THE STOLARSKY TYPE FUNCTIONS
AND THEIR MONOTONICITIES

V. Lokesha*, Zhi-Gang Wang†, Zhi-Hua Zhang‡ and S. Padmanabhan*  

Received 04:07:2008 : Accepted 23:03:2009

Abstract
In this paper, we give the definition of a Stolarsky type function, and obtain its monotonicity. By using these results, we establish a series of means and their monotonicities in n variables.

Keywords: Two-parameter, Monotonicity, Mean, Vandermonde determinant.

2000 AMS Classification: 26 D 15.

1. Introduction
The so-called Stolarsky means \(S(a, b; \alpha)\) were defined first by Stolarsky in [9] as follows:

\[
S(a, b; \alpha) = \left[ \frac{a^{\alpha+1} - b^{\alpha+1}}{(\alpha + 1)(a - b)} \right]^{1/\alpha}, \quad \alpha(a + 1)(a - b) \neq 0; \tag{1.1}
\]

\[
S(a, b; -1) = \frac{a - b}{\ln a - \ln b}, \quad \alpha(a - b) \neq 0, \quad \alpha = -1; \tag{1.2}
\]

\[
S(a, b; 0) = \exp \left( -1 + \frac{a \ln a - b \ln b}{a - b} \right), \quad (\alpha + 1)(a - b) \neq 0, \quad \alpha = 0; \tag{1.3}
\]

\[
S(a, a; \alpha) = a, \quad a = b. \tag{1.4}
\]

The monotonicity of \(S(a, b; \alpha)\) has been discussed by Leach and Sholander [3, 4], and by Qi [7, 8] also using different ideas and simpler methods.

*Department of Mathematics, Acharya institute of Technology, Soldevahnalli, Hesaragatta Road, Karanataka Bangalore-90, India. E-mail: lokiv@yahoo.com  
†Corresponding author  
‡School of Mathematics and Computing Science, Changsha University of Science and Technology, Changsha 410076, Hunan, People’s Republic of China. E-mail: zhigwang@163.com  
§Department of Mathematics, Zixing Educational Research Section, Chenzhou 423400, Hunan, People’s Republic of China. E-mail: zzxh12340163.com  
¶Department of Mathematics, R.N.S. Institute of Technology, Karanataka, Bangalore, India. E-mail: padmanabhanapsce@rediffmail.com
In [7], Qi studied the following generalized weighted Stolarsky type mean values $E_{f,p}(a,b;\alpha)$ with parameter $\alpha$, and proved that $E_{f,p}(x,y;\alpha)$ is an increasing function in $\alpha$:

\begin{equation}
E_{f,p}(a,b;\alpha) = \left(\frac{\int_a^b p(u)f^\alpha(u)du}{\int_a^b p(u)du}\right)^{\frac{1}{\alpha}}, \quad (\alpha - \beta)(a - b) \neq 0;
\end{equation}

\begin{equation}
E_{f,p}(a,b;0) = \exp\left(\frac{\int_a^b p(u)\ln f(u)du}{\int_a^b p(u)du}\right), \quad \alpha = 0, a - b \neq 0;
\end{equation}

\begin{equation}
E_{f,p}(a,a;\alpha) = f(a), \quad \alpha = \beta, a = b;
\end{equation}

where $a, b, \alpha, \beta \in \mathbb{R}$, $p \geq 0$, and $f > 0$ is an integrable function on the interval $[a,b] \subset \mathbb{R}$.

We know by the definition of the power mean that

\begin{equation}
M(x;\alpha) = \left(\frac{\sum_{k=1}^n x_k^\alpha}{n}\right)^{\frac{1}{\alpha}}, \quad \alpha \neq 0;
\end{equation}

\begin{equation}
M(x;0) = \exp\left(\frac{\sum_{k=1}^n \ln x_k}{n}\right), \quad \alpha = 0;
\end{equation}

where $a_k \in \mathbb{R}_+$, and $\alpha \in \mathbb{R}$.

We note that for each of these two means the one-parameter means are of the type $(F(\alpha)/F(0))^{1/\alpha}$ if $\alpha \neq 0$, and $\exp(F'(\alpha)/F(\alpha))$ if $\alpha = 0$, where $F(\alpha)$ is a certain univariate function involving an $\alpha$-order power.

In this paper, we define a Stolarsky type function and obtain its monotonicity. By using these results, we establish a series of means and their monotonicities in $n$ variables.

**2. Main results**

Throughout the paper we assume $\mathbb{R}$ to be the set of real numbers, $\mathbb{R}_+$ the set of strictly positive real numbers, $\mathbb{R}^n$ the $n$-dimensional Euclidean Space,

\[ \mathbb{R}^n_+ = \{(x_1, x_2, \ldots, x_n) : x_i > 0, \ i = 1, 2, \ldots, n\}, \]

and

\[ \alpha x = (\alpha x_1, \alpha x_2, \ldots, \alpha x_n), \quad e^x = (e^{x_1}, e^{x_2}, \ldots, e^{x_n}), \]

\[ x^\alpha = (x_1^\alpha, x_2^\alpha, \ldots, x_n^\alpha), \quad \ln x = (\ln x_1, \ln x_2, \ldots, \ln x_n), \]

where $\alpha \in \mathbb{R}$, $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n_+$, and $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n_+$.

**2.1. Definition.** Let $\alpha, \beta \in \mathbb{R}$, and $f$ be continuous involving an $(\alpha\beta)$-order power function on $I \subseteq \mathbb{R}^n$. If $F(\alpha) = f(x;\alpha\beta)$, $\beta \neq 0$, and $f$ is a differentiable function with respect to $\alpha \in \mathbb{R}$, then the Stolarsky type function $S_f(x;\alpha,\beta)$ is defined as follows,

\begin{equation}
S_f(x;\alpha) = \left(\frac{f(x;\alpha\beta)}{f(x;0)}\right)^{\frac{1}{\alpha}}, \quad (\alpha \neq 0),
\end{equation}

\begin{equation}
S_f(x;0) = \lim_{\alpha \to 0} \exp\left(\frac{f'_\alpha(x;\alpha\beta)}{f(x;\alpha\beta)}\right), \quad (\alpha = 0),
\end{equation}

where $f'_\alpha$ is the partial derivative with respect to $\alpha$ of $f(x;\alpha\beta)$.

**2.2. Remark.** For convenience, we write

\begin{equation}
S_f(x;\alpha) = S_f(x) = S_f(\alpha) = S_f,
\end{equation}

shifting notation to suit the context.
2.3. Theorem. Let $\alpha, \beta \in \mathbb{R}, \beta \neq 0$, and $f$ be continuous involving an $(\alpha\beta)$-order power function on $I \subseteq \mathbb{R}^+_+$. If
\[
(2.4) \quad f(x; \alpha\beta)f''_{\alpha\beta}(x; \alpha\beta) > |f'_{\alpha\beta}(x; \alpha\beta)|^2,
\]
then $S_f(x; \alpha)$ is a monotonic increasing function in $\alpha$, and monotonic decreasing if the inequality (2.4) is reversed.

Proof. Suppose the inequality (2.4) holds. Setting $T(\alpha) = \ln |f(x; \alpha\beta)|$, then $T'(\alpha) = f'_{\alpha\beta}(x; \alpha\beta)/f(x; \alpha\beta)$, and
\[
T''(\alpha) = \frac{f(x; \alpha\beta)f''_{\alpha\beta}(x; \alpha\beta) - |f'_{\alpha\beta}(x; \alpha\beta)|^2}{|f(x; \alpha\beta)|^2} > 0.
\]
When $\alpha = 0$, $\ln S_f = f'_{\alpha}(x; \alpha\beta)/f(x; \alpha\beta) = T'(\alpha)$, and $\partial \ln S_f/\partial \alpha = T''(\alpha) > 0$, which implies that $S_f(x; \alpha)$ is a monotonic increasing function in $\alpha$.

When $\alpha \neq 0$, using the mean value theorem, we find
\[
\frac{\partial \ln S_f}{\partial \alpha} = \frac{T'(\alpha)}{\alpha} - \frac{T(\alpha)}{\alpha^2} = \frac{T'(\alpha) - T(\alpha)/\alpha}{\alpha} = \frac{T'(\alpha) - T''(\alpha)}{\alpha} = \frac{\alpha - \eta T''(\eta)}{\alpha} > 0,
\]
where $\zeta$ is between 0 and $\alpha$, and $\eta$ is between $\alpha$ and $\zeta$. That is to say, $S_f(x; \alpha)$ is a monotonic increasing function in $\alpha$. Theorem 2.3 is thus proved. $\square$

3. The generalized weighted Stolarsky type functional mean

3.1. Theorem. The generalized weighted Stolarsky type functional mean values $S_{f,p}(x; \alpha)$ are monotonic increasing functions with $\alpha$ in $\mathbb{R}$, where
\[
(3.1) \quad S_{f,p}(x; \alpha) = \left( \frac{\int_E p(t)f^\alpha(A(x; t))dt}{\int_E p(t)dt} \right)^{1/\alpha}, \quad \alpha \neq 0,
\]
\[
(3.2) \quad S_{f,p}(x; 0) = \exp \left( \frac{\int_E p(t)\ln f(A(x; t))dt}{\int_E p(t)dt} \right), \quad \alpha = 0,
\]
and $E = \{ (t_1, t_2, \ldots, t_n) | \sum_{i=1}^n t_i \leq 1, \sum_{i=1}^n t_i \geq 0, i = 1, 2, \ldots, n \}, t_0 = 1 - \sum_{i=1}^n t_i, A(x; t) = x_0 + \sum_{i=1}^n (x_i - x_0) t_i = \sum_{i=0}^{n-1} x_i t_i, x_i \in I \subseteq \mathbb{R}^+_+$, and $p \geq 0, f > 0$ integrable functions respectively on $E$ and $I$.

Proof. By taking $T(x; \alpha) = \int_E p(t)f^\alpha(A(x; t))dt$, and using Cauchy’s integral inequality, we have
\[
T(x; \alpha \beta)T''_{\alpha\beta}(x; \alpha\beta) - [T'_{\alpha\beta}(x; \alpha\beta)]^2
\]
\[
= \int_E p(t)f^\alpha(A(x; t))dt \cdot \int_E p(t)f^\alpha(A(x; t)) \ln^2 f(A(x; t))dt
\]
\[
- \left( \int_E p(t)f^\alpha(A(x; t)) \ln f(A(x; t))dt \right)^2 > 0,
\]
which implies Theorem 3.1 from Theorem 2.3. $\square$

3.2. Corollary. The generalized weighted Stolarsky type functional mean values $E_{f,p}(a, b; \alpha)$ are monotonic increasing functions with $\alpha$ in $\mathbb{R}$, where $E_{f,p}(a, b; \alpha)$ is given by (1.5)–(1.7).

Proof. Setting $u = x_0 + (x_1 - x_0)t_1$, then $du = (x_1 - x_0)dt_1$. Setting $a = x_0$ and $b = x_1$, from Theorem 3.1, we immediately obtain Corollary 3.2. The proof is completed. $\square$
4. The generalized weighted Stolarsky type functional mean with two parameters

4.1. Definition. Let \( \alpha, \beta \in \mathbb{R}, \ E, \ t_0 \) and \( p, f \) be defined as in Theorem 3.1. If

\[
M_\beta(x; t) = \left( x_0^\beta + \sum_{i=1}^n (x_i^\beta - x_0^\beta) t_i \right)^{1/\beta},
\]

and \( M_0(x; t) = G(x; t) = \prod_{i=0}^n x_i^{t_i}, \) then the first generalized weighted Stolarsky type functional mean values, \( S_{f,p}^{[1]}(x; \alpha, \beta), \) with two parameters \( \alpha \) and \( \beta \) are as follows

\[
(4.1) \quad S_{f,p}^{[1]}(x; \alpha, \beta) = \left( \frac{\int_E p(t) f^\alpha(M_\beta(x; t)) dt}{\int_E p(t) dt} \right)^{1/\alpha}, \quad \alpha \beta \neq 0;
\]

\[
(4.2) \quad S_{f,p}^{[1]}(x; 0, \beta) = \exp \left( \frac{\int_E p(t) \ln f(M_\beta(x; t)) dt}{\int_E p(t) dt} \right), \quad \alpha = 0, \beta \neq 0;
\]

\[
(4.3) \quad S_{f,p}^{[1]}(x; \alpha, 0) = \left( \frac{\int_E p(t) f^\alpha(G(x; t)) dt}{\int_E p(t) dt} \right)^{1/\alpha}, \quad \alpha \neq 0, \beta = 0;
\]

\[
(4.4) \quad S_{f,p}^{[1]}(x; 0, 0) = \exp \left( \frac{\int_E p(t) \ln f(G(x; t)) dt}{\int_E p(t) dt} \right), \quad \alpha = \beta = 0.
\]

In a manner similar to Section 3, from Definition 4.1 we obtain the following theorem.

4.2. Theorem. The first generalized weighted Stolarsky type functional mean values \( S_{f,p}^{[1]}(x; \alpha, \beta) \) are monotonic increasing functions in \( \alpha \in \mathbb{R}. \)

4.3. Theorem. The first generalized weighted Stolarsky type functional mean values \( S_{f,p}^{[1]}(x; \alpha, \beta) \) are monotonic increasing functions with \( \beta \in \mathbb{R} \) if \( f \) is a monotonic increasing function.

Proof. This follows from the weighted power mean inequality, Definition 4.1 and the fact that \( f \) is a monotonic increasing function. \( \square \)

4.4. Remark. We have \( S_{f,p}^{[1]}(x; 1, 1) = S_{f,p}(x; \alpha). \)

4.5. Definition. Let \( \alpha, \beta \in \mathbb{R}, \ E, \ t_0 \) and \( p, f \) be defined as in Theorem 3.1. If

\[
M_\beta(x^{\alpha}; t) = \left[ x_0^{\alpha \beta} + \sum_{i=1}^n (x_i^{\alpha \beta} - x_0^{\alpha \beta}) t_i \right]^{1/\beta},
\]

and \( M_0(x^{\alpha}; t) = G(x^{\alpha}; t) = \prod_{i=0}^n x_i^{t_i}, \) and \( f'(1) \) exists, then the second generalized weighted Stolarsky type functional mean values \( S_{f,p}^{[2]}(x; \alpha, \beta) \) with two parameters \( \alpha \) and \( \beta \) are as follows

\[
(4.5) \quad S_{f,p}^{[2]}(x; \alpha, \beta) = \left( \frac{\int_E p(t) f^\alpha(M_\beta(x^{\alpha}; t)) dt}{\int_E p(t) dt} \right)^{1/\alpha}, \quad \alpha \beta \neq 0;
\]

\[
(4.6) \quad S_{f,p}^{[2]}(x; 0, \beta) = \exp \left( \frac{\int_E p(t) f'(1) \left( \sum_{k=1}^n t_k \ln x_k \right) dt}{\int_E p(t) dt} \right), \quad \alpha = 0, \beta \in \mathbb{R};
\]

\[
(4.7) \quad S_{f,p}^{[2]}(x; \alpha, 0) = \left( \frac{\int_E p(t) f^\alpha(G(x^{\alpha}; t)) dt}{\int_E p(t) dt} \right)^{1/\alpha}, \quad \alpha \neq 0, \beta = 0.
\]

4.6. Theorem. The second generalized weighted Stolarsky type functional mean values \( S_{f,p}^{[2]}(x; \alpha, \beta) \) are monotonic increasing functions with \( \alpha \) in \( \mathbb{R} \) if \( f' > 0, ff'' > (f')^2 \) and \( \beta > 0. \)
Proof. By taking \( T(x; \alpha \beta) = \int_E \rho(t) f(M_\beta(x^\alpha; t)) dt \), if \( \beta \neq 0 \), then

\[
(4.8) \quad T'_\alpha(x; \alpha \beta) = \int_E \rho(t) f'(M_\beta(x^\alpha; t)) \left[ \sum_{k=0}^{n} x_k^{\alpha \beta} t_k \right]^{1/\beta-1} \left( \sum_{k=0}^{n} x_k^{\alpha \beta} t_k \ln x_k \right) dt,
\]

\[
(4.9) \quad T''_\alpha(x; \alpha \beta) = \int_E \rho(t) f''(M_\beta(x^\alpha; t)) \left[ \sum_{k=0}^{n} x_k^{\alpha \beta} t_k \right]^{1/\beta-1} \left( \sum_{k=0}^{n} x_k^{\alpha \beta} t_k \ln x_k \right) dt
\]

\[
+ \int_E \rho(t) f''(M_\beta(x^\alpha; t)) \left(1 - \beta \right) \left( \sum_{k=0}^{n} x_k^{\alpha \beta} t_k \right) \left( \sum_{k=0}^{n} x_k^{\alpha \beta} t_k \ln x_k \right)^2 dt
\]

\[
+ \beta \left( \sum_{k=0}^{n} x_k^{\alpha \beta} t_k \right)^{1/\beta-1} \left( \sum_{k=0}^{n} x_k^{\alpha \beta} t_k \ln x_k \right) dt.
\]

Using Cauchy’s integral inequality, from (4.8)-(4.9), and \( f' > 0, f f'' > (f')^2 \), \( \beta > 0 \), yields

\[
T(x; \alpha \beta) T''_\alpha(x; \alpha \beta) - [T'_\alpha(x; \alpha \beta)]^2
\]

\[
= \int_E \rho(t) f(M_\beta(x^\alpha; t)) dt \cdot \int_E \rho(t) f''(M_\beta(x^\alpha; t))
\]

\[
\cdot \left[ \sum_{k=0}^{n} x_k^{\alpha \beta} t_k \right]^{1/\beta-1} \left( \sum_{k=0}^{n} x_k^{\alpha \beta} t_k \ln x_k \right)^2 dt
\]

\[
- \left[ \int_E \rho(t) f'(M_\beta(x^\alpha; t)) \left( \sum_{k=0}^{n} x_k^{\alpha \beta} t_k \right)^{1/\beta-1} \left( \sum_{k=0}^{n} x_k^{\alpha \beta} t_k \ln x_k \right) dt \right]^2
\]

\[
+ \int_E \rho(t) f'(M_\beta(x^\alpha; t)) dt \cdot \int_E \rho(t) f''(M_\beta(x^\alpha; t)) \left( \sum_{k=0}^{n} x_k^{\alpha \beta} t_k \right)^{1/\beta-2}
\]

\[
\cdot \left( \sum_{k=0}^{n} x_k^{\alpha \beta} t_k \ln x_k \right)^2 + \beta \left( \sum_{k=0}^{n} x_k^{\alpha \beta} t_k \right) \left( \sum_{k=0}^{n} x_k^{\alpha \beta} t_k \ln x_k \right)^2
\]

\[
- \left( \sum_{k=0}^{n} x_k^{\alpha \beta} t_k \ln x_k \right)^2 \right) dt > 0
\]

which implies Theorem 4.6 from Theorem 2.3. If \( \beta = 0 \) we can obtain Theorem 4.6 similarly.

\[\square\]

5. Some mean values in \( n \) variables

5.1. Notation and lemmas. Throughout this section we assume \( x = (x_0, x_1, \ldots, x_n) \in \mathbb{R}_{n+1}^n \), and that \( \varphi \) is a function in \( \mathbb{R} \). Put

\[
(5.1) \quad V(x; \varphi) = \begin{vmatrix}
1 & x_0 & x_0^2 & \cdots & x_0^{n-1} \\
1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\
\vdots & \vdots & \ddots & \cdots & \vdots \\
1 & x_n & x_n^2 & \cdots & x_n^{n-1}
\end{vmatrix}.
\]

Assuming \( \varphi(t) = t^{n+r} \ln^r t \), then
where

\[ V(x; r, k) = \begin{vmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^{n-1} & x_0^n \ln^k x_0 \\ 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} & x_1^n \ln^k x_1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} & x_n^n \ln^k x_n \end{vmatrix} . \]

Note the case \( r = 0 \) and \( k = 0 \) is just the determinant of Van der Monde’s matrix of the \( n \)-th order:

\[ V(x; 0, 0) = \prod_{0 \leq i < j \leq n} (x_j - x_i). \]

Write \( \ln x = (\ln x_0, \ln x_1, \ldots, \ln x_n) \), then

\[ V(\ln x; r, k) = \begin{vmatrix} 1 & \ln x_0 & \ln^2 x_0 & \cdots & \ln^{n-1} x_0 & x_0^r \ln^k x_0 \\ 1 & \ln x_1 & \ln^2 x_1 & \cdots & \ln^{n-1} x_1 & x_1^r \ln^k x_1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \ln x_n & \ln^2 x_n & \cdots & \ln^{n-1} x_n & x_n^r \ln^k x_n \end{vmatrix} . \]

Also, let \( 0 \leq i \leq n \), and set

\[ V_{[i]}(x; \varphi) = \begin{vmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^{n-i} & \varphi(x_0) \\ 1 & x_1 & x_1^2 & \cdots & x_1^{n-i} & \varphi(x_1) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & x_i & x_i^2 & \cdots & x_i^{n-i} & \varphi(x_i) \\ 0 & 1 & 2x_i \cdots & (n-1)x_i^{n-2} & \varphi'(x_i) \\ 1 & x_{i+1} & x_{i+1}^2 & \cdots & x_{i+1}^{n-i} & \varphi(x_{i+1}) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-i} & \varphi(x_n) \end{vmatrix} . \]

and for \( \varphi(t) = t^{n+r+1} \) in (5.5), we have

\[ V_{[i]}(x; r) = \begin{vmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^{n-i} & x_0^{n+r+1} \\ 1 & x_1 & x_1^2 & \cdots & x_1^{n-i} & x_1^{n+r+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & x_i & x_i^2 & \cdots & x_i^{n-i} & x_i^{n+r+1} \\ 0 & 1 & 2x_i \cdots & nx_i^{n-i} & (n+r+1)x_i^{n+r} \\ 1 & x_{i+1} & x_{i+1}^2 & \cdots & x_{i+1}^{n-i} & x_{i+1}^{n+r+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-i} & x_n^{n+r+1} \end{vmatrix} , \quad (i \leq i \leq n), \]

and

\[ V_{[i]}(x; 0) = (-1)^{i+1} V(x; 0, 0) \prod_{j=0, j \neq i}^{n} (x_j - x_i) = (-1)^{i+1} V^2(x; 0, 0)/V_i(x), \]

where

\[ V_i(x) = \begin{vmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^{n-i} \\ 1 & x_1 & x_1^2 & \cdots & x_1^{n-i} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{i-1} & x_{i-1}^2 & \cdots & x_{i-1}^{n-i} \\ 1 & x_{i+1} & x_{i+1}^2 & \cdots & x_{i+1}^{n-i} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-i} \end{vmatrix} , \quad (0 \leq i \leq n). \]

(5.8)
5.1. Lemma. (see [12, 13, 14]) If \( n \in \mathbb{N} \), and \( \varphi \) is a \( n \)-order differentiable function on an interval \( I \subset \mathbb{R}_+ \), then

\[
(5.9) \quad V(x; \varphi) = V(x; 0, 0) \int_E \varphi^{(n)}(A(x; t)) dt,
\]

\[
(5.10) \quad \sum_{i=0}^{n} (-1)^{i+1} \lambda_i V_i(x; \varphi) V_i(x) = V^2(x; 0, 0) \int_E A(\lambda; t) \varphi^{(n)}(A(x; t)) dx,
\]

where \( dt = dt_1 dt_2 \cdots dt_n \), and \( E, A(x; t) \) are as in Theorem 3.1.

5.2. Lemma. (see [10]) Let \( r \) be an integer, then

\[
(5.11) \quad V(a; r, 0) = V(a; 0, 0) \cdot \sum_{i_0, i_1, \ldots, i_n \geq 0} \prod_{k=0}^{n} a_{i_k}^{i_k}, \quad r > 0;
\]

\[
(5.12) \quad V(a; r, 0) = 0, \quad r = 0, 1, \ldots, -(n - 1);
\]

\[
(5.13) \quad V(a; r, 0) = (-1)^n V(a; 0, 0) \cdot \sum_{i_0, i_1, \ldots, i_n \geq 1} \prod_{k=0}^{n} a_{i_k}^{-i_k}, \quad r < -n.
\]

5.2. The Stolarsky type mean with one parameter in \( n \) variables.

5.3. Definition. (see [11]) The Stolarsky type generalized mean values \( S_\alpha(x) \) with parameter \( \alpha \) in \( n \) variables are

\[
(5.14) \quad S_\alpha(x) = \left[ n! \int_E \varphi_1^{(n)}(\alpha, A(x; t)) dt \right]^{1/\alpha}, \quad \alpha \neq 0,
\]

\[
(5.15) \quad S_0(x) = \exp \left( n! \int_E \varphi_2^{(n)}(0, A(x; t)) dt \right), \quad \alpha = 0,
\]

where \( \varphi_1^{(n)}(\alpha, t) = t^\alpha \) and \( \varphi_2^{(n)}(\alpha, t) = t^\alpha \ln t \).

5.4. Theorem. If the generalized mean values \( S_\alpha(x) \) with two parameters \( \alpha \) and \( \beta \), in \( n \) variables are as given by Definition 5.3, then

\[
(5.16) \quad S_\alpha(x) = \left[ \prod_{k=1}^{n} \frac{V(x; \alpha, 0)}{k + \alpha} \right]^{\frac{1}{\alpha}}, \quad \alpha \neq 0, -1, -2, \ldots, -n;
\]

\[
(5.17) \quad S_0(x) = \exp \left( \frac{V(x; 0, 1)}{V(x; 0, 0)} \sum_{k=1}^{n} \frac{1}{k} \right), \quad \alpha = 0;
\]

\[
(5.18) \quad S_\alpha(x) = \left[ \frac{n! \cdot V(x; \alpha, 1)}{(-1)^{n+1}(\alpha - 1)! \cdot (n + \alpha) \cdot V(x; 0, 0)} \right]^{\frac{1}{\alpha}}, \quad \alpha = -1, \ldots, -n;
\]

where \( S_\alpha(x) \) are monotonic increasing functions with \( \alpha \) in \( R \).

Proof. Consider the following two functions:

\[
(5.19) \quad \varphi_1(\alpha, t) = \prod_{k=1}^{n} (k + \alpha)^{-1} t^{\alpha + \alpha},
\]

if \( \alpha \neq 0, -1, -2, \ldots, -n; \) and

\[
(5.20) \quad \varphi_2(0, t) = (n!)^{-1} t^n \left( \ln t - \sum_{k=1}^{n} \frac{1}{k} \right),
\]

if \( \alpha = 0; \) and

\[
(5.21) \quad \varphi_1(\alpha, t) = [(-1)^{n+1}(\alpha - 1)! (n + \alpha)]^{-1} t^{\alpha + \alpha} \ln t,
\]

if \( \alpha \neq 0, -1, -2, \ldots, -n; \) and

\[
(5.22) \quad \varphi_2(0, t) = (n!)^{-1} t^n \left( \ln t - \sum_{k=1}^{n} \frac{1}{k} \right),
\]

if \( \alpha = 0; \) and

\[
(5.23) \quad \varphi_1(\alpha, t) = [(-1)^{n+1}(\alpha - 1)! (n + \alpha)]^{-1} t^{\alpha + 1} \ln t,
\]

if \( \alpha \neq 0, -1, -2, \ldots, -n; \) and

\[
(5.24) \quad \varphi_2(0, t) = (n!)^{-1} t^n \left( \ln t - \sum_{k=1}^{n} \frac{1}{k} \right),
\]

if \( \alpha = 0; \) and
if \( \alpha = -1, -2, \ldots, -n \). Then \( \varphi_1^{(n)}(\alpha, t) = t^\alpha \) and \( \varphi_2^{(n)}(0, t) = \ln t \).

According to Lemma 5.1 and (5.19)–(5.21), we know that the expressions (5.16)–(5.18) hold true.

Let \( f^n(A; x; t) = \varphi_1^{(n)}(\alpha, A; x; t) \). Then \( \ln f(A; x; t) = \varphi_2^{(n)}(0; A; x; t) \). Taking \( p(x) \equiv 1 \), we find from Theorem 3.1 that the \( S_n(x) \) are monotonic increasing functions with \( \alpha \) in \( R \). The proof of Theorem 5.4 is completed. \( \square \)

5.5. Remark. (see [10]–[17]) \( S_0(x) \) is the so-called identric mean in \( n \) variables, and \( S_{-1}(x) \) the first logarithmic mean \( L(x) \). It is noted that \( S_0(x) := I(x) \), and

\[
L(x) := S_{-1}(x) = \frac{V(x; 0; 0)}{n V(x; -1, 1)}.
\]

5.6. Remark. (see [1]) If \( \alpha \) is a nonnegative integer, from Lemma 5.2, \( [S_n(x)]^\alpha \) is the \( r \)-th generalized elementary symmetric mean in \( n \) variables, i.e.

\[
\sum \alpha \bigg( \begin{array}{l}
\end{array} \bigg) := [S_n(x)]^\alpha = \left( \begin{array}{c}
n + \alpha \\
\alpha 
\end{array} \right)^{-1} \sum_{i_0 + i_1 + \cdots + i_n = \alpha, i_0, i_1, \ldots, i_n \in N_0} n \cdot x^{i_k}.
\]

5.3. The Stolarsky type mean with two parameters in \( n \) variables.

5.7. Definition. (see [12]) The Stolarsky type generalized mean values \( S_{\alpha, \beta}(x) \) with two parameters \( \alpha \) and \( \beta \) in \( n \) variables are

\[
S_{\alpha, \beta}(x) = \left[ n! \int_E \varphi_1^{(n)}(\alpha, M_\beta(x; t))dt \right]^{1/\alpha}, \quad \alpha \neq 0, \beta \neq 0;
\]

\[
S_{0, \beta}(x) = \exp \left( n! \int_E \varphi_2^{(n)}(0, M_\beta(x; t))dt \right), \quad \alpha = 0, \beta \neq 0;
\]

\[
S_{\alpha, 0}(x) = \left[ n! \int_E \varphi_1^{(n)}(\alpha, G(x; t))dt \right]^{1/\beta}, \quad \alpha \neq 0, \beta = 0;
\]

\[
S_{0, 0}(x) = \left( \prod_{i=0}^{n} a_i \right)^{1/(n+1)}, \quad \alpha = 0, \beta = 0;
\]

where \( \varphi_1^{(n)}(\alpha, t) = t^\alpha \) and \( \varphi_2^{(n)}(\alpha, t) = t^\alpha \ln t \).

5.8. Theorem. We have that \( S_{\alpha, \beta}(x) \) are monotonic increasing functions with \( \alpha \) in \( R \), and

\[
S_{\alpha, \beta}(x) = \left[ \prod_{k=1}^{n} \frac{\beta_k^{\beta}}{k^{\beta + \alpha}} \cdot \frac{V(x^{\beta}; 0; 0)}{V(x^{\beta}; 0; 0)} \right]^{1/\alpha}, \quad \beta \neq 0, \alpha \neq -k\beta, 0 \leq k \leq n;
\]

\[
S_{\alpha, 0}(x) = \left[ (-1)^{k + 1} k^{\beta} \frac{n}{k} \cdot \frac{V(x^{\beta}; -k, 1)}{V(x^{\beta}; 0, 0)} \right]^{-1/(k\beta)}, \quad \beta \neq 0, \alpha = -k\beta, 1 \leq k \leq n;
\]

\[
S_{0, 0}(x) = \left[ \prod_{i=0}^{n} \frac{1}{\alpha_i} \cdot \frac{V(\ln x; 0, 0)}{V(\ln x; 0, n)} \right]^{1/\alpha}, \quad \beta = 0, \alpha \neq 0;
\]

\[
S_{0, 0}(x) = \exp \left( \frac{1}{\beta} \sum_{k=1}^{n} \frac{1}{k} \right), \quad \alpha = 0, \beta \neq 0;
\]

\[
S_{0, 0}(x) = \left( \prod_{i=0}^{n} x_i \right)^{1/(n+1)}, \quad \alpha = \beta = 0.
\]
5.9. Remark. Replacing $\alpha$ by $\alpha - \beta$, the generalized Stolarsky type mean $S_{\alpha-\beta,\beta}(x)$ is the Pečarić-Šimić mean in [6].

5.10. Remark. (see [15] and also [5, 16]) If $\alpha = 1$, then $S_{1,0}(x)$ is the second logarithmic mean in $n$ variables:

\[(5.33) \quad l(x) := S_{1,0}(x) = \frac{n!V(\ln x; 1, 0)}{V(\ln x; 0, n)}.
\]

5.11. Remark. (see [15] and also [5]) Change $\beta$ to $1/\beta$, and set $\alpha = 1$. If $\beta$ is a nonnegative integer, from Lemma 5.2 we see that $S_{1,1/\beta}(x)$ is the generalized Heron’s mean in $n$ variables:

\[(5.34) \quad H_{\beta}(x) := S_{1,1/\beta}(x) = \frac{(n+\beta)^{\beta} - 1}{\beta} \sum_{i_0+i_1+\cdots+i_n=\beta, i_0,i_1,\ldots,i_n\in\mathbb{N}_0}^n \prod_{k=1}^n x_k^{i_k/\beta},
\]

5.4. The $r$-th weighted elementary symmetric mean in $n$ variables.

5.12. Definition. (see [17]) Let $x$ be a tuple of $n$ non-negative real numbers and the weight $\lambda$ a tuple of $n$ positive real numbers, then

\[(5.35) \quad E_{\alpha}(x, \lambda) = \sum_{i_0+i_1+\cdots+i_n=\alpha, i_0,i_1,\ldots,i_n\in\mathbb{N}_0} \prod_{k=1}^n x_k^{i_k/\alpha},
\]

is called the $\alpha$-th weighted elementary symmetric function of $x$ for the positive weight $\lambda$, where the sum is over all \(\binom{n+\alpha+1}{\alpha}\)-tuples of non-negative integers such that $i_1 + i_2 + \cdots + i_n = \alpha$; In addition, $E_0(x, \lambda) = \sum_{i=1}^n \lambda_i$. The $\alpha$-th weighted elementary symmetric mean of $x$ for $\lambda$ is defined by

\[(5.36) \quad \sum \alpha(x, \lambda) = \frac{E_{\alpha}(x, \lambda)}{\binom{n+\alpha+1}{\alpha} \sum_{i=1}^n \lambda_i}.
\]

5.13. Theorem. (see [17]) If $r \in \mathbb{N}$, then

\[(5.37) \quad \sum \alpha(x, \lambda) = \int E \left( \sum_{i=0}^n \lambda_i x_i \right) \left( \sum_{i=0}^n a_i x_i \right)^{\alpha} dx
\]

is a monotonic increasing function with $\alpha$ in $\mathbb{N}$.

Acknowledgement. We are thankful to the referee for some valuable suggestions.

References


