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COMMON FIXED POINT THEOREMS IN CONE BANACH SPACES

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Abstract

Recently, E. Karapınar (*Fixed Point Theorems in Cone Banach Spaces*, Fixed Point Theory Applications, Article ID 609281, 9 pages, 2009) presented some fixed point theorems for self-mappings satisfying certain contraction principles on a cone Banach space. Here we will give some generalizations of this theorem.

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1. Introduction and Preliminaries

It is quite natural to consider generalization of the notion of metric $d : X \times X \to [0, \infty)$. The question was, what must $[0, \infty)$ be replaced by. In 1980 Bogdan Rzepecki [17], in 1987 Shy-Der Lin [14] and in 2007 Huang and Zhang [5] gave the same answer: Replace the real numbers with a Banach space ordered by a cone, resulting in the so called cone metric. In this setting, Bogdan Rzepecki [17] generalized the fixed point theorems of Maia type [15] and Shy-Der Lin [14] considered some results of Khan and Imdad [13]. Also, Huang and Zhang [5] discussed some properties of convergence of sequences and proved a fixed point theorem of contractive mapping for cone metric spaces: Any mapping T of a complete cone metric space X into itself that satisfies, for some $0 \leq k < 1$, the inequality $d(Tx, Ty) \leq kd(x, y)$ for all $x, y \in X$, has a unique fixed point.

Following Huang and Zhang [5], many results on fixed point theorems have been extended from metric spaces to cone metric spaces (see e.g. [1, 2, 3, 5, 7, 8, 9, 10, 11, 12, 16, 18, 19, 20]).

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Recently, E.Karapınar [7] presented some fixed point theorems for self-mappings satisfying some contraction principles on a cone Banach space. More precisely, he proved that for a closed and convex subset C of a cone Banach space with the norm $\|\cdot\|_P$, and letting $d : X \times X \to E$ with $d(x, y) = \|x - y\|_P$, if there exist a, b, s and $T : C \to C$ satisfies the conditions $0 \leq s + |a| - 2b < 2(a + b)$ and $ad(Tx, Ty) + b\big(d(x, Tx) + d(y, Ty)\big) \leq sd(x, y)$ for all $x, y \in C$, then T has at least one fixed point.

Here we will give some generalization of this theorem. Throughout this paper $E := (E, \|\cdot\|)$ stands for a real Banach space and $P := P_E$ will always denote a closed non-empty subset of E. Then P is called a *cone* if $ax + by \in P$ for all $x, y \in P$, and non-negative real numbers a, b where $P \cap (-P) = \{0\}$ and $P \neq \{0\}$.

For a given cone P, one can define a partial ordering (denoted by $\leq$ or $\leq_P$) with respect to P by $x \leq y$ if and only if $y - x \in P$. The notation $x < y$ indicates that $x \leq y$ and $x \neq y$, while $x \ll y$ will denote $y - x \in \operatorname{int}P$, where $\operatorname{int}P$ denotes the interior of P. From now on, it is assumed that $\operatorname{int}P \neq \emptyset$.

The cone P is called *normal* if there is a number $K \geq 1$ such that for all $x, y \in E$: $0 \leq x \leq y \implies \|x\| \leq K\|y\|$. Here, the least positive integer K satisfying this equation is called the *normal constant* of P. P is said to be *regular* if every increasing sequence which is bounded from above is convergent. That is, if $\{x_n\}_{n\geq 1}$ is a sequence such that $x_1 \leq x_2 \leq \cdots \leq y$ for some $y \in E$, then there is $x \in E$ such that $\lim_{n\to\infty} \|x_n - x\| = 0$.

1.1. Lemma. (see [4],[16])

- (i) *Every regular cone is normal.*
- (ii) *For each $k > 1$, there is a normal cone with normal constant $K > k$.*
- (iii) *The cone P is regular if every decreasing sequence which is bounded from below is convergent.* □

1.2. Definition. (see [5]) Let X be a non-empty set. Suppose the mapping $d : X \times X \to E$ satisfies:

- $(M1)$ $0 \leq d(x, y)$ for all $x, y \in X$,
- $(M2)$ $d(x, y) = 0$ if and only if $x = y$,
- $(M3)$ $d(x, y) \leq d(x, z) + d(z, y)$, for all $x, y \in X$.
- $(M4)$ $d(x, y) = d(y, x)$ for all $x, y \in X$

then d is called a *cone metric* on X, and the pair (X, d) is a *cone metric space* (CMS).

It is quite natural to consider cone normed spaces (CNS):

1.3. Definition. ([1, 21]) Let X be a vector space over $\mathbb{R}$. Suppose the mapping $\|\cdot\|_P : X \to E$ satisfies:

- $(N1)$ $\|x\|_P \geq 0$ for all $x \in X$,
- $(N2)$ $\|x\|_P = 0$ if and only if $x = 0$,
- $(N3)$ $\|x + y\|_P \leq \|x\|_P + \|y\|_P$, for all $x, y \in X$.
- $(N4)$ $\|kx\|_P = |k|\|x\|_P$ for all $k \in \mathbb{R}$,

then $\|\cdot\|_P$ is *called a cone norm on* X, and the pair $(X, \|\cdot\|_P)$ *a cone normed space* (CNS).

Note that each CNS is a CMS. Indeed, $d(x, y) = \|x - y\|_P$.

1.4. Definition. Let $(X, \|\cdot\|_P)$ be a CNS, $x \in X$ and $\{x_n\}_{n\geq 1}$ a sequence in X. Then:

- (i) $\{x_n\}_{n\geq 1}$ *converges to* x whenever for every $c \in E$ with $0 \ll c$ there is a natural number N such that $\|x_n - x\|_P \ll c$ for all $n \geq N$. It is denoted by $\lim_{n\to\infty} x_n = x$, or $x_n \to x$.

(*ii*) $\{x_n\}_{n\geq 1}$ is a *Cauchy sequence* whenever for every $c \in E$ with $0 \ll c$ there is a natural number N such that $\|x_n - x\|_P \ll c$ for all $n, m \geq N$.

(*iii*) $(X, \|\cdot\|_P)$ is a *complete cone normed space* if every Cauchy sequence is convergent.

As expected, complete cone normed spaces will be called cone Banach spaces.

1.5. Lemma. (see [7]) *Let $(X, \|\cdot\|_P)$ be a CNS, P a normal cone with normal constant K, and $\{x_n\}$ a sequence in X. Then,*

(*i*) *The sequence $\{x_n\}$ converges to x if and only if $\|x_n - x\|_P \to 0$, as $n \to \infty$,*

(*ii*) *The sequence $\{x_n\}$ is Cauchy if and only if $\|x_n - x_m\|_P \to 0$ as $n, m \to \infty$,*

(*iii*) *If the sequence $\{x_n\}$ converges to x and the sequence $\{y_n\}$ converges to y then $\|x_n - y_n\|_P \to \|x - y\|_P$.*

Proof. Immediate by applying Lemma 1, Lemma 4 and Lemma 5 in [5] to the cone metric space (X, d), where $d(x, y) = \|x - y\|_P$ for all $x, y \in X$. □

1.6. Lemma. (see [19, 20, 7]) *Let $(X, \|\cdot\|_P)$ be a CNS over a cone P in E. Then*

(1) $\mathrm{int}(P) + \mathrm{int}(P) \subseteq \mathrm{int}(P)$ *and* $\lambda \mathrm{int}(P) \subseteq \mathrm{int}(P), \lambda > 0$.

(2) *If $c \gg 0$ then there exists $\delta > 0$ such that $\|b\| < \delta$ implies $b \ll c$.*

(3) *For any given $c \gg 0$ and $c_0 \gg 0$ there exists $n_0 \in \mathbb{N}$ such that $\frac{c_0}{n_0} \ll c$.*

(4) *If a_n, b_n are sequences in E such that $a_n \to a$, $b_n \to b$ and $a_n \leq b_n, \forall n$, then $a \leq b$.* □

2. Main Results

From now on, $X = (X, \|\cdot\|_P)$ will be a cone Banach space, P a normal cone with normal constant K, and T a self-mapping operator defined on a subset C of X.

2.1. Theorem. *Let C be a closed and convex subset of a cone Banach space X with norm $\|x\|_P$, and let $d : X \times X \to E$ be such that $d(x, y) = \|x - y\|_P$. If there exist a, b, c, s and $T : C \to C$ satisfying the conditions*

$$0 \leq \frac{s + a - 2b - c}{2(a + b)} < 1, \ a + b \neq 0, \ a + b + c > 0 \text{ and } s \geq 0, \tag{2.1}$$

$$ad(Tx, Ty) + b\big[d(x, Tx) + d(y, Ty)\big] + cd(y, Tx) \leq sd(x, y) \tag{2.2}$$

hold for all $x, y \in C$. Then, T has at least one fixed point.

Proof. Let $x_0 \in C$ be arbitrary. Define a sequence $\{x_n\}$ in the following way:

$$x_{n+1} := \frac{x_n + Tx_n}{2}, \ n = 0, 1, 2, \ldots. \tag{2.3}$$

Notice that

$$x_n - Tx_n = 2\Big(x_n - \Big(\frac{x_n + Tx_n}{2}\Big)\Big) = 2(x_n - x_{n+1}), \tag{2.4}$$

which yields that

$$d(x_n, Tx_n) = \|x_n - Tx_n\|_P = 2\|x_n - x_{n+1}\|_P = 2d(x_n, x_{n+1}) \tag{2.5}$$

for $n = 0, 1, 2, \ldots$. Analogously, for $n = 0, 1, 2, \ldots$, one can get

$$\begin{aligned} &d(x_{n-1}, Tx_{n-1}) = 2d(x_{n-1}, x_n), \text{ and} \\ &d(x_n, Tx_{n-1}) = \frac{1}{2}d(x_{n-1}, Tx_{n-1}) = d(x_{n-1}, x_n), \end{aligned} \tag{2.6}$$

and by the triangle inequality

$$d(x_n, Tx_n) - d(x_n, Tx_{n-1}) \leq d(Tx_{n-1}, Tx_n). \tag{2.7}$$

When we substitute $x = x_{n-1}$ and $y = x_n$ in the inequality (2.2), it implies that

$$ad(Tx_{n-1}, Tx_n) + b\big[d(x_{n-1}, Tx_{n-1}) + d(x_n, Tx_n)\big] + cd(x_n, Tx_{n-1}) \leq sd(x_{n-1}, x_n) \tag{2.8}$$

for all a, b, c, s that satisfy (2.1). Taking into account (2.5) and (2.6), one can observe

$$ad(Tx_{n-1}, Tx_n) + b\big[2d(x_{n-1}, x_n) + 2d(x_n, x_{n+1})\big] + cd(x_{n-1}, x_n) \leq sd(x_{n-1}, x_n), \tag{2.9}$$

which is equivalent to

$$ad(Tx_{n-1}, Tx_n) \leq sd(x_{n-1}, x_n) - 2b\big[d(x_{n-1}, x_n) + d(x_n, x_{n+1})\big] - cd(x_{n-1}, x_n). \tag{2.10}$$

By using (2.7), the statement (2.10) turns into

$$\begin{aligned} & a\big[d(x_n, Tx_n) - d(x_n, Tx_{n-1})\big] \\ & \qquad \leq sd(x_{n-1}, x_n) - 2b\big[d(x_{n-1}, x_n) + d(x_n, x_{n+1})\big] - cd(x_{n-1}, x_n). \end{aligned} \tag{2.11}$$

Regarding (2.5) and (2.6) again, simple calculations yield (2.11), that is

$$2(a + b)d(x_n, x_{n+1}) \leq (s + a - 2b - c)d(x_{n-1}, x_n).$$

Since $a + b \neq 0$, we get

$$d(x_n, x_{n+1}) \leq \frac{s + a - 2b - c}{2(a + b)} d(x_{n-1}, x_n).$$

Thus, the sequence $\{x_n\}$ is a Cauchy sequence that converges to some element of C, say z.

To show z is a fixed point of T, it is sufficient to substitute $x = z$ and $y = x_n$ in the inequality (2.2). Indeed, due to the equation (2.3) and $x_n \to z$, we have $Tx_n \to z$. Thus,

$$ad(Tz, Tx_n) + b\big[d(z, Tz) + d(x_n, Tx_n)\big] + cd(x_n, Tz) \leq sd(z, x_n),$$

which implies $ad(Tz, z) + bd(z, Tz) + cd(z, Tz) \leq 0$ as $n \to \infty$. Thus, $Tz = z$ as $a + b + c > 0$. □

2.2. Definition. [6] Let S, T be self-mappings on a CMS (X, d). A point $z \in X$ is called a *coincidence point of* S, T if $Sz = Tz$, and it is called a *common fixed point of* S, T if $Sz = z = Tz$. Moreover, a pair of self-mappings (S, T) is called *weakly compatible on* X if they commute at their coincidence points, in other words,

$$z \in X,\ Sz = Tz \implies STz = TSz.$$

2.3. Theorem. *Let C be a closed and convex subset of a cone Banach space X with norm $\|\cdot\|_P$, and let $d : X \times X \to E$ with $d(x, y) = \|x - p\|_P$. If T and S are self-mappings on C that satisfy the conditions*

$$T(C) \subset S(C) \tag{2.12}$$

$$S(C) \textit{ is a complete subspace} \tag{2.13}$$

$$\begin{aligned} & ad(Tx, Ty) + b\big[d(Sx, Tx) + d(Sy, Ty)\big] \leq rd(Sx, Sy), \\ & \textit{for } a + b \neq 0,\ 0 \leq r < a + 2b,\ r < b,\ a \neq r, \end{aligned} \tag{2.14}$$

hold for all $x, y \in C$, then, S and T have a common coincidence point. Furthermore, if S and T are weakly compatible, then they have a unique common fixed point in C.

Proof. Let $x_0 \in C$ be arbitrary. Regarding (2.12), we can find a point in C, say x_1, such that $Tx_0 = Sx_1$. Since S, T are self-mappings, there is a point in C, say y_0, such that $y_0 = Tx_0 = Sx_1$. Inductively we can define a sequence $\{y_n\}$ and a sequence $\{x_n\} \subset C$ in the following way:

$$y_n = Sx_{n+1} = Tx_n,\ n = 0, 1, 2, \ldots. \tag{2.15}$$

When we substitute $x = x_n$ and $y = x_{n+1}$ in the inequality (2.14), it implies that

$$ad(Tx_n, Tx_{n+1}) + b\big[d(Sx_n, Tx_n) + d(Sx_{n+1}, Tx_{n+1})\big] \le rd(Sx_n, Sx_{n+1}), \tag{2.16}$$

which is equivalent to

$$ad(y_n, y_{n+1}) + b\big[d(y_{n-1}, y_n) + d(y_n, y_{n+1})\big] \le rd(y_{n-1}, y_n). \tag{2.17}$$

By simple calculations, (2.17) turns into

$$d(y_n, y_{n+1}) \le \frac{r-b}{a+b} d(y_{n-1}, y_n). \tag{2.18}$$

Analogously, one can observe that

$$d(y_{n-1}, y_n) \le kd(y_{n-2}, y_{n-1}), \tag{2.19}$$

where $k = \frac{r-b}{a+b}$. Since $0 \le r < a + 2b$, $r < b$, then $0 \le k < 1$. Combining (2.18) and (2.19), we have

$$d(y_n, y_{n+1}) \le kd(y_{n-1}, y_n) \le k^2 d(y_{n-2}, y_{n-1}). \tag{2.20}$$

By routine calculations,

$$d(y_n, y_{n+1}) \le k^n d(y_0, y_1). \tag{2.21}$$

To show $\{y_n\}$ is a Cauchy sequence, let $n > m$. Then by (2.21) and the triangle inequality, one can obtain

$$\begin{aligned} d(y_n, y_m) &\le d(y_n, y_{n-1}) + d(y_{n-1}, y_{n-2}) + \cdots + d(y_{m+1}, y_m) \\ &\le k^{n-1} d(y_0, y_1) + k^{n-2} d(y_0, y_1) + \cdots + k^m d(y_0, y_1) \\ &\le \frac{k^m}{1-k} d(y_0, y_1), \end{aligned} \tag{2.22}$$

which concludes the proof that $\{y_n\}$ is a Cauchy sequence. Since $S(C)$ is complete, then $\{y_n = Sx_{n+1} = Tx_n\}$ converges to some point in $S(C)$, say z. In other words, there is a point $p \in C$ such that $Sp = z$. Now, by replacing x with p and y with x_{n+1} in the inequality (2.14), we get

$$ad(Tp, Tx_{n+1}) + b\big[d(Sp, Tp) + d(Sx_{n+1}, Tx_{n+1})\big] \le rd(Sp, Sx_{n+1}),$$

which is equivalent to

$$ad(Tp, y_{n+1}) + b\big[d(z, Tp) + d(y_n, y_{n+1})\big] \le rd(z, y_n).$$

As $n \to \infty$, it becomes

$$ad(Tp, z) + bd(z, Tp) \le 0.$$

Since $a + b \ne 0$, then $Tp = z$. Hence $Tp = z = Sp$, in other words, p is a coincidence point of S and T.

If S and T are weakly compatible, then they commute at a coincidence point. Therefore, $Tp = z = Sp \implies STp = TSp$ for some $p \in C$, which implies $Tz = Sz$.

Claim: z is common fixed point of S and T. To show this, substitute $x = p$ and $y = Tp = z$ in the inequality (2.14) to give

$$ad(Tp, TTp) + b\big[d(Sp, Tp) + d(STp, TTp)\big] \le rd(Sp, STp),$$

which is equivalent to

$$ad(z, Tz) + b\big[d(z, z) + d(Sz, Tz)\big] \le rd(Tp, TSp) = rd(z, Tz).$$

So we have $(a - r)d(z, Tz) \le 0$. Since $a \ne r$, then $z = Tz = Sz$.

We use reductio ad absurdum to prove uniqueness. Suppose the contrary, that w is another common fixed point of S and T. Substituting x by z and y by w in the inequality (2.14), one can get

$$ad(Tz, Tw) + b\big[d(Sz, Tz) + d(Sw, Tw)\big] \leq rd(Sz, Sw),$$

which is equivalent to

$$ad(z, w) \leq rd(z, w) \iff (a - r)d(z, w) \leq 0,$$

which is a contradiction since $a \neq r$. Therefore, the common fixed point of S and T is unique. □

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