Abstract

In the present paper, we establish several new Hardy-Hilbert integral inequalities, and give some applications to other integral inequalities.

Keywords: Hardy-Hilbert’s inequality, Integral inequality.

2000 AMS Classification: 26 D 15.

1. Introduction

If \( f, g \) are measurable real functions such that
\[
0 < \int_0^\infty f^2(x) \, dx < \infty \quad \text{and} \quad 0 < \int_0^\infty g^2(x) \, dx < \infty,
\]
then we have the following well known Hilbert integral inequality [2],
\[
\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} \, dx \, dy < \pi \left( \int_0^\infty f^2(x) \, dx \right)^{1/2} \left( \int_0^\infty g^2(x) \, dx \right)^{1/2},
\]
where \( \pi \) is the best possible.

If \( f, g \geq 0, p > 1, \frac{1}{p} + \frac{1}{q} = 1, \) and
\[
0 < \int_0^\infty f^p(x) \, dx < \infty \quad \text{and} \quad 0 < \int_0^\infty g^q(x) \, dx < \infty,
\]
then the following Hardy-Hilbert integral inequality (see [2]), which is important in analysis and applications, holds
\[
\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} \, dx \, dy < \pi \left( \int_0^\infty f^p(x) \, dx \right)^{1/p} \left( \int_0^\infty g^q(x) \, dx \right)^{1/q},
\]
where the constant factor \( \frac{\pi}{\sin(\pi/p)} \) is the best possible.
Other mathematicians have presented generalizations or new kinds of the above Hardy-Hilbert inequalities, as follows:

1.1. Theorem. [7] Let \( f, g > 0 \). If \( p > 1, \ q > 1, \ \frac{1}{p} + \frac{1}{q} \geq 1 \), and \( 0 < \lambda = 2 - \frac{1}{p} + \frac{1}{q} \leq 1 \), then one has

\[
\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x + y)^\lambda} \, dx \, dy \leq k \left( \int_0^\infty f^p(x) \, dx \right)^{1/p} \left( \int_0^\infty g^q(x) \, dx \right)^{1/q}.
\]

Here, \( k \) depends on \( p \) and \( q \); only if \( \frac{1}{p} + \frac{1}{q} = 1 \), \( \lambda = 2 - \frac{1}{p} + \frac{1}{q} = 1 \), \( k \) is the best possible.

1.2. Theorem. [6] If \( f, g > 0, \lambda > 0, \ p > 1 \) and \( \frac{1}{p} + \frac{1}{q} = 1 \) are such that

\[
0 < \int_0^\infty t^{p-1-\lambda} f^p(x) \, dx < \infty \quad \text{and} \quad 0 < \int_0^\infty t^{q-1-\lambda} g^q(x) \, dx < \infty,
\]

then one has

\[
\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\max\{x^\lambda, y^\lambda\}} \, dx \, dy < \frac{pq}{\lambda} \left( \int_0^\infty t^{p-1-\lambda} f^p(x) \, dx \right)^{1/p} \left( \int_0^\infty t^{q-1-\lambda} g^q(x) \, dx \right)^{1/q},
\]

where the constant factor \( \frac{pq}{\lambda} \) is the best possible.

Recently, Du and Miao [1, 4] have studied the function \( \frac{1}{\alpha x + \beta y + \min\{x, y\}} \) with positive numbers \( \alpha, \beta, \gamma \), and given the following extended analogue of Hilbert’s inequalities,

1.3. Theorem. [1] Let \( f, g \) be real functions such that \( 0 < \int_0^\infty f^2(x) \, dx < \infty \) and \( 0 < \int_0^\infty g^2(x) \, dx < \infty \). Furthermore, let \( A \in (0, \infty) \). Then we have

\[
\int_0^\infty \int_0^\infty \frac{|\log x - \log y|^\gamma}{\alpha x + \beta y + \min\{x, y\}} f(x)g(y) \, dx \, dy < A \left( \int_0^\infty f^2(x) \, dx \right)^{1/2} \left( \int_0^\infty g^2(x) \, dx \right)^{1/2},
\]

where \( A \) is defined as

\[
A := \int_0^1 \frac{2^{\gamma+1}|\log t|^\gamma}{t^2(1 + \alpha) + \alpha} \, dt + \int_0^1 \frac{2^{\gamma+1}|\log t|^\gamma}{t^2(1 + \beta) + \beta} \, dt.
\]

Here \( \alpha, \beta, \gamma \) are any positive real numbers.

In the present paper, based on the above works, we establish several new Hardy-Hilbert integral inequalities. What’s more, as applications, some specific integral inequalities are deduced.

2. Main results

Before starting our work, we recall some results and definitions about the Gamma function \( \Gamma(p) \) and Beta function \( B(p, q) \) as follows,

\[
\Gamma(p) = \int_0^\infty t^{p-1} e^{-t} \, dt, \quad p > 0
\]

(2.1)

\[
B(p, q) = \int_0^1 t^{p-1} (1 - t)^{q-1} \, dt = \int_0^\infty \frac{t^{p-1}}{(1 + t)^{p+q}} \, dt, \quad p, q > 0.
\]
2.1. Lemma. [5] Let \( p, q > 0 \). Then

\[
\Gamma(p) = e \int_0^1 x^{-p-1} e^{- \frac{x}{t}} \, dx = e \int_1^\infty \frac{e^{-x}}{(x-1)^{1-p}} \, dx
\tag{2.2}
\]

\[
B(p, q) = \int_1^\infty \frac{x^{-p-q}}{(x-1)^{1-p}} \, dx.
\]

Furthermore, for convenience, we state the definition of homogeneous function: The function \( F(x, y) \) is said to be homogeneous of degree \( \lambda \), \((\lambda > 0)\), if \( F(tx, ty) = t^\lambda F(x, y) \) for all \((x, y) \in D\) and \((tx, ty) \in D\), where \( D\) denotes the domain of the function \( F(x, y) \).

Now we can give the following main results in this paper.

2.2. Theorem. Assume that \( f, g, h, k \geq 0 \), \( h = h(x, y) : R_+ \times R_+ \to R_+ \), \( k = k(t) : R_+ \to R_+ \), \( h \) is homogeneous of degree \( \lambda \) and \( k \) is nondecreasing; \( p > 1, \frac{1}{p} + \frac{1}{q} = 1 \). Then

(a) For \( k(t) \neq 1 \) (or, in general, \( k(t) \neq c \), where \( c \) is some constant),

\[
\int_0^\infty y^{(p-1)(\lambda-1)} \left( \int_0^\infty \frac{f(x)}{h(x, y) \max\{k(x), k(y)\}} \, dx \right)^p \, dy
\leq C p \int_0^\infty x^{1-\lambda} f^p(x) \, dx,
\tag{2.3}
\]

where \( C = I_1 + I_2 \),

\[
I_1 = \int_0^1 \frac{dx}{h(x, 1)k(x^{-1})}, \quad I_2 = \int_1^\infty \frac{dx}{h(x, 1)k(x)}.
\]

(b) For \( k(t) = 1 \) (in general \( a \) and \( b \) are both arbitrary constants),

\[
\int_0^\infty y^{b(q-1) + \lambda - 1-a(p-1)} \left( \int_0^\infty \frac{f(x)}{h(x, y)} \, dx \right)^p \, dy
\leq C p \int_0^\infty x^{1+b-\lambda-a(p-1)} f^p(x) \, dx,
\tag{2.4}
\]

where \( C = K_1^{\frac{1}{q}} K_2^{\frac{1}{p}} \),

\[
K_1 = \int_0^\infty \frac{t^b \, dt}{h(1,t)}, \quad K_2 = \int_0^\infty \frac{t^a \, dt}{h(t,1)}.
\]

Here we assume that all the integrals on the RHS do exist.

Proof. (a) According to Holder’s inequality, it is easy to see that

\[
\int_0^\infty \frac{f(x)}{h(x, y) \max\{k(x), k(y)\}} \, dx
\leq \left( \int_0^\infty \frac{f^p(x)}{h(x, y) \max\{k(x), k(y)\}} \right)^\frac{1}{p} \left( \int_0^\infty \frac{dx}{h(x, y) \max\{k(x), k(y)\}} \right)^\frac{1}{q},
\]

which yields

\[
\left( \int_0^\infty \frac{f(x)}{h(x, y) \max\{k(x), k(y)\}} \right)^p
\leq \int_0^\infty \frac{f^p(x)}{h(x, y) \max\{k(x), k(y)\}} \left( \int_0^\infty \frac{dx}{h(x, y) \max\{k(x), k(y)\}} \right)^\frac{p}{q}.
\tag{2.5}
\]
We first consider the following integral
\[
\int_0^\infty \frac{dx}{h(x, y) \max\{k\left(\frac{x}{y}\right), k\left(\frac{y}{x}\right)\}} = \int_0^y \frac{dx}{h(x, y) \max\{k\left(\frac{x}{y}\right), k\left(\frac{y}{x}\right)\}} + \int_y^\infty \frac{dx}{h(x, y) \max\{k\left(\frac{x}{y}\right), k\left(\frac{y}{x}\right)\}} = M_1 + M_2.
\]

For the case \(M_1\), since \(x \leq y\) implies \(\frac{x}{y} \leq \frac{y}{x}\), hence \(k\left(\frac{x}{y}\right) \leq k\left(\frac{y}{x}\right)\), then we have
\[
M_1 = \int_0^y \frac{dx}{h(x, y) k\left(\frac{y}{x}\right)}.
\]

Let \(u = \frac{x}{y}\), then
\[
M_1 = \int_0^y \frac{dx}{h(xy, y) k\left(\frac{y}{x}\right)} = y^{1-\lambda} \int_0^1 \frac{du}{h(u, 1) k(u^{-1})} = I_1 y^{1-\lambda},
\]

and
\[
M_2 = \int_y^\infty \frac{dx}{h(x, y) \max\{k\left(\frac{x}{y}\right), k\left(\frac{y}{x}\right)\}} = \int_y^\infty \frac{dx}{h(x, y) k\left(\frac{x}{y}\right)} = y^{1-\lambda} \int_1^\infty \frac{du}{h(u, 1) k(u)} = I_2 y^{1-\lambda},
\]

which implies
\[
(2.6) \quad \int_0^\infty \frac{dx}{h(x, y) \max\{k\left(\frac{x}{y}\right), k\left(\frac{y}{x}\right)\}} = (I_1 + I_2) y^{1-\lambda} = Cy^{1-\lambda}.
\]

Therefore, from (2.5) and (2.6), we have
\[
\left(\int_0^\infty \frac{f(x)dx}{h(x, y) \max\{k\left(\frac{x}{y}\right), k\left(\frac{y}{x}\right)\}}\right)^p \leq C_p y^{p(1-\lambda)} \int_0^\infty \frac{f^p(x)dx}{h(x, y) \max\{k\left(\frac{x}{y}\right), k\left(\frac{y}{x}\right)\}}.
\]

Now since
\[
\int_0^\infty y^{p-1}(\lambda-1) C_p y^{p(1-\lambda)} \int_0^\infty \frac{f^p(x)dx}{h(x, y) \max\{k\left(\frac{x}{y}\right), k\left(\frac{y}{x}\right)\}} dy = C_p \int_0^\infty \int_0^\infty \frac{f^p(x)dx}{h(x, y) \max\{k\left(\frac{x}{y}\right), k\left(\frac{y}{x}\right)\}} dy = C_p^p \int_0^\infty f^p(x)dx \int_0^\infty \frac{dy}{h(x, y) \max\{k\left(\frac{x}{y}\right), k\left(\frac{y}{x}\right)\}} = C_p^p \int_0^\infty f^p(x)dx \int_0^\infty \frac{dy}{h(x, y) \max\{k\left(\frac{x}{y}\right), k\left(\frac{y}{x}\right)\}} = C_p^p \int_0^\infty x^{1-\lambda} f^p(x) dx \]
then the inequality (2.3) holds.
(b) Similarly, according to Holder’s inequality, we have

\[ \int_0^\infty \frac{f(x) dx}{h(x, y)} = \int_0^\infty \frac{f(x) y^p}{h^p(x, y) x^q} dx \]

\[ \leq \left( \int_0^\infty \frac{f^p(x) y^q dx}{h^p(x, y) x^q} \right)^{\frac{1}{p}} \left( \int_0^\infty \frac{x^a dx}{h(x, y) y^{\frac{b}{p}}} \right)^{\frac{1}{q}}. \]

It is easy to check

\[ \int_0^\infty \frac{x^a dx}{h(x, y) y^{\frac{b}{p}}} = \frac{y^{b+\frac{a}{p}}}{\frac{b}{p} + a} \int_0^\infty \frac{u^a du}{h(u, 1)} = K_2 y^{b+\frac{a}{p}}, \]

then we can obtain

\[ \left( \int_0^\infty \frac{f(x) dx}{h(x, y)} \right)^p \leq K^p_2 y^{b+\frac{a}{p}} \int_0^\infty \frac{f^p(x) y^q dx}{h(x, y) x^q}. \]

Therefore, by the inequality (2.8), we have

\[ \int_0^\infty \frac{f(x) dx}{h(x, y)} \leq K^p_2 y^{b+\frac{a}{p}} \int_0^\infty \frac{f^p(x) y^q dx}{h(x, y) x^q} \]

\[ \leq C^p \int_0^\infty \frac{f^p(x) y^q dx}{h(x, y) x^q}, \]

where \( C = K_1^{1/p} K_2^{-q/p}. \) By now, we have completed the proof of the theorem. \( \square \)

3. Applications

Firstly, we recall the fact: if \( 0 < p < 1, \) then one has

\( \Gamma(p) \Gamma(1-p) = \frac{\pi}{\sin(p\pi)} \) and \( B(p, 1-p) = \frac{\pi}{\sin(p\pi)}. \)

3.1. Corollary. Assume that \( f \geq 0, \) \( p > 1 \) and \( \frac{1}{p} + \frac{1}{q} = 1, \) then

\[ \int_0^\infty \left( \int_0^\infty \frac{f(x) dx}{x+y} \right)^p dy \leq \frac{\pi}{\sin(\pi/p)} \int_0^\infty f^p(x) dx \]

provided the integrals on the RHS exist.
Proof. The result is obtained from result (b) in Theorem 2.2 by putting
\[ h(x, y) = x + y, \quad a = \frac{1}{q} - 1, \quad b = \frac{1}{p} - 1. \]
Thus we have
\[
\int_0^\infty \left( \int_0^\infty \frac{f(x)}{x+y} \, dy \right)^p dx \\
\leq \left[ \left( \int_0^\infty \frac{x^{\frac{1}{p}-1}}{1+t} \, dt \right)^{\frac{1}{p}} \cdot \left( \int_0^\infty \frac{y^{\frac{1}{q}-1}}{1+t} \, dt \right)^{\frac{1}{q}} \right]^p \int_0^\infty f^p(x) \, dx \\
\leq \left[ \frac{\Gamma \left( \frac{1}{p} \right) \Gamma \left( \frac{1}{q} \right)}{\sin(\pi/p)} \right]^p \int_0^\infty f^p(x) \, dx.
\]

\[ \square \]

3.2. Corollary. Assume that \( f, g \geq 0, \lambda > 0, p > 1 \) and \( \frac{1}{p} + \frac{1}{q} = 1 \). Then
\[
\int_0^\infty y^{(p-1)(\lambda-1)} \left( \int_0^\infty \frac{f(x)}{|x-y|^{1-\lambda}} \max\{f^{\lambda}(x), g^{\lambda}(x)\} \right)^p dy \\
\leq [2B(\lambda, \lambda)]^p \int_0^\infty x^{1-\lambda} f^p(x) \, dx.
\]

Proof. The result is obtained from result (a) in Theorem 2.2 by putting
\[ h(x, y) = |x - y|^{1-\lambda}, \quad k(x) = x^{2\lambda}. \]
So we get
\[
I_1 = \int_1^\infty \frac{t^{2\lambda}}{(1-t)^{1-\lambda}} dt = \int_0^1 \frac{t^{\lambda+1} t^{\lambda-1}}{(1-t)^{1-\lambda}} dt \leq \int_0^1 \frac{t^{\lambda-1}}{(1-t)^{1-\lambda}} dt = B(\lambda, \lambda), \\
I_2 = \int_1^\infty \frac{t^{-2\lambda}}{(t-1)^{1-\lambda}} dt = B(\lambda, \lambda).
\]
The desired result can now be obtained. \[ \square \]

3.3. Corollary. Assume that \( f, g \geq 0, \lambda < 2, p > 1 \) and \( \frac{1}{p} + \frac{1}{q} = 1 \). Then
\[
\int_0^\infty y^{(p-1)(\lambda-1)} \left( \int_0^\infty \frac{f(x)}{(x^\lambda + y^\lambda) \max\{f^{\lambda}(x), g^{\lambda}(x)\}} \right)^p dy \\
\leq \left[ \frac{\pi}{2\lambda} + \frac{2}{\lambda} \int_0^1 \frac{y^{\frac{1}{\lambda}-2}}{y^2+1} dy \right]^p \int_0^\infty x^{1-\lambda} f^p(x) \, dx.
\]
In particular, when \( m := \frac{1}{\lambda} - 2 \) is a positive integer, we have
\[
\int_0^\infty y^{(p-1)(\lambda-1)} \left( \int_0^\infty \frac{f(x)}{(x^\lambda + y^\lambda) \max\{f^{\lambda}(x), g^{\lambda}(x)\}} \right)^p dy \\
\leq \left[ \frac{\pi}{2\lambda} + \frac{2}{\lambda} \sum_{k=0}^{m} (-1)^k \frac{1}{m+2k+1} \right]^p \int_0^\infty x^{1-\lambda} f^p(x) \, dx.
\]
Proof. The result is obtained from result (a) in Theorem 2.2 by putting
\[ h(x, y) = x^\lambda + y^\lambda, \quad k(x) = x^{1 - \frac{2}{\lambda}}. \]
Therefore, we can obtain (by letting \( y = u^{\frac{1}{\lambda}} \)),
\[
I_1 = \int_0^1 \frac{u^{1 - \frac{2}{\lambda}}}{u^\lambda + 1} \, du = \frac{2}{\lambda} \int_0^1 \frac{y^{\frac{2}{\lambda} - 2}}{y^{2\lambda} + 1} \, dy,
\]
\[
I_2 = \int_1^\infty \frac{u^{\frac{2}{\lambda} - 1}}{u^\lambda + 1} \, du = \frac{2}{\lambda} \int_1^\infty \frac{1}{y^{2\lambda} + 1} \, dy = \frac{\pi}{2\lambda},
\]
which yields \( C = \frac{\pi}{2\lambda} + \frac{2}{\lambda} \int_0^1 \frac{y^{\frac{2}{\lambda} - 2}}{y^{2\lambda} + 1} \, dy \). In particular, when \( m := \frac{2}{\lambda} - 2 \) is a positive integer, on the basis of the table of integrals, we have
\[
I_1 = \frac{2}{\lambda} \int_0^1 \frac{y^m}{y^{2\lambda} + 1} \, dy = \frac{2}{\lambda} \sum_{k=0}^{\infty} (-1)^k \frac{1}{m + 2k + 1}.
\]
So the proof of the desired result can be completed. \( \square \)

Acknowledgement

The authors are very grateful to the referee for his/her valuable report which improved the presentation of this work.

References