

A CLASS OF MULTIVARIABLE POLYNOMIALS ASSOCIATED WITH HUMBERT POLYNOMIALS

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Received 20:06:2012 : Accepted 08:08:2012

Abstract

In this paper, we present a generalization (and unification) of a class of Humbert polynomials which include well known families of Chan-Chyan-Srivastava, Lagrange-Hermite and Erkus-Srivastava multivariable polynomials. We derive various families of multilateral and multilinear generating functions for these polynomials. We also obtain other miscellaneous properties of these polynomials. Furthermore, for some special cases of these polynomials, we present hypergeometric representations and give expansions of these polynomials in series of some orthogonal polynomials.

Keywords: Humbert polynomials; Chan-Chyan-Srivastava multivariable polynomials; Lagrange-Hermite multivariable polynomials; Erkus-Srivastava multivariable polynomials; Multilinear and multilateral generating function; Recurrence relation; Hypergeometric function

2000 AMS Classification: 33C45

1. Introduction

An interesting generalization of Humbert, Gegenbauer, Legendre, Tchebycheff, Pincherle and Kinney polynomials, which is called generalized Humbert polynomials, was presented by Gould [10] and it is generated by

$$(1.1) \quad (c - mx + yt^m)^p = \sum_{n=0}^{\infty} P_n(m, x, y, p, c) t^n$$

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where m is a positive integer and other parameters are unrestricted in general (see also [20, p. 77, 86] and [19, 21]). For $c = 1$, $p = -\nu$ and $y = 1$, (1.1) reduces to Humbert polynomials $h_{n,m}^\nu(x)$, generated by

$$(1 - mxt + t^m)^{-\nu} = \sum_{n=0}^{\infty} h_{n,m}^\nu(x) t^n.$$

The polynomials $\{p_{n,m}^\lambda\}_{n=0}^\infty$ considered by Milovanović and Dordević [14, 15] are defined by

$$(1.2) \quad (1 - 2xt + t^m)^{-\lambda} = \sum_{n=0}^{\infty} p_{n,m}^\lambda(x) t^n$$

where m is a positive integer and $\lambda > -1/2$. For the special cases of (1.2), including Horadam polynomials, Horadam-Pethe polynomials and Gegenbauer polynomials, see [11, 12, 17]. Sinha [18] introduced the polynomials $S_n^\nu(x)$, generated by

$$(1.3) \quad [1 - 2xt + t^2(2x - 1)]^{-\nu} = \sum_{n=0}^{\infty} S_n^\nu(x) t^n.$$

Pathan and Khan [16] considered a generalization of polynomials mentioned above, defined by

$$(1.4) \quad [c - ax + bt^m(2x - 1)^d]^{-\nu} = \sum_{n=0}^{\infty} p_{n,m,a,b,c,d}^\nu(x) t^n.$$

In [1], Aktaş *et al* presented a multivariable generalization of Humbert polynomials including Chan-Chyan-Srivastava, Lagrange-Hermite and Erkus-Srivastava multivariable polynomials, generated by

$$(1.5) \quad \prod_{i=1}^r \left\{ (c_i - m_i x_i t + y_i t^{m_i})^{-\alpha_i} \right\} = \sum_{n=0}^{\infty} P_n^{(\alpha_1, \dots, \alpha_r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{c}) t^n$$

$$(|m_i x_i t - y_i t^{m_i}| < |c_i|, \quad c_i \neq 0, \quad \alpha_i \in \mathbb{C}; \quad i = 1, 2, \dots, r)$$

where $\mathbf{x} = (x_1, \dots, x_r)$, $\mathbf{y} = (y_1, \dots, y_r)$, $\mathbf{c} = (c_1, \dots, c_r)$, $\mathbf{m} = (m_1, \dots, m_r)$. For $i = 1, 2, \dots, r$; m_i is a positive integer and other parameters are unrestricted. (1.5) yields the following explicit representation:

$$(1.6) \quad P_n^{(\alpha_1, \dots, \alpha_r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{c}) = \sum_{m_1 k_1 + \dots + m_r k_r + n_1 + \dots + n_r = n} \frac{(\alpha_1)_{n_1+k_1} \dots (\alpha_r)_{n_r+k_r}}{n_1! \dots n_r! k_1! \dots k_r!} c_1^{-\alpha_1 - n_1 - k_1} \dots c_r^{-\alpha_r - n_r - k_r}$$

$$\times m_1^{n_1} \dots m_r^{n_r} (-1)^{k_1 + \dots + k_r} x_1^{n_1} \dots x_r^{n_r} y_1^{k_1} \dots y_r^{k_r}$$

where, as usual, $(\lambda)_k$ denotes the Pochhammer symbol given by

$$(\lambda)_k := \frac{\Gamma(\lambda + k)}{\Gamma(\lambda)} \quad (k \in \mathbb{N}_0 := \{0, 1, 2, 3, \dots\}).$$

Note that the case

$$c_i = 1, \quad m_i = 1, \quad y_i = 0, \quad i = 1, 2, \dots, r$$

of the polynomials given by (1.5) is reduced to the Chan-Chyan-Srivastava multivariable polynomials which is a multivariable extension of the Lagrange polynomials (see [8, p.

267]), generated by [4]

$$\prod_{i=1}^r \{(1 - x_i t)^{-\alpha_i}\} = \sum_{n=0}^{\infty} g_n^{(\alpha_1, \dots, \alpha_r)}(x_1, \dots, x_r) t^n$$

(1.7) $(\alpha_i \in \mathbb{C} \ (i = 1, 2, \dots, r) ; |t| < \min \{|x_1|^{-1}, \dots, |x_r|^{-1}\}) .$

Getting $c_i = 1, m_i = i, x_i = 0, y_i = -x_i, i = 1, 2, \dots, r$ in (1.5) gives the multivariable Lagrange-Hermite polynomials presented by Altın *et al.* [3]

$$\prod_{i=1}^r \left\{ \left(1 - x_i t^i \right)^{-\alpha_i} \right\} = \sum_{n=0}^{\infty} h_n^{(\alpha_1, \dots, \alpha_r)}(x_1, \dots, x_r) t^n$$

(1.8) $(\alpha_i \in \mathbb{C} \ (i = 1, 2, \dots, r) ; |t| < \min \{|x_1|^{-1}, |x_2|^{-1/2}, \dots, |x_r|^{-1/r}\})$

which is a multivariable generalization of the familiar (two-variable) Lagrange-Hermite polynomials considered by Dattoli *et al.* [5, 6].

Moreover, the special case

$$c_i = 1, x_i = 0, y_i = -x_i, i = 1, 2, \dots, r$$

is reduced to the Erkus-Srivastava multivariable polynomials generated by [9]

$$\prod_{i=1}^r \left\{ \left(1 - x_i t^{m_i} \right)^{-\alpha_i} \right\} = \sum_{n=0}^{\infty} u_n^{(\alpha_1, \dots, \alpha_r)}(x_1, \dots, x_r) t^n$$

(1.9) $(\alpha_i \in \mathbb{C} \ (i = 1, 2, \dots, r) ; |t| < \min \{|x_1|^{-1/m_1}, |x_2|^{-1/m_2}, \dots, |x_r|^{-1/m_r}\}) ,$

where $m_i \ (i = 1, 2, \dots, r)$ are positive integers. A generalization (and unification) of various polynomials given above is provided by the definition

$$\prod_{i=1}^r \left\{ \left(c_i - a_i x_i t^{p_i} + b_i y_i^{d_i} t^{m_i} \right)^{-\alpha_i} \right\} = \sum_{n=0}^{\infty} P_{n, \mathbf{p}, \mathbf{m}, \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}}^{(\alpha_1, \dots, \alpha_r)}(\mathbf{x}, \mathbf{y}) t^n$$

$$= \sum_{n=0}^{\infty} \Phi_n^{(\alpha_1, \dots, \alpha_r)}(\mathbf{x}, \mathbf{y}) t^n$$

(1.10) $(|a_i x_i t^{p_i} - b_i y_i^{d_i} t^{m_i}| < |c_i|, c_i \neq 0, \alpha_i \in \mathbb{C} ; i = 1, 2, \dots, r)$

where $\mathbf{x} = (x_1, \dots, x_r), \mathbf{y} = (y_1, \dots, y_r), \mathbf{a} = (a_1, \dots, a_r), \mathbf{b} = (b_1, \dots, b_r), \mathbf{c} = (c_1, \dots, c_r), \mathbf{d} = (d_1, \dots, d_r), \mathbf{p} = (p_1, \dots, p_r), \mathbf{m} = (m_1, \dots, m_r)$. For $i = 1, 2, \dots, r; m_i, p_i$ and d_i are positive integers and other parameters are unrestricted. (1.10) yields

$$\Phi_n^{(\alpha_1, \dots, \alpha_r)}(\mathbf{x}, \mathbf{y}) = \sum_{m_1 k_1 + \dots + m_r k_r + p_1 n_1 + \dots + p_r n_r = n} \frac{(\alpha_1)_{n_1 + k_1} \dots (\alpha_r)_{n_r + k_r}}{n_1! \dots n_r! k_1! \dots k_r!} c_1^{-\alpha_1 - n_1 - k_1} \dots c_r^{-\alpha_r - n_r - k_r}$$

(1.11) $\times a_1^{n_1} \dots a_r^{n_r} b_1^{k_1} \dots b_r^{k_r} (-1)^{k_1 + \dots + k_r} x_1^{n_1} \dots x_r^{n_r} y_1^{d_1 k_1} \dots y_r^{d_r k_r} .$

In this paper, we give some basic relations for the generalized (unified) multivariable polynomials given explicitly by (1.11). We derive various families of multilinear and multilateral generating functions for $\Phi_n^{(\alpha_1, \dots, \alpha_r)}(\mathbf{x}, \mathbf{y})$ similar to method given in [1, 2, 3, 7, 9] and obtain several recurrence relations. We also give hypergeometric representations for some special cases of $\Phi_n^{(\alpha_1, \dots, \alpha_r)}(\mathbf{x}, \mathbf{y})$ and expansions of these polynomials in series of some orthogonal polynomials. Furthermore, we present some special cases of our

results and give some new results for Chan-Chyan-Srivastava, Lagrange-Hermite, Erkus-Srivastava multivariable polynomials and Humbert multivariable polynomials given by (1.6).

2. Bilinear and Bilateral Generating Functions

In this section, we consider many general families of bilinear and bilateral generating functions for the generalized (unified) multivariable polynomials $\Phi_n^{(\alpha_1, \dots, \alpha_r)}(\mathbf{x}, \mathbf{y})$ which are generated by (1.10) and given explicitly by (1.11).

We begin by stating the following theorem.

2.1. Theorem. *For a non-vanishing function $\Omega_\mu(\mathbf{z})$ of s complex variables z_1, \dots, z_s ($s \in \mathbb{N}$) and of complex order μ , let*

$$(2.1) \quad \Lambda_{\mu, \nu}(\mathbf{z}; w) := \sum_{k=0}^{\infty} a_k \Omega_{\mu+\nu k}(\mathbf{z}) w^k$$

where $(a_k \neq 0, \mu, \nu \in \mathbb{C})$; $\mathbf{z} = (z_1, \dots, z_s)$ and

$$(2.2) \quad \Theta_{n,p,\mu,\nu}(\mathbf{x}, \mathbf{y}; \mathbf{z}; \zeta) := \sum_{k=0}^{[n/p]} a_k \Phi_{n-pk}^{(\alpha_1, \dots, \alpha_r)}(\mathbf{x}, \mathbf{y}) \Omega_{\mu+\nu k}(\mathbf{z}) \zeta^k$$

where $n, p \in \mathbb{N}$; $\mathbf{x} = (x_1, \dots, x_r)$; $\mathbf{y} = (y_1, \dots, y_r)$. Then we have

$$(2.3) \quad \sum_{n=0}^{\infty} \Theta_{n,p,\mu,\nu}(\mathbf{x}, \mathbf{y}; \mathbf{z}; \frac{\eta}{t^p}) t^n = \prod_{i=1}^r \left\{ \left(c_i - a_i x_i t^{p_i} + b_i y_i^{d_i} t^{m_i} \right)^{-\alpha_i} \right\} \Lambda_{\mu, \nu}(\mathbf{z}; \eta)$$

provided that each member of (2.3) exists.

Proof. For convenience, let S denote the first member of the assertion (2.3) of Theorem 2.1. Then, upon substituting for the polynomials

$$\Theta_{n,p,\mu,\nu}(\mathbf{x}, \mathbf{y}; \mathbf{z}; \frac{\eta}{t^p})$$

from the definition (2.2) into the left-hand side of (2.3), we obtain

$$(2.4) \quad S = \sum_{n=0}^{\infty} \sum_{k=0}^{[n/p]} a_k \Phi_{n-pk}^{(\alpha_1, \dots, \alpha_r)}(\mathbf{x}, \mathbf{y}) \Omega_{\mu+\nu k}(\mathbf{z}) \eta^k t^{n-pk}.$$

Using the equality

$$\sum_{n=0}^{\infty} \sum_{k=0}^{[n/p]} A(k, n) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k, n + pk),$$

we may write

$$\begin{aligned} S &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_k \Phi_n^{(\alpha_1, \dots, \alpha_r)}(\mathbf{x}, \mathbf{y}) \Omega_{\mu+\nu k}(\mathbf{z}) \eta^k t^n \\ &= \sum_{n=0}^{\infty} \Phi_n^{(\alpha_1, \dots, \alpha_r)}(\mathbf{x}, \mathbf{y}) t^n \sum_{k=0}^{\infty} a_k \Omega_{\mu+\nu k}(\mathbf{z}) \eta^k \\ &= \prod_{i=1}^r \left\{ \left(c_i - a_i x_i t^{p_i} + b_i y_i^{d_i} t^{m_i} \right)^{-\alpha_i} \right\} \Lambda_{\mu, \nu}(\mathbf{z}; \eta), \end{aligned}$$

which completes the proof. □

In a similar manner, we can easily prove the following result.

2.2. Theorem. *Corresponding to an identically non-vanishing function $\Omega_\mu(\mathbf{z})$ of s complex variables z_1, \dots, z_s ($s \in \mathbb{N}$) and for $p \in \mathbb{N}$, $\mu, \nu \in \mathbb{C}$, $\mathbf{z} = (z_1, \dots, z_s)$, $\alpha := (\alpha_1, \dots, \alpha_r)$, $\beta := (\beta_1, \dots, \beta_r)$, let*

$$(2.5) \quad \Lambda_{\mu, \nu, \alpha, \beta}^{n, p}(\mathbf{x}, \mathbf{y}; \mathbf{z}; w) := \sum_{k=0}^{[n/p]} a_k \Phi_{n-pk}^{(\alpha_1 + \beta_1, \dots, \alpha_r + \beta_r)}(\mathbf{x}, \mathbf{y}) \Omega_{\mu + \nu k}(\mathbf{z}) w^k$$

where $a_k \neq 0$; $n, k \in \mathbb{N}_0$; $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Then

$$(2.6) \quad \sum_{k=0}^n \sum_{l=0}^{[k/p]} a_l \Phi_{n-k}^{(\alpha_1, \dots, \alpha_r)}(\mathbf{x}, \mathbf{y}) \Phi_{k-pl}^{(\beta_1, \dots, \beta_r)}(\mathbf{x}, \mathbf{y}) \Omega_{\mu + \nu l}(\mathbf{z}) w^l \\ = \Lambda_{\mu, \nu, \alpha, \beta}^{n, p}(\mathbf{x}, \mathbf{y}; \mathbf{z}; w)$$

provided that each member of (2.6) exists.

3. Special cases and miscellaneous properties

When the multivariable function $\Omega_{\mu + \nu k}(\mathbf{z})$, $\mathbf{z} = (z_1, \dots, z_s)$, $k \in \mathbb{N}_0$, $s \in \mathbb{N}$ is expressed in terms of simpler functions of one and more variables, then we can give further applications of the above theorems. For example, if we set

$$s = r \text{ and } \Omega_{\mu + \nu k}(\mathbf{z}) = u_{\mu + \nu k}^{(\beta_1, \dots, \beta_r)}(\mathbf{z})$$

in Theorem 2.1, where Erkus-Srivastava multivariable polynomials

$$u_n^{(\alpha_1, \dots, \alpha_r)}(\mathbf{x})$$

are generated by (1.9), then we obtain the following result which provides a class of bilateral generating functions for Erkus-Srivastava multivariable polynomials and the generalized (unified) multivariable polynomials $\Phi_n^{(\alpha_1, \dots, \alpha_r)}(\mathbf{x}, \mathbf{y})$ given explicitly by (1.11).

3.1. Corollary. *If $\Lambda_{\mu, \nu}(\mathbf{z}; w) := \sum_{k=0}^{\infty} a_k u_{\mu + \nu k}^{(\beta_1, \dots, \beta_r)}(\mathbf{z}) w^k$, $a_k \neq 0$, $\mu, \nu \in \mathbb{C}$, $\mathbf{z} = (z_1, \dots, z_r)$ and*

$$\Theta_{n, p, \mu, \nu}(\mathbf{x}, \mathbf{y}; \mathbf{z}; \zeta) := \sum_{k=0}^{[n/p]} a_k \Phi_{n-pk}^{(\alpha_1, \dots, \alpha_r)}(\mathbf{x}, \mathbf{y}) u_{\mu + \nu k}^{(\beta_1, \dots, \beta_r)}(\mathbf{z}) \zeta^k$$

where $n \in \mathbb{N}_0$; $p \in \mathbb{N}$; $\mathbf{x} = (x_1, \dots, x_r)$; $\mathbf{y} = (y_1, \dots, y_r)$, then

$$(3.1) \quad \sum_{n=0}^{\infty} \Theta_{n, p, \mu, \nu}(\mathbf{x}, \mathbf{y}; \mathbf{z}; \frac{\eta}{t^p}) t^n = \prod_{i=1}^r \left\{ \left(c_i - a_i x_i t^{p_i} + b_i y_i^{d_i} t^{m_i} \right)^{-\alpha_i} \right\} \Lambda_{\mu, \nu}(\mathbf{z}; \eta)$$

provided that each member of (3.1) exists.

3.2. Remark. Using the generating relation (1.9) for Erkus-Srivastava multivariable polynomials and getting $a_k = 1$, $\mu = 0$, $\nu = 1$, we find that

$$\sum_{n=0}^{\infty} \sum_{k=0}^{[n/p]} \Phi_{n-pk}^{(\alpha_1, \dots, \alpha_r)}(\mathbf{x}, \mathbf{y}) u_k^{(\beta_1, \dots, \beta_r)}(\mathbf{z}) \eta^k t^{n-pk} \\ = \prod_{i=1}^r \left\{ \left(c_i - a_i x_i t^{p_i} + b_i y_i^{d_i} t^{m_i} \right)^{-\alpha_i} (1 - z_i \eta^{n_i})^{-\beta_i} \right\}, \\ \left(|\eta| < \min \left\{ |z_1|^{-1/n_1}, \dots, |z_r|^{-1/n_r} \right\} ; \left| a_i x_i t^{p_i} - b_i y_i^{d_i} t^{m_i} \right| < |c_i| ; i = 1, 2, \dots, r \right)$$

where n_i, m_i, p_i and d_i ($i = 1, 2, \dots, r$) are positive integers.

By choosing $s = 2r$ and $\Omega_{\mu+\nu k}(\mathbf{z}) = \Phi_{\mu+\nu k}^{(\gamma_1, \dots, \gamma_r)}(\mathbf{t}, \omega)$, $\mu, \nu \in \mathbb{N}_0$, $\mathbf{t} = (t_1, \dots, t_r)$, $\omega = (\omega_1, \dots, \omega_r)$ in Theorem 2.2, we obtain the following class of bilinear generating functions for the polynomials $\Phi_n^{(\alpha_1, \dots, \alpha_r)}(\mathbf{x}, \mathbf{y})$ given explicitly by (1.11).

3.3. Corollary. *Let*

$$\begin{aligned} & \Xi_{\mu, \nu, \alpha, \beta, \gamma}^{n, p}(\mathbf{x}, \mathbf{y}; \mathbf{t}, \omega; w) \\ & := \sum_{k=0}^{[n/p]} a_k \Phi_{n-pk}^{(\alpha_1 + \beta_1, \dots, \alpha_r + \beta_r)}(\mathbf{x}, \mathbf{y}) \Phi_{\mu+\nu k}^{(\gamma_1, \dots, \gamma_r)}(\mathbf{t}, \omega) w^k \end{aligned}$$

where $a_k \neq 0$; $p \in \mathbb{N}$; $n, k, \mu, \nu \in \mathbb{N}_0$; $\alpha := (\alpha_1, \dots, \alpha_r)$; $\beta := (\beta_1, \dots, \beta_r)$; $\gamma := (\gamma_1, \dots, \gamma_r)$. Then

$$\begin{aligned} & \sum_{k=0}^n \sum_{l=0}^{[k/p]} a_l \Phi_{n-k}^{(\alpha_1, \dots, \alpha_r)}(\mathbf{x}, \mathbf{y}) \Phi_{k-pl}^{(\beta_1, \dots, \beta_r)}(\mathbf{x}, \mathbf{y}) \Phi_{\mu+\nu l}^{(\gamma_1, \dots, \gamma_r)}(\mathbf{t}, \omega) w^l \\ (3.2) \quad & = \Xi_{\mu, \nu, \alpha, \beta, \gamma}^{n, p}(\mathbf{x}, \mathbf{y}; \mathbf{t}, \omega; w) \end{aligned}$$

provided that each member of (3.2) exists.

Furthermore, for every suitable choice of the coefficients a_k ($k \in \mathbb{N}_0$), if the multivariable function $\Omega_{\mu+\nu k}(\mathbf{z})$, $\mathbf{z} = (z_1, \dots, z_s)$, ($s \in \mathbb{N}$), is expressed as an appropriate product of several simpler functions, the assertions of Theorem 2.1 and Theorem 2.2 can be applied to derive various families of multilinear and multilateral generating functions for the generalized (unified) multivariable polynomials $\Phi_n^{(\alpha_1, \dots, \alpha_r)}(\mathbf{x}, \mathbf{y})$ given explicitly by (1.11).

We now give some further properties of the generalized (unified) multivariable polynomials $\Phi_n^{(\alpha_1, \dots, \alpha_r)}(\mathbf{x}, \mathbf{y})$ given by (1.11). The generating function (1.10) yields the following addition formula for these multivariable polynomials:

$$\Phi_n^{(\alpha_1 + \beta_1, \dots, \alpha_r + \beta_r)}(\mathbf{x}, \mathbf{y}) = \sum_{k=0}^n \Phi_{n-k}^{(\alpha_1, \dots, \alpha_r)}(\mathbf{x}, \mathbf{y}) \Phi_k^{(\beta_1, \dots, \beta_r)}(\mathbf{x}, \mathbf{y}).$$

On the other hand, the polynomials $\Phi_n^{(\alpha_1, \dots, \alpha_r)}(\mathbf{x}, \mathbf{y})$ satisfy the following differential equation:

$$(3.3) \quad \sum_{j=1}^r \left(p_j x_j \frac{\partial}{\partial x_j} + \frac{m_j}{d_j} y_j \frac{\partial}{\partial y_j} \right) \Phi_n^{(\alpha_1, \dots, \alpha_r)}(\mathbf{x}, \mathbf{y}) = n \Phi_n^{(\alpha_1, \dots, \alpha_r)}(\mathbf{x}, \mathbf{y}).$$

If we differentiate each member of the generating function (1.10) with respect to x_j and y_j ($j = 1, 2, \dots, r$), we arrive at the following (differential) recurrence relations for $\Phi_n^{(\alpha_1, \dots, \alpha_r)}(\mathbf{x}, \mathbf{y})$, respectively:

$$\begin{aligned} & \frac{\partial}{\partial x_j} \Phi_n^{(\alpha_1, \dots, \alpha_r)}(\mathbf{x}, \mathbf{y}) \\ & = \sum_{k=0}^{\left[\frac{n-p_j}{p_j} \right]} \sum_{l=0}^{\left[\frac{n-(k+1)p_j}{m_j} \right]} \frac{(-1)^l (k+l)! \alpha_j b_j^l (a_j)^{k+1}}{k! l! c_j^{k+l+1}} x_j^k y_j^{ld_j} \\ (3.4) \quad & \times \Phi_{n-lm_j-(k+1)p_j}^{(\alpha_1, \dots, \alpha_r)}(\mathbf{x}, \mathbf{y}) \end{aligned}$$

for $n \geq p_j$, and

$$\begin{aligned}
 & \frac{\partial}{\partial y_j} \Phi_n^{(\alpha_1, \dots, \alpha_r)}(\mathbf{x}, \mathbf{y}) \\
 &= - \sum_{l=0}^{\left[\frac{n-m_j}{m_j} \right]} \sum_{k=0}^{\left[\frac{n-(l+1)m_j}{p_j} \right]} \frac{(-1)^l (k+l)! \alpha_j d_j a_j^k b_j^{l+1}}{k!! c_j^{k+l+1}} x_j^k y_j^{d_j(l+1)-1} \\
 (3.5) \quad & \times \Phi_{n-(l+1)m_j-kp_j}^{(\alpha_1, \dots, \alpha_r)}(\mathbf{x}, \mathbf{y})
 \end{aligned}$$

where $n \geq m_j$, and m_j, p_j and d_j ($j = 1, 2, \dots, r$) are positive integers. By applying (3.3), (3.4) and (3.5), we can easily derive the following recurrence relation for the generalized (unified) multivariable polynomials $\Phi_n^{(\alpha_1, \dots, \alpha_r)}(\mathbf{x}, \mathbf{y})$ given explicitly by (1.11):

$$\begin{aligned}
 & \sum_{j=1}^r \sum_{k=0}^{\left[\frac{n-p_j}{p_j} \right]} \sum_{l=0}^{\left[\frac{n-(k+1)p_j}{m_j} \right]} \frac{(-1)^l (k+l)! p_j \alpha_j b_j^l a_j^{k+1}}{k!! c_j^{k+l+1}} x_j^{k+1} y_j^{l d_j} \Phi_{n-lm_j-(k+1)p_j}^{(\alpha_1, \dots, \alpha_r)}(\mathbf{x}, \mathbf{y}) \\
 & - \sum_{j=1}^r \sum_{l=0}^{\left[\frac{n-m_j}{m_j} \right]} \sum_{k=0}^{\left[\frac{n-(l+1)m_j}{p_j} \right]} \frac{(-1)^l (k+l)! \alpha_j m_j a_j^k b_j^{l+1}}{k!! c_j^{k+l+1}} x_j^k y_j^{d_j(l+1)} \Phi_{n-(l+1)m_j-kp_j}^{(\alpha_1, \dots, \alpha_r)}(\mathbf{x}, \mathbf{y}) \\
 & = n \Phi_n^{(\alpha_1, \dots, \alpha_r)}(\mathbf{x}, \mathbf{y})
 \end{aligned}$$

where $n \geq \max\{p_j, m_j\}$ ($j = 1, 2, \dots, r$) and, m_j and p_j ($j = 1, 2, \dots, r$) are positive integers.

4. Hypergeometric Representations for the special cases of

$$\Phi_n^{(\alpha_1, \dots, \alpha_r)}(\mathbf{x}, \mathbf{y})$$

In this section, we give a hypergeometric representation for the special case $p_i = 1$ ($i = 1, 2, \dots, r$) of the generalized (unified) multivariable polynomials $\Phi_n^{(\alpha_1, \dots, \alpha_r)}(\mathbf{x}, \mathbf{y})$ generated by (1.10). In this special case, we denote the polynomials $\Phi_n^{(\alpha_1, \dots, \alpha_r)}(\mathbf{x}, \mathbf{y})$ by $\Psi_n^{(\alpha_1, \dots, \alpha_r)}(\mathbf{x}, \mathbf{y})$. From (1.10), the polynomials $\Psi_n^{(\alpha_1, \dots, \alpha_r)}(\mathbf{x}, \mathbf{y})$ are generated by

$$\begin{aligned}
 & \prod_{i=1}^r \left(c_i - a_i x_i t + b_i y_i^{d_i} t^{m_i} \right)^{-\alpha_i} = \sum_{n=0}^{\infty} \Psi_n^{(\alpha_1, \dots, \alpha_r)}(\mathbf{x}, \mathbf{y}) t^n \\
 (4.1) \quad & \left(\left| a_i x_i t - b_i y_i^{d_i} t^{m_i} \right| < |c_i|, \quad c_i \neq 0, \quad \alpha_i \in \mathbb{C} \quad ; \quad i = 1, 2, \dots, r \right)
 \end{aligned}$$

The generalized hypergeometric function ${}_pF_q$ is defined by [17]

$$(4.2) \quad {}_pF_q \left[\begin{matrix} \alpha_1, \dots, \alpha_p; \\ \beta_1, \dots, \beta_q; \end{matrix} \quad z \right] = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \dots (\alpha_p)_k}{(\beta_1)_k \dots (\beta_q)_k} \frac{z^k}{k!},$$

from which, we can give the following hypergeometric representation for $\Psi_n^{(\alpha_1, \dots, \alpha_r)}(\mathbf{x}, \mathbf{y})$.

4.1. Theorem. *The polynomials $\Psi_n^{(\alpha_1, \dots, \alpha_r)}(\mathbf{x}, \mathbf{y})$ have the following hypergeometric representation*

$$(4.3) \quad \Psi_n^{(\alpha_1, \dots, \alpha_r)}(\mathbf{x}, \mathbf{y}) = \sum_{n_1 + \dots + n_r = n} \prod_{i=1}^r \frac{(\alpha_i)_{n_i} (a_i x_i)^{n_i}}{n_i! c_i^{n_i + \alpha_i}} \times {}_{m_i} F_{m_i-1} \left[\begin{matrix} -\frac{n_i}{m_i}, \frac{-n_i+1}{m_i}, \dots, \frac{-n_i+m_i-1}{m_i}, \\ -\frac{\alpha_i-n_i+1}{m_i-1}, \frac{-\alpha_i-n_i+2}{m_i-1}, \dots, \frac{-\alpha_i-n_i+m_i-1}{m_i-1}, \end{matrix} \middle| \frac{m_i^{m_i} b_i c_i^{m_i-1} y_i^{d_i}}{(a_i x_i)^{m_i} (m_i-1)^{m_i-1}} \right]$$

where $m_i \geq 2$ ($i = 1, 2, \dots, r$).

Proof. With the help of the result

$$(1-z)^{-\alpha} = \sum_{n=0}^{\infty} \frac{(\alpha)_n z^n}{n!},$$

from (4.1), we get

$$\prod_{i=1}^r (c_i - a_i x_i t + b_i y_i^{d_i} t^{m_i})^{-\alpha_i} = \prod_{i=1}^r \left\{ \sum_{n_i=0}^{\infty} \sum_{k_i=0}^{n_i} \frac{c_i^{-\alpha_i-n_i} (\alpha_i)_{n_i}}{(n_i - k_i)! k_i!} (a_i x_i t)^{n_i - k_i} (-b_i y_i^{d_i} t^{m_i})^{k_i} \right\}.$$

Replacing n_i by $n_i + k_i$ ($i = 1, 2, \dots, r$), we have

$$(4.4) \quad \prod_{i=1}^r (c_i - a_i x_i t + b_i y_i^{d_i} t^{m_i})^{-\alpha_i} = \prod_{i=1}^r \left\{ \sum_{n_i=0}^{\infty} \sum_{k_i=0}^{\infty} \frac{c_i^{-\alpha_i-n_i-k_i} (\alpha_i)_{n_i+k_i}}{n_i! k_i!} (a_i x_i t)^{n_i} (-b_i y_i^{d_i} t^{m_i})^{k_i} \right\}$$

Getting $n_i - m_i k_i$ instead of n_i ($i = 1, 2, \dots, r$) in (4.4) gives

$$\begin{aligned} & \prod_{i=1}^r (c_i - a_i x_i t + b_i y_i^{d_i} t^{m_i})^{-\alpha_i} \\ &= \prod_{i=1}^r \left\{ \sum_{n_i=0}^{\infty} \sum_{k_i=0}^{[n_i/m_i]} \frac{c_i^{-\alpha_i-n_i+(m_i-1)k_i} (\alpha_i)_{n_i-(m_i-1)k_i}}{(n_i - m_i k_i)! k_i!} (a_i x_i)^{n_i - m_i k_i} (-b_i y_i^{d_i})^{k_i} t^{n_i} \right\} \\ &= \sum_{n=0}^{\infty} \sum_{n_1 + \dots + n_r = n} \left\{ \prod_{i=1}^r \sum_{k_i=0}^{[n_i/m_i]} \frac{c_i^{-\alpha_i-n_i+(m_i-1)k_i} (\alpha_i)_{n_i-(m_i-1)k_i}}{(n_i - m_i k_i)! k_i!} (a_i x_i)^{n_i - m_i k_i} (-b_i y_i^{d_i})^{k_i} \right\} t^n. \end{aligned}$$

Since it is known that [17, p.58(2)]

$$(\alpha)_{n-k} = \frac{(-1)^k (\alpha)_n}{(1-\alpha-n)_k}, \quad 0 \leq k \leq n,$$

using

$$(n - mk)! = \frac{(-1)^{mk} n!}{(-n)_{mk}}, \quad 0 \leq mk \leq n,$$

$$(-n)_{mk} = m^{mk} \prod_{s=1}^m \binom{-n+s-1}{m}_k$$

and

$$(1 - \alpha - n)_{(m-1)k} = (m - 1)^{(m-1)k} \prod_{p=1}^{m-1} \left(\frac{-\alpha - n + p}{m - 1} \right)_k, \quad k = 0, 1, 2, \dots,$$

we find

$$\begin{aligned} \sum_{n=0}^{\infty} \Psi_n^{(\alpha_1, \dots, \alpha_r)}(\mathbf{x}, \mathbf{y}) t^n &= \prod_{i=1}^r \left(c_i - a_i x_i t + b_i y_i^{d_i} t^{m_i} \right)^{-\alpha_i} \\ &= \sum_{n=0}^{\infty} \sum_{n_1 + \dots + n_r = n} \frac{(\alpha_1)_{n_1} \dots (\alpha_r)_{n_r} (a_1 x_1)^{n_1} \dots (a_r x_r)^{n_r}}{c_1^{n_1 + \alpha_1} \dots c_r^{n_r + \alpha_r} n_1! \dots n_r!} \\ &\times \left\{ \prod_{i=1}^r \sum_{k_i=0}^{[n_i/m_i]} \frac{\left(\frac{-n_i}{m_i} \right)_{k_i} \left(\frac{-n_i+1}{m_i} \right)_{k_i} \dots \left(\frac{-n_i+m_i-1}{m_i} \right)_{k_i}}{\left(\frac{-\alpha_i - n_i + 1}{m_i - 1} \right)_{k_i} \left(\frac{-\alpha_i - n_i + 2}{m_i - 1} \right)_{k_i} \dots \left(\frac{-\alpha_i - n_i + m_i - 1}{m_i - 1} \right)_{k_i} k_i!} \right. \\ &\times \left. \left(\frac{m_i^{m_i} b_i c_i^{m_i-1} y_i^{d_i}}{(a_i x_i)^{m_i} (m_i - 1)^{m_i-1}} \right)^{k_i} \right\} t^n. \end{aligned}$$

Considering (4.2) and then comparing the coefficients of t^n from both sides, we obtain the desired hypergeometric representation. \square

We can choose some special cases of this theorem here.

4.2. Corollary. *If we set $a_i = m_i$, $b_i = d_i = 1$ ($i = 1, 2, \dots, r$) in (4.3), we get the following hypergeometric representation of the multivariable Humbert polynomials $P_n^{(\alpha_1, \dots, \alpha_r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{c})$ given explicitly by (1.6)*

$$(4.5) \quad P_n^{(\alpha_1, \dots, \alpha_r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{c}) = \sum_{n_1 + \dots + n_r = n} \prod_{i=1}^r \frac{(\alpha_i)_{n_i} (m_i x_i)^{n_i}}{n_i! c_i^{n_i + \alpha_i}} \times_{m_i} F_{m_i-1} \left[\begin{matrix} -\frac{n_i}{m_i}, \frac{-n_i+1}{m_i}, \dots, \frac{-n_i+m_i-1}{m_i}; & \frac{c_i^{m_i-1} y_i}{x_i^{m_i} (m_i-1)^{m_i-1}} \end{matrix} \right].$$

4.3. Remark. For $r = 1$ and $\alpha_1 = -p$, (4.5) is reduced to the hypergeometric representation for the generalized Humbert polynomials $P_n(m, x, y, p, c)$

$$(4.6) \quad P_n(m, x, y, p, c) = \frac{(-p)_n (mx)^n}{n! c^{n-p}} \times_m F_{m-1} \left[\begin{matrix} -\frac{n}{m}, \frac{-n+1}{m}, \dots, \frac{-n+m-1}{m}; & \frac{c^{m-1} y}{x^m (m-1)^{m-1}} \end{matrix} \right].$$

4.4. Remark. If we get $c = 1$, $y = 1$ and $p = -\nu$ in (4.6), we have the following hypergeometric representation of Humbert polynomials $h_{n,m}^\nu(x)$

$$(4.7) \quad h_{n,m}^\nu(x) = \frac{(\nu)_n (mx)^n}{n!} \times_m F_{m-1} \left[\begin{matrix} -\frac{n}{m}, \frac{-n+1}{m}, \dots, \frac{-n+m-1}{m}; & \frac{1}{x^m (m-1)^{m-1}} \end{matrix} \right]$$

which is a known result (see [16]).

4.5. Remark. For $m = 2$, (4.7) gives the following hypergeometric representation for Gegenbauer polynomials [16]

$$C_n^\nu(x) = \frac{(\nu)_n (2x)^n}{n!} {}_2F_1 \left[\begin{matrix} -\frac{n}{2}, \frac{-n+1}{2}; \\ 1 - \nu - n; \end{matrix} \quad \frac{1}{x^2} \right]$$

which is a generalization of a known result [17, p. 166(4)].

4.6. Remark. Setting $r = 1$, $y_1 = 2x - 1$ and $\alpha_1 = \nu$ in (4.3), we have the following known result [16] for the polynomials $p_{n,m,a,b,c,d}^\nu(x)$ given by (1.4)

$$p_{n,m,a,b,c,d}^\nu(x) = \frac{(\nu)_n (ax)^n}{n! c^{n+\nu}} \times_m F_{m-1} \left[\begin{matrix} -\frac{n}{m}, \frac{-n+1}{m}, \dots, \frac{-n+m-1}{m}; \\ \frac{-\nu-n+1}{m-1}, \frac{-\nu-n+2}{m-1}, \dots, \frac{-\nu-n+m-1}{m-1}; \end{matrix} \quad \frac{m^m bc^{m-1} (2x-1)^d}{(ax)^m (m-1)^{m-1}} \right]$$

which gives the hypergeometric representation for $S_n^\nu(x)$ for $b = c = d = 1$ and $a = m = 2$ [18, p. 442(12)].

5. Series expansions for the special cases of $\Phi_n^{(\alpha_1, \dots, \alpha_r)}(\mathbf{x}, \mathbf{y})$

We give expansions of $\Psi_n^{(\alpha_1, \dots, \alpha_r)}(\mathbf{x}, \mathbf{y})$, which is the special case $p_i = 1$ ($i = 1, 2, \dots, r$) of $\Phi_n^{(\alpha_1, \dots, \alpha_r)}(\mathbf{x}, \mathbf{y})$ given explicitly by (1.11), in series of some orthogonal polynomials.

5.1. Theorem. *Some expansions of $\Psi_n^{(\alpha_1, \dots, \alpha_r)}(\mathbf{x}, \mathbf{y})$ in series of Legendre, Gegenbauer, Hermite and Laguerre polynomials are given by*

$$\begin{aligned} & \Psi_n^{(\alpha_1, \dots, \alpha_r)}(\mathbf{x}, \mathbf{y}) \\ &= \sum_{n_1 + \dots + n_r = n} \prod_{i=1}^r \left\{ \sum_{k_i=0}^{[n_i/m_i]} \sum_{s_i=0}^{[\frac{n_i - m_i k_i}{2}]} \frac{(2n_i - 2m_i k_i - 4s_i + 1) (\alpha_i)_{n_i - (m_i - 1)k_i}}{k_i! s_i! (3/2)_{n_i - m_i k_i - s_i}} \right. \\ & \times C_i^{-\alpha_i - n_i + (m_i - 1)k_i} \left(-b_i y_i^{d_i} \right)^{k_i} P_{n_i - m_i k_i - 2s_i} \left(\frac{a_i x_i}{2} \right) \left. \right\}, \end{aligned}$$

$$\begin{aligned} & \Psi_n^{(\alpha_1, \dots, \alpha_r)}(\mathbf{x}, \mathbf{y}) \\ &= \sum_{n_1 + \dots + n_r = n} \prod_{i=1}^r \left\{ \sum_{k_i=0}^{[n_i/m_i]} \sum_{s_i=0}^{[\frac{n_i - m_i k_i}{2}]} \frac{(\nu_i + n_i - m_i k_i - 2s_i) (\alpha_i)_{n_i - (m_i - 1)k_i}}{k_i! s_i! (\nu_i)_{n_i - m_i k_i - s_i + 1}} \right. \\ & \times C_i^{-\alpha_i - n_i + (m_i - 1)k_i} \left(-b_i y_i^{d_i} \right)^{k_i} C_{n_i - m_i k_i - 2s_i}^{\nu_i} \left(\frac{a_i x_i}{2} \right) \left. \right\}, \end{aligned}$$

$$\begin{aligned} & \Psi_n^{(\alpha_1, \dots, \alpha_r)}(\mathbf{x}, \mathbf{y}) \\ &= \sum_{n_1 + \dots + n_r = n} \prod_{i=1}^r \left\{ \sum_{k_i=0}^{[n_i/m_i]} \sum_{s_i=0}^{[\frac{n_i - m_i k_i}{2}]} \frac{(\alpha_i)_{n_i - (m_i - 1)k_i}}{k_i! s_i! (n_i - m_i k_i - 2s_i)!} \right. \\ & \times C_i^{-\alpha_i - n_i + (m_i - 1)k_i} \left(-b_i y_i^{d_i} \right)^{k_i} H_{n_i - m_i k_i - 2s_i} \left(\frac{a_i x_i}{2} \right) \left. \right\} \end{aligned}$$

and

$$\begin{aligned} & \Psi_n^{(\alpha_1, \dots, \alpha_r)}(\mathbf{x}, \mathbf{y}) \\ &= \sum_{n_1 + \dots + n_r = n} \prod_{i=1}^r \left\{ \sum_{k_i=0}^{\lfloor n_i/m_i \rfloor} \sum_{s_i=0}^{n_i - m_i k_i} \frac{(\alpha_i)_{n_i - (m_i - 1)k_i} (\beta_i + 1)_{n_i - m_i k_i}}{k_i! (n_i - m_i k_i - s_i)! (\beta_i + 1)_{s_i}} \right. \\ & \quad \left. \times 2^{n_i - m_i k_i} (-1)^{s_i} c_i^{-\alpha_i - n_i + (m_i - 1)k_i} \left(-b_i y_i^{d_i}\right)^{k_i} L_{s_i}^{(\beta_i)}\left(\frac{a_i x_i}{2}\right) \right\}. \end{aligned}$$

Proof. From (4.4), we have

$$\begin{aligned} & \sum_{n=0}^{\infty} \Psi_n^{(\alpha_1, \dots, \alpha_r)}(\mathbf{x}, \mathbf{y}) t^n \\ &= \prod_{i=1}^r \left(c_i - a_i x_i t + b_i y_i^{d_i} t^{m_i} \right)^{-\alpha_i} \\ &= \prod_{i=1}^r \left\{ \sum_{n_i=0}^{\infty} \sum_{k_i=0}^{\infty} \frac{c_i^{-\alpha_i - n_i - k_i} (\alpha_i)_{n_i + k_i}}{n_i! k_i!} (a_i x_i t)^{n_i} \left(-b_i y_i^{d_i} t^{m_i}\right)^{k_i} \right\}. \end{aligned}$$

Using the result [17, p. 181]

$$\frac{(ax)^n}{n!} = \sum_{s=0}^{\lfloor n/2 \rfloor} \frac{(2n - 4s + 1) P_{n-2s}(ax/2)}{s! (3/2)_{n-s}},$$

we can write

$$\begin{aligned} & \sum_{n=0}^{\infty} \Psi_n^{(\alpha_1, \dots, \alpha_r)}(\mathbf{x}, \mathbf{y}) t^n \\ &= \prod_{i=1}^r \left\{ \sum_{n_i=0}^{\infty} \sum_{k_i=0}^{\infty} \sum_{s_i=0}^{\lfloor n_i/2 \rfloor} \frac{(2n_i - 4s_i + 1) c_i^{-\alpha_i - n_i - k_i} (\alpha_i)_{n_i + k_i}}{s_i! k_i! (3/2)_{n_i - s_i}} \times \right. \\ & \quad \left. \times P_{n_i - 2s_i}\left(\frac{a_i x_i}{2}\right) \left(-b_i y_i^{d_i} t^{m_i}\right)^{k_i} t^{n_i} \right\}. \end{aligned}$$

Replacing n_i by $n_i - m_i k_i$ in the last equality, we get

$$\begin{aligned} & \sum_{n=0}^{\infty} \Psi_n^{(\alpha_1, \dots, \alpha_r)}(\mathbf{x}, \mathbf{y}) t^n \\ &= \prod_{i=1}^r \left\{ \sum_{n_i=0}^{\infty} \sum_{k_i=0}^{\lfloor n_i/m_i \rfloor} \sum_{s_i=0}^{\lfloor \frac{n_i - m_i k_i}{2} \rfloor} \frac{(2n_i - 2m_i k_i - 4s_i + 1) c_i^{-\alpha_i - n_i + (m_i - 1)k_i}}{s_i! k_i! (3/2)_{n_i - m_i k_i - s_i}} \right. \\ & \quad \left. \times (\alpha_i)_{n_i - (m_i - 1)k_i} P_{n_i - m_i k_i - 2s_i}\left(\frac{a_i x_i}{2}\right) \left(-b_i y_i^{d_i}\right)^{k_i} t^{n_i} \right\} \\ &= \sum_{n=0}^{\infty} \sum_{n_1 + \dots + n_r = n} \left\{ \prod_{i=1}^r \sum_{k_i=0}^{\lfloor n_i/m_i \rfloor} \sum_{s_i=0}^{\lfloor \frac{n_i - m_i k_i}{2} \rfloor} \frac{(2n_i - 2m_i k_i - 4s_i + 1)}{s_i! k_i! (3/2)_{n_i - m_i k_i - s_i}} \right. \\ & \quad \left. \times c_i^{-\alpha_i - n_i + (m_i - 1)k_i} (\alpha_i)_{n_i - (m_i - 1)k_i} P_{n_i - m_i k_i - 2s_i}\left(\frac{a_i x_i}{2}\right) \left(-b_i y_i^{d_i}\right)^{k_i} \right\} t^n. \end{aligned}$$

If we compare the coefficients of t^n from the both sides, we find the desired relation.

In a similar manner, in (4.4), using the following results respectively [17, p. 283 (36), p. 194 (4), p. 207 (2)]

$$\frac{(ax)^n}{n!} = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(\nu + n - 2k) C_{n-2k}^\nu (ax/2)}{k! (\nu)_{n+1-k}},$$

$$\frac{(ax)^n}{n!} = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{H_{n-2k} (ax/2)}{k! (n - 2k)!}$$

and

$$\frac{(ax)^n}{n!} = 2^n \sum_{k=0}^n \frac{(-1)^k (\alpha + 1)_n L_k^{(\alpha)} (ax/2)}{(n - k)! (\alpha + 1)_k},$$

we can easily give the other expansions of $\Psi_n^{(\alpha_1, \dots, \alpha_r)}(\mathbf{x}, \mathbf{y})$ in series of Gegenbauer, Hermite and Laguerre polynomials. \square

We can give some special cases of this theorem.

5.2. Corollary. *Setting $a_i = m_i$, $b_i = d_i = 1$ ($i = 1, 2, \dots, r$) in Theorem 5.1, for the multivariable Humbert polynomials $P_n^{(\alpha_1, \dots, \alpha_r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{c})$ given explicitly by (1.6), expansions in series of Legendre, Gegenbauer, Hermite and Laguerre polynomials are given by*

$$P_n^{(\alpha_1, \dots, \alpha_r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{c})$$

$$= \sum_{n_1 + \dots + n_r = n} \prod_{i=1}^r \left\{ \sum_{k_i=0}^{\lfloor n_i/m_i \rfloor} \sum_{s_i=0}^{\lfloor \frac{n_i - m_i k_i}{2} \rfloor} \frac{(2n_i - 2m_i k_i - 4s_i + 1) (\alpha_i)_{n_i - (m_i - 1)k_i}}{k_i! s_i! (3/2)_{n_i - m_i k_i - s_i}} \right.$$

$$\left. \times c_i^{-\alpha_i - n_i + (m_i - 1)k_i} (-y_i)^{k_i} P_{n_i - m_i k_i - 2s_i} \left(\frac{m_i x_i}{2} \right) \right\},$$

$$P_n^{(\alpha_1, \dots, \alpha_r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{c})$$

$$= \sum_{n_1 + \dots + n_r = n} \prod_{i=1}^r \left\{ \sum_{k_i=0}^{\lfloor n_i/m_i \rfloor} \sum_{s_i=0}^{\lfloor \frac{n_i - m_i k_i}{2} \rfloor} \frac{(\nu_i + n_i - m_i k_i - 2s_i) (\alpha_i)_{n_i - (m_i - 1)k_i}}{k_i! s_i! (\nu_i)_{n_i - m_i k_i - s_i + 1}} \right.$$

$$\left. \times c_i^{-\alpha_i - n_i + (m_i - 1)k_i} (-y_i)^{k_i} C_{n_i - m_i k_i - 2s_i}^{\nu_i} \left(\frac{m_i x_i}{2} \right) \right\},$$

$$P_n^{(\alpha_1, \dots, \alpha_r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{c})$$

$$= \sum_{n_1 + \dots + n_r = n} \prod_{i=1}^r \left\{ \sum_{k_i=0}^{\lfloor n_i/m_i \rfloor} \sum_{s_i=0}^{\lfloor \frac{n_i - m_i k_i}{2} \rfloor} \frac{(\alpha_i)_{n_i - (m_i - 1)k_i}}{k_i! s_i! (n_i - m_i k_i - 2s_i)!} \right.$$

$$\left. \times c_i^{-\alpha_i - n_i + (m_i - 1)k_i} (-y_i)^{k_i} H_{n_i - m_i k_i - 2s_i} \left(\frac{m_i x_i}{2} \right) \right\}$$

and

$$P_n^{(\alpha_1, \dots, \alpha_r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{c})$$

$$= \sum_{n_1 + \dots + n_r = n} \prod_{i=1}^r \left\{ \sum_{k_i=0}^{\lfloor n_i/m_i \rfloor} \sum_{s_i=0}^{n_i - m_i k_i} \frac{(\alpha_i)_{n_i - (m_i - 1)k_i} (\beta_i + 1)_{n_i - m_i k_i}}{k_i! (n_i - m_i k_i - s_i)! (\beta_i + 1)_{s_i}} \right.$$

$$\left. \times 2^{n_i - m_i k_i} (-1)^{s_i} c_i^{-\alpha_i - n_i + (m_i - 1)k_i} (-y_i)^{k_i} L_{s_i}^{(\beta_i)} \left(\frac{m_i x_i}{2} \right) \right\}.$$

5.3. Remark. If we get $r = 1$ and $\alpha_1 = -p$ in Corollary 5.2, we have similar results for the generalized Humbert polynomials $P_n(m, x, y, p, c)$.

5.4. Corollary. Setting $r = 1, y_1 = 2x - 1$ and $\alpha_1 = \nu$ in Theorem 5.1, then we get the known results for the polynomials $p_{n,m,a,b,c,d}^\nu(x)$ given by (1.4) [16].

Similar to Theorem 5.1, we can give series expansions for Chan-Chan-Srivastava, Lagrange-Hermite and Erkus-Srivastava multivariable polynomials.

5.5. Theorem. Some expansions of Erkus-Srivastava multivariable polynomials in series of Legendre, Gegenbauer, Hermite and Laguerre polynomials are as follows

$$\begin{aligned} & u_n^{(\alpha_1, \dots, \alpha_r)}(x_1, \dots, x_r) \\ &= \sum_{m_1 n_1 + \dots + m_r n_r = n} \prod_{i=1}^r \sum_{k_i=0}^{\lfloor \frac{n_i}{2} \rfloor} \frac{(2n_i - 4k_i + 1) (\alpha_i)_{n_i}}{k_i! (3/2)_{n_i - k_i}} P_{n_i - 2k_i} \left(\frac{x_i}{2} \right), \\ & u_n^{(\alpha_1, \dots, \alpha_r)}(x_1, \dots, x_r) \\ &= \sum_{m_1 n_1 + \dots + m_r n_r = n} \prod_{i=1}^r \sum_{k_i=0}^{\lfloor \frac{n_i}{2} \rfloor} \frac{(\nu_i + n_i - 2k_i) (\alpha_i)_{n_i}}{k_i! (\nu_i)_{n_i - k_i + 1}} C_{n_i - 2k_i}^{\nu_i} \left(\frac{x_i}{2} \right), \\ & u_n^{(\alpha_1, \dots, \alpha_r)}(x_1, \dots, x_r) \\ &= \sum_{m_1 n_1 + \dots + m_r n_r = n} \prod_{i=1}^r \sum_{k_i=0}^{\lfloor \frac{n_i}{2} \rfloor} \frac{(\alpha_i)_{n_i}}{k_i! (n_i - 2k_i)!} H_{n_i - 2k_i} \left(\frac{x_i}{2} \right) \end{aligned}$$

and

$$\begin{aligned} & u_n^{(\alpha_1, \dots, \alpha_r)}(x_1, \dots, x_r) \\ &= \sum_{m_1 n_1 + \dots + m_r n_r = n} \prod_{i=1}^r \sum_{k_i=0}^{n_i} \frac{(-1)^{k_i} 2^{n_i} (\alpha_i)_{n_i} (\beta_i + 1)_{n_i}}{(n_i - k_i)! (\beta_i + 1)_{k_i}} L_{k_i}^{(\beta_i)} \left(\frac{x_i}{2} \right). \end{aligned}$$

5.6. Corollary. In Theorem 5.5, getting $m_i = i$ and $m_i = 1$ ($i = 1, 2, \dots, r$) respectively gives the similar results for Lagrange-Hermite and Chan-Chyan-Srivastava multivariable polynomials.

References

- [1] Aktaş, R., Şahin, R. and Altın, A. *On a multivariable extension of Humbert polynomials*, Appl. Math. Comp. **218**, 662–666, 2011.
- [2] Altın, A., Aktaş, R. and Çekim, B. *On a multivariable extension of the Hermite and related polynomials*, Ars Combinatoria **110**, 487–503, 2013.
- [3] Altın, A. and Erkuş, E. *On a multivariable extension of the Lagrange-Hermite polynomials*, Integral Transforms Spec. Funct. **17**, 239–244, 2006.
- [4] Chan, W.-C. C., Chyan, C.-J. and Srivastava, H. M. *The Lagrange polynomials in several variables*, Integral Transforms Spec. Funct. **12**, 139–148, 2001.
- [5] Dattoli, G., Ricci, P.E. and Cesarano, C. *The Lagrange polynomials, the associated generalizations, and the umbral calculus*, Integral Transforms Spec. Funct. **14**, 181–186, 2003.
- [6] Dattoli, G., Ricci, P.E. and Cesarano, C. *Operational and umbral methods for the solution of partial differential equations*, J. Concr. Appl. Math. **2**(3), 281–288, 2004.
- [7] Dattoli, G., Ricci, P. E., Cesarano, C. and Khomasuridze, I. *Bilateral generating functions and operational methods*, South East Asian J. Math. Math. Sci. **4**(2), 1–6, 2006.
- [8] Erdélyi, A., Magnus, W., Oberhettinger, F. and Tricomi, F. G. *Higher Transcendental Functions*, **III**, McGraw-Hill Book Company, New York, Toronto, London, 1955.

- [9] Erkuş, E. and Srivastava, H.M. *A unified presentation of some families of multivariable polynomials*, Integral Transforms Spec. Funct. **17**, 267–273, 2006.
- [10] Gould, H. W. *Inverse series relation and other expansions involving Humbert polynomials*, Duke Math. J. **32**, 697–711, 1965.
- [11] Horadam, A.F. *Gegenbauer polynomials revisited*, Fibonacci Quart. **23**, 294–299, 1985.
- [12] Horadam, A. F. and Pethe, S. *Polynomials associated with Gegenbauer polynomials*, Fibonacci Quart. **19**, 393–398, 1981.
- [13] Humbert, P. *Some extensions of Pincherle's polynomials*, Proc. Edinburgh Math. Soc. **39**, 21–24, 1921.
- [14] Milovanović, G. V. and Dordević, G. P. *On some properties of Humbert's polynomials*, Fibonacci Quart. **25**, 356–360, 1987.
- [15] Milovanović, G. V. and Dordević, G. P. *On some properties of Humbert's polynomials II*, Facta Univ. Ser. Math. Inform. **6**, 23–30, 1991.
- [16] Pathan, M. A. and Khan, M. A. *On polynomials associated with Humbert's polynomials*, Publ. Inst. Math. (Beograd) (N.S.) **62**(76), 53–62, 1997.
- [17] Rainville, E. D. *Special Functions*, The Macmillan Company, New York, 1960.
- [18] Sinha, S. K. *On a polynomial associated with Gegenbauer polynomial*, Proc. Nat. Acad. Sci. India Sect. A **59**, 439–455, 1989.
- [19] Singhal, J. P. *A note on generalized Humbert polynomials*, Glasnik Mat. Ser III **5**(25), 241–245, 1970.
- [20] Srivastava, H. M. and Manocha, H. L. *A Treatise on Generating Functions*, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, 1984.
- [21] Zeitlin, D. *On a class of polynomials obtained from generalized Humbert polynomials*, Proc. Amer. Math. Soc. **18**, 28–34, 1967.