FIXED POINT THEOREMS FOR A THIRD POWER TYPE CONTRACTION MAPPINGS IN G-METRIC SPACES

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Abstract

In this paper, we introduce a new third power type contractive condition in the G-metric spaces, and several new fixed point theorems are established in complete G-metric space. The obtained results in this paper extend the recent relative results.

Keywords: G-metric space, third power type contraction mappings, fixed point.


1. Introduction

Metric fixed point theory is an important mathematical discipline because of its applications in areas as variational and linear inequalities, optimization theory. In 1992, Dhage[2] introduced the concept of D-metric space. Unfortunately, it was shown that certain theorems involving Dhage’s D-metric spaces are flawed, and most of the results claimed by Dhage and others are invalid. These errors are pointed out by Mustafa and Sims[7]. In 2006, a new structure of generalized metric spaces was introduced by Mustafa and Sims[8] as appropriate notion of generalized metric space called G-metric spaces. Some other papers dealing with G-metric spaces are those in[1], [3]-[6], [9]-[11]. In this paper, we will prove some general fixed point theorems for third power type contractions mapping in complete G-metric spaces.

Throughout the paper, we mean by \( \mathbb{N} \) the set of all natural numbers.

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1.1. **Definition** (see[8]). Let $X$ be a nonempty set, and let $G : X \times X \times X \rightarrow R^+$ be a function satisfying the following axioms:

- $(G1)$ $G(x, y, z) = 0$ if $x = y = z$;
- $(G2)$ $0 < G(x, x, y)$, for all $x, y \in X$ with $x \neq y$;
- $(G3)$ $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$;
- $(G4)$ $G(x, y, z) = G(x, z, y) = G(y, z, x) = \ldots$ (symmetry in all three variables);
- $(G5)$ $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$. (rectangle inequality)

Then the function $G$ is called a generalized metric, or, more specifically a $G$-metric on $X$ and the pair $(X, G)$ is called a $G$-metric space.

1.2. **Definition** (see[8]). Let $(X, G)$ be a $G$-metric space, and let $\{x_n\}$ be a sequence of points in $X$, a point $x$ in $X$ is said to be the limit of the sequence $\{x_n\}$ if $\lim_{n \to \infty} G(x, x_n, x_m) = 0$, and one says that sequence $\{x_n\}$ is $G$-convergent to $x$.

Thus, if $x_n \to x$ in a $G$-metric space $(X, G)$, then for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x, x_n, x_m) < \epsilon$, for all $n, m \geq N$.

1.3. **Proposition** (see[8]). Let $(X, G)$ be a $G$-metric space, then the followings are equivalent:

- $(1)$ $x_n$ is $G$-convergent to $x$.
- $(2)$ $G(x_n, x, x) \to 0$ as $n \to \infty$.
- $(3)$ $G(x, x_n, x) \to 0$ as $n \to \infty$.
- $(4)$ $G(x_n, x_n, x) \to 0$ as $n, m \to \infty$.

1.4. **Definition** (see[8]). Let $(X, G)$ be a $G$-metric space. A sequence $\{x_n\}$ is called $G$-Cauchy sequence if, for each $\epsilon > 0$ there exists a positive integer $N \in \mathbb{N}$ such that $G(x_n, x_m, x_l) < \epsilon$ for all $n, m, l \geq N$; i.e. if $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to \infty$.

1.5. **Definition** (see[8]). A $G$-metric space $(X, G)$ is said to be $G$-complete if every $G$-Cauchy sequence in $(X, G)$ is $G$-convergent in $X$.

1.6. **Proposition** (see[8]). Let $(X, G)$ be a $G$-metric space. Then the followings are equivalent.

- $(1)$ The sequence $\{x_n\}$ is $G$-Cauchy.
- $(2)$ For every $\epsilon > 0$, there is $k \in \mathbb{N}$ such that $G(x_n, x_m, x_k) < \epsilon$, for all $n, m \geq k$.

1.7. **Proposition** (see[8]). Let $(X, G)$ be a $G$-metric space. Then the function $G(x, y, z)$ is jointly continuous in all three of its variables.

1.8. **Definition** (see[8]). Let $(X, G)$ and $(X', G')$ be $G$-metric space, and $f : (X, G) \to (X', G')$ be a function. Then $f$ is said to be $G$-continuous at a point $a \in X$ if and only if for every $\epsilon > 0$, there is $\delta > 0$ such that $x, y \in X$ and $G(a, x, y) < \delta$ implies $G(f(a), f(x), f(y)) < \epsilon$. A function $f$ is $G$-continuous at $X$ if and only if it is $G$-continuous at all $a \in X$.

1.9. **Proposition** (see[8]). Let $(X, G)$ and $(X', G')$ be $G$-metric space. Then $f : X \to X'$ is $G$-continuous at $x \in X$ if and only if it is $G$-sequentially continuous at $x$, that is, whenever $\{x_n\}$ is $G$-convergent to $x$, $\{f(x_n)\}$ is $G$-convergent to $f(x)$.

1.10. **Proposition** (see[8]). Let $(X, G)$ be a $G$-metric space. Then, for any $x, y, z, a$ in $X$ it follows that:

- $(i)$ if $G(x, y, z) = 0$, then $x = y = z$,
- $(ii)$ $G(x, y, z) \leq G(x, x, y) + G(x, x, z)$,
- $(iii)$ $G(x, y, z) \leq 2G(y, x, x)$,
- $(iv)$ $G(x, y, z) \leq G(x, a, z) + G(a, y, z)$,
- $(v)$ $G(x, y, z) \leq \frac{1}{2}(G(x, y, a) + G(x, a, z) + G(a, y, z))$,
- $(vi)$ $G(x, y, z) \leq (G(x, a, a) + G(y, a, a) + G(z, a, a))$. 


2. Main Results

2.1. Theorem. Let \((X,G)\) be a complete \(G\)-metric space. Suppose the map \(T: X \to X\) satisfies

\[
G^3(Tx,Ty,Tz) \leq qG(x,Ty,Tz)G(y, Tx, Tx)G(z, Tx, Tx)
\]

for all \(x, y, z \in X\), where \(0 \leq q < 1\). Then \(T\) has a unique fixed point (say \(u\)) and \(T\) is \(G\)-continuous at \(u\).

Proof. Let \(x_0 \in X\) be arbitrary point, and define the sequence \(\{x_n\} \) by \(x_n = T^n x_0 = Tx_{n-1}\), \(n \in \mathbb{N}\). Assume \(x_n \neq x_{n+1}\), for each \(n \in \mathbb{N}\).

First, we prove the sequence \(\{x_n\}\) is a \(G\)-Cauchy sequence. In fact, by (2.1), we have

\[
G^3(x_n, x_{n+1}, x_{n+1}) \leq qG(x_{n-1}, x_n, x_n)G(x_n, x_{n+1}, x_{n+1})G(x_n, x_{n+1}, x_{n+1}) = G^3(Tx_{n-1}, Tx_n, Tx_n).
\]

Thus, we have

\[
G(x_n, x_{n+1}, x_{n+1}) \leq qG(x_{n-1}, x_n, x_n) \leq \cdots \leq q^n G(x_0, x_1, x_1).
\]

For every \(m, n \in \mathbb{N}\), \(m > n\), using (G5) and (2.2), we have

\[
G(x_n, x_m, x_m) \leq G(x_{n+1}, x_{n+1}) + G(x_{n+2}, x_{n+2}) + \cdots + G(x_{m-1}, x_m, x_m) \leq (q^n + q^{n+1} + \cdots + q^{m-1})G(x_0, x_1, x_1)
\]

and so \(G(x_n, x_m, x_n) \to 0\), as \(n, m \to \infty\). Thus \(\{x_n\}\) is \(G\)-Cauchy sequence. Due to the completeness of \((X,G)\), there exists \(u \in X\) such that \(\{x_n\}\) is \(G\)-convergent to \(u\).

On the other hand, using (2.1), we have

\[
G^3(x_n, x_n, Tu) = G^3(Tx_{n-1}, Tx_{n-1}, Tu)
\]

\[
\leq qG(x_{n-1}, x_n, G(x_n, x_{n+1}, x_{n+1})G(u, Tu, Tu)
\]

Letting \(n \to \infty\), and using the fact that \(G\) is continuous on its variable, we get that

\[
G^3(u, u, Tu) = 0.
\]

Therefore, \(Tu = u\), hence \(u\) is a fixed point of \(T\). Now, let \(v\) be an another fixed point of \(T\), then we have

\[
G^3(u, u, v) = G^3(Tu, Tu, Tv)
\]

\[
\leq qG(u, Tu, Tu)G(u, Tu, Tu)G(v, Tv, Tv)
\]

\[
= 0.
\]

Thus, \(u = v\). Then we know the fixed point of \(T\) is unique.

To show that \(T\) is \(G\)-continuous at \(u\), let \(\{y_n\}\) be any sequence in \(X\) such that \(\{y_n\}\) is \(G\)-convergent to \(u\). For \(n \in \mathbb{N}\), we have

\[
G^3(u, u, Ty_n) = G^3(Tu, Tu, Ty_n) \leq qG(u, Tu, Tu)G(u, Tu, Tu)G(y_n, Ty_n, Ty_n)
\]

Letting \(n \to \infty\), we get \(\lim_{n \to \infty} G(u, u, Ty_n) = 0\). Hence \(\{Ty_n\}\) is \(G\)-convergent to \(u = Tu\). So \(T\) is \(G\)-continuous at \(u\).

\[
\square
\]

2.2. Corollary. Let \((X,G)\) be a complete \(G\)-metric space. Suppose the map \(T: X \to X\) satisfies

\[
G^3(T^p x, T^p y, T^p z) \leq qG(x, T^p x, T^p x)G(y, T^p y, T^p y)G(z, T^p z, T^p z)
\]

for all \(x, y, z \in X\), where \(0 \leq q < 1\), \(p \in \mathbb{N}\). Then \(T\) has a unique fixed point (say \(u\)) and \(T^p\) is \(G\)-continuous at \(u\).
2.4. Theorem. Let $x_0 \in X$ be arbitrary point, and define the sequence \{x_n\} by $x_n = T^n x_0 = T^{n-1} x_{n-1}$, $n \in \mathbb{N}$. Assume $x_n \neq x_{n+1}$, for each $n \in \mathbb{N}$.

First, we prove the sequence \{x_n\} is a $G$-Cauchy sequence. In fact, by (2.3), we have

\[
G^3(x_n, x_{n+1}, x_{n+2}) = G^3(Tx_{n-1}, T^2 x_{n-2}, T^3 x_{n-3}) 
\leq qG(x_{n-1}, x_n) G(x_n, x_{n+1}) G(x_{n+1}, x_{n+2}).
\]

On the other hand, using (G3), we have

\[
G(x_{n-1}, x_n, x_n) \leq G(x_{n-1}, x_n, x_{n+1}), \\
G(x_n, x_{n+1}, x_{n+1}) \leq G(x_n, x_{n+1}, x_{n+2}), \\
G(x_{n+1}, x_{n+2}, x_{n+2}) \leq G(x_{n+1}, x_{n+1}, x_{n+2}).
\]

Thus, we have

\[
G^3(x_n, x_{n+1}, x_{n+2}) \leq qG(x_{n-1}, x_n, x_{n+1}) G^2(x_n, x_{n+1}, x_{n+2}).
\]

Therefore, we can get

\[
(2.4) \quad G(x_n, x_{n+1}, x_{n+2}) \leq qG(x_{n-1}, x_n, x_{n+1}) \leq \cdots \leq q^n G(x_0, x_1, x_2).
\]

Moreover, for all $n, m \in \mathbb{N}$, $n < m$, by (G3), (G5) and (2.4) we have

\[
G(x_n, x_m, x_m) \leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) + \cdots + G(x_{m-1}, x_m, x_m), \\
\leq G(x_{n+1}, x_{n+2}) + G(x_{n+1}, x_{n+2}) + \cdots + G(x_{m-1}, x_m, x_{m+1}), \\
\leq (q^n + q^{n+1} + \cdots + q^{m-1}) G(x_0, x_1, x_2) \\
\leq \frac{q^n}{1-q} G(x_0, x_1, x_2).
\]

That means the sequence \{x_n\} is a $G$-Cauchy sequence. Due to the completeness of $(X, G)$, there exists $u \in X$ such that \{x_n\} is $G$ -converge to $u$. Furthermore, since $T$ is $G$-continuous, from $x_{n+1} = Tx_n$, letting $n \to \infty$ at both sides, we have $u = T u$. Thus, $u$ is a fixed point of $T$. □

2.4. Theorem. Let $(X, G)$ be a complete $G$-metric space and let $T : X \to X$ be a $G$-continuous mapping, which satisfies the following condition:

\[
G^3(Tx, T^2 y, T^3 z) \leq qG(x, Tx, Tx) G(Tx, T^2 y, T^2 y) G(Ty, T^3 z, T^3 z)
\]

for all $x, y, z \in X$, where $0 \leq q < 1$. Then $T$ has a unique fixed point (say $u$) and $T$ is $G$-continuous at $u$.

Proof. Let $x_0 \in X$ be arbitrary point, and define the sequence \{x_n\} by $x_n = T^n x_0 = T^{n-1} x_{n-1}$, $n \in \mathbb{N}$. Assume $x_n \neq x_{n+1}$, for each $n \in \mathbb{N}$.

First, we prove the sequence \{x_n\} is a $G$-Cauchy sequence. In fact, by (2.5), we have

\[
G^3(x_n, x_{n+1}, x_{n+1}) = G^3(Tx_{n-1}, T^2 x_{n-1}, T^3 x_{n-2}) \\
\leq qG(x_{n-1}, x_n, x_n) G(x_n, x_{n+1}, x_{n+1}) G(x_{n+1}, x_{n+1}, x_{n+1}).
\]
Therefore, we can get
\[(2.6) \quad G(x_n, x_{n+1}) \leq qG(x_{n-1}, x_n) \leq \cdots \leq q^n G(x_0, x_1).
\]
Moreover, for all \(n, m \in N, n < m\), by (G3), (G5) and (2.6) we have
\[
G(x_n, x_m) \leq G(x_n, x_{n+1}) + G(x_{n+1}, x_{n+2}) + \cdots + G(x_m, x_m),
\]
\[
\leq G(x_n, x_{n+1}) + G(x_{n+1}, x_{n+2}) + G(x_{n+2}, x_{n+3}) + \cdots + G(x_m, x_m),
\]
\[
\leq (q^n + q^{n+1} + \cdots + q^{m-1}) G(x_0, x_1).
\]
That means the sequence \(\{x_n\}\) is a \(G\)-Cauchy sequence. Due to the completeness of \((X, G)\), there exists \(u \in X\) such that \(\{x_n\}\) is \(G\)-convergent to \(u\).

On the other hand, using (2.5), we have
\[
G^3(Tu, x_{n+1}, x_{n+1}) = G^3(Tu, T^2 x_{n-1}, T^3 x_{n-2})
\]
\[
\leq qG(u, Tu, Tu) G(Tu, x_{n+1}, x_{n+1}) G(x_n, x_{n+1}, x_{n+1})
\]
Letting \(n \to \infty\), and using the fact that \(G\) is continuous on its variable, we get that
\[
G^3(Tu, u, u) = 0.
\]
Therefore, \(Tu = u\), hence \(u\) is a fixed point of \(T\). Now, let \(v\) be another fixed point of \(T\), then we have
\[
G^3(u, u, v) = G^3(Tu, T^2 u, T^3 v)
\]
\[
\leq qG(u, Tu, Tu) G(Tu, T^2 u, T^3 v) G(Tu, T^2 u, T^3 v)
\]
\[
= qG(u, u, u) G(u, u, v) = 0.
\]
Thus, \(u = v\). Then we know the fixed point of \(T\) is unique.

To show that \(T\) is \(G\)-continuous at \(u\), let \(\{y_n\}\) be any sequence in \(X\) such that \(\{y_n\}\) is \(G\)-convergent to \(u\). For \(n \in N\), we have
\[
G^3(Ty_n, u, u) = G^3(Ty_n, T^2 u, T^3 u) \leq qG(y_n, Ty_n, Ty_n) G(Ty_n, T^2 u, T^3 u) G(Tu, T^3 u, T^3 u)
\]
Letting \(n \to \infty\), we get \(\lim_{n \to \infty} G(Ty_n, u, u) = 0\). Hence \(\{Ty_n\}\) is \(G\)-convergent to \(u = Tu\). So \(T\) is \(G\)-continuous at \(u\). \(\square\)

References


