \). We can define \( X = \bigcup_{\alpha < \omega_1} E_\alpha \) and \( Y = \bigcup_{\alpha < \omega_1} F_\alpha \), where \( E_\alpha \) and \( F_\alpha \) are the subsets of \( \beta D \) which are defined inductively so as to satisfy the following conditions (1), (2) and (3):

1. \( E_\alpha \cap F_\beta = D \) if \( \alpha \neq \beta \);
2. \( |E_\alpha| \leq c \) and \( |F_\beta| \leq c \);
3. every infinite subset of \( E_\alpha \) (resp., \( F_\alpha \)) has an accumulation point in \( E_{\alpha+1} \) (resp., \( F_{\alpha+1} \)).

These sets \( E_\alpha \) and \( F_\alpha \) are well-defined since every infinite closed set in \( \beta D \) has cardinality at least \( 2^c \) (see [7]). Then \( X \times Y \) is not star-K-Menger, because the diagonal \( \{(d, d) : d \in D\} \) is a discrete open and closed subset of \( X \times Y \) with cardinality \( c \) and the open and closed subsets of star-K-Menger spaces are star-K-Menger. \( \square \)

In [1, Example 3.3.3], van Douwen-Reed-Roscoe-Tree gave an example showing that there exist a countably compact space \( X \) and a Lindelöf space \( Y \) such that \( X \times Y \) is not strongly star-Lindelöf. Now, we shall show that the product space \( X \times Y \) is not star-K-Menger.

**3.7. Example.** There exist a countably compact (and hence star-K-Menger) space \( X \) and a Lindelöf space \( Y \) such that \( X \times Y \) is not star-K-Menger.
Proof. Let $X = [0, \omega_1)$ with the usual order topology and $Y = \omega_1 + 1$ with the following topology: each point $\alpha$ with $\alpha < \omega_1$ is isolated and a set $U$ containing $\omega_1$ is open if and only if $Y \setminus U$ is countable. Then $X$ is countably compact and $Y$ is Lindelöf. Now, we show that $X \times Y$ is not star-$K$-Menger. For each $\alpha < \omega_1$, let

$$U_\alpha = [0, \alpha] \times [\alpha, \omega_1) \quad \text{and} \quad V_\alpha = (\alpha, \omega_1) \times \{\alpha\}.$$ 

For each $n \in \mathbb{N}$, let

$$U_n = \{U_\alpha : \alpha < \omega_1\} \cup \{V_\alpha : \alpha < \omega_1\}.$$ 

Then $U_n$ is an open cover of $X \times Y$. Let us consider the sequence $(U_n : n \in \mathbb{N})$ of the open covers of $X \times Y$. It suffices to show that $\bigcup_{n \in \mathbb{N}} St(K_n, U_n) \neq X \times Y$ for any sequence $(K_n : n \in \mathbb{N})$ of compact subsets of $X \times Y$. For each $n \in \mathbb{N}$, since $K_n$ is compact, then $\pi(K_n)$ is a compact subset of $X$, where $\pi : X \times Y \to X$ is the projection. Thus there exists $\alpha_n < \omega_1$ such that

$$(K_n \cap ((\alpha_n, \omega_1) \times Y) = \emptyset.$$ 

Let $\beta = \sup\{\alpha_n : n \in \mathbb{N}\}$. Then $\beta < \omega_1$ and

$$(\bigcup_{n \in \mathbb{N}} K_n) \cap ((\beta, \omega_1) \times Y) = \emptyset.$$ 

If we pick $\alpha > \beta$. Then $\langle \alpha + 1, \alpha \rangle \notin St(K_n, U_n)$ for each $n \in \mathbb{N}$, since $V_\alpha$ is the only element of $U_n$ containing the point $\langle \alpha + 1, \alpha \rangle$ for each $n \in \mathbb{N}$, which shows that $X \times Y$ is not star-$K$-Menger. \qed

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