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SOME IDENTITIES FOR GENERALIZED FIBONACCI AND LUCAS SEQUENCES

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Abstract

In this study, we define a generalization of Lucas sequence $\{p_n\}$. Then we obtain Binet formula of sequence $\{p_n\}$. Also, we investigate relationships between generalized Fibonacci and Lucas sequences.

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1. Introduction

For $n \geq 2$, the Fibonacci and Lucas numbers are defined by following recurrence relations

$$F_0 = 0, \quad F_1 = 1, \quad F_n = F_{n-1} + F_{n-2}$$

and

$$L_0 = 2, \quad L_1 = 1, \quad L_n = L_{n-1} + L_{n-2}.$$

And Fibonacci and Lucas numbers' Binet formulas are known as,

$$F_n = \frac{\tau^n - \gamma^n}{\tau - \gamma} \qquad \text{and} \qquad L_n = \tau^n + \gamma^n$$

where $n \geq 0$ and τ, γ are roots of $x^2 - x - 1 = 0$.

These sequences have been generalized in many ways. For example, in [1], the author generalized the sequences $\{F_n\}$ and $\{L_n\}$ as follows,

$$W_n = AW_{n-1} + BW_{n-1}, \quad W_0 = a, \quad W_1 = b \quad \text{for} \quad n \geq 2,$$

where a, b, A and B are arbitrary integers.

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In [2] and [3], the authors introduced and studied a new kind generalized Fibonacci sequence and its properties that depends on two real parameters as defined below, for $n > 1$

$$q_0 = 0, \quad q_1 = 1 \quad q_n = \begin{cases} aq_{n-1} + q_{n-2} & \text{if } n \text{ is even,} \\ bq_{n-1} + q_{n-2} & \text{if } n \text{ is odd .} \end{cases}$$

Its extended Binet's formula was given by

$$q_n = \left(\frac{a^{1-\xi(n)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \right) \frac{\alpha^n - \beta^n}{\alpha - \beta},$$

where $\alpha = \frac{ab+\sqrt{a^2b^2+4ab}}{2}$, $\beta = \frac{ab-\sqrt{a^2b^2+4ab}}{2}$ and $\xi(n) := n - 2\left\lfloor \frac{n}{2} \right\rfloor$. Note that α and β are roots of the quadratic equation $x^2 - abx - ab = 0$ and $\xi(n) = 0$ when n is even, $\xi(n) = 1$ when n is odd. Also, authors generalized some identities as follows;

- Cassini Identity

  $a^{1-\xi(n)} b^{\xi(n)} q_{n-1} q_{n+1} - a^{\xi(n)} b^{1-\xi(n)} q_n^2 = a(-1)^n$

- Catalan's Identity

  $a^{\xi(n-r)} b^{1-\xi(n-r)} q_{n-r} q_{n+r} - a^{\xi(n)} b^{1-\xi(n)} q_n^2 = a^{\xi(r)} b^{1-\xi(r)} (-1)^{n+1-r} q_r^2$

- d'Ocagne's Identity

  $a^{\xi(mn+m)} b^{\xi(mn+m)} q_m q_{n+1} - a^{\xi(mn+m)} b^{\xi(mn+m)} q_{m+1} q_n = (-1)^n a^{\xi(m-n)} q_{m-n}$

- Additional Identities

  $a^{\xi(mn+m)} b^{\xi(mn+m)} q_m q_{n+1} + a^{\xi(mn+m)} b^{\xi(mn+m)} q_{m-1} q_n = a^{\xi(mn+m)} q_{m+n}$

  $a^{\xi(km)} b^{\xi(km+k)} q_m q_{k-m+1} + a^{\xi(km+k)} b^{\xi(km)} q_{m-1} q_{k-m} = a^{\xi(k)} q_k$

  $a^{1-\xi(n+k)} b^{\xi(n+k)} q_{n+k+1}^2 + a^{\xi(n-k)} b^{1-\xi(n-k)} q_{n-k}^2 = a q_{2n+1} q_{2k+1}$

  For more details, we refer to [2]. Also, in [7], author gave the Gelin-Cesaro identity as

  $$a^{2\xi(n)-1} b^{1-2\xi(n)} q_n^4 - q_{n-2} q_{n-1} q_{n+1} q_{n+2} = (-1)^{n+1} \left(\frac{a}{b}\right)^{\xi(n)} q_n^2 (ab-1) + a^2$$

In [6], author defined $k-$periodic second order linear recurrence as;

$$q_n = \begin{cases} a_0 q_{n-1} + b_0 q_{n-2} & \text{if } n \equiv 0 \,(mod\ k) \\ a_1 q_{n-1} + b_1 q_{n-2} & \text{if } n \equiv 0 \,(mod\ k) \\ \vdots & \vdots \\ a_{k-1} q_{n-1} + b_{k-1} q_{n-2} & \text{if } n \equiv 0 \,(mod\ k) \end{cases} \tag{1.1}$$

and investigated the combinatorial interpretation of the coefficients A_k and B_k appearing in the recurrence relation $q_n = A_k q_{n-k} + B_k q_{n-2k}$. And in [8], we found (1.1)'s exclipt formula for arbitrary coefficient and arbitrary initial conditions. The generalized Fibonacci and Lucas sequences have word combinatorial interpretation and they are closely related to continued expansion of quadratic irrationals (see in [2]).

There are lots of combinatorial identities between Fibonacci and Lucas numbers. For example,

· $F_n L_n = F_{2n}$
· $F_m F_n - F_{m+k} F_{n-k} = (-1)^{n-k} F_{m+k-n} F_k$
· $F_n = F_m F_{n-m+1} + F_{m-1} F_{n-m}$
· $L_n = L_m F_{n-m+1} + L_{m-1} F_{n-m}$
· $F_m L_n + F_n L_m = 2F_{m+n}$

· $L_n = F_{n+1} + F_{n-1}$
· $5F_n = L_{n+1} + L_{n-1}$
For more identities, they can be found in [4]. (page 87-93).

Up to now, authors gave some identities which are only contains Fibonacci generalizations. In this study, we define generalized Lucas sequences and give extended Binet's formula for generalized Lucas sequences. Moreover we investigate some properties which are involving generalized Fibonacci and Lucas numbers.

2. Main Results

2.1. Definition. For any two nonpozitive real numbers a and b, the generalized Lucas sequence $\{p_n\}$ is defined as follows;

$$p_0 = 2, \quad p_1 = 1, \quad p_n = \begin{cases} ap_{n-1} + p_{n-2} & \text{if } n \text{ is even,} \\ bp_{n-1} + p_{n-2} & \text{if } n \text{ is odd .} \end{cases}$$

We note that, these new generalizations is in the fact of a family sequences where each new choise of a and b produces a distinct sequences. For example, when we take $a = b = 1$ in $\{q_n\}$, the sequence produce Fibonacci numbers. When taking $a = b = 1$ in $\{p_n\}$, it produces Lucas numbers. When we take $a = b = 2$ in $\{p_n\}$, it produces Pell-Lucas numbers.

We derive some identities involving the generalized Fibonacci and Lucas sequences. From the definitions of α and β, we note that

$$(\alpha+1)(\beta+1) = 1, \quad \alpha+\beta = ab, \quad \alpha\beta = -ab, \quad ab(\alpha+1) = \alpha^2, \quad -\beta(\alpha+1) = \alpha.$$

Now we give the generalized Binet formula for the generalized Lucas sequences $\{p_n\}$:

2.2. Theorem. *For* $n > 1$,

$$p_n = \frac{a^{1-\xi(n)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \left(\frac{\alpha^n + \beta^n}{a} + (1-b)\frac{\alpha^n - \beta^n}{\alpha - \beta} \right)$$

where $\alpha = \frac{ab+\sqrt{a^2b^2+4ab}}{2}$, $\beta = \frac{ab-\sqrt{a^2b^2+4ab}}{2}$ *and* $\xi(n) := n - 2\lfloor \frac{n}{2} \rfloor$.

Proof. In order to prove the theorem, we use following equation given in [2] :

$$Q_n = Dq_n + C\left(\frac{b}{a}\right)^{\xi(n)} q_{n-1}$$

where

$$Q_n = \begin{cases} aQ_{n-1} + Q_{n-2} & \text{if } n \text{ is even,} \\ bQ_{n-1} + Q_{n-2} & \text{if } n \text{ is odd,} \end{cases}$$

$Q_0 = C$ and $Q_1 = D$ are initial conditions of the sequence $\{Q_n\}$. When $C = 2$ and $D = 1$, we obtain

$$\begin{aligned} p_n &= q_n + 2\left(\frac{b}{a}\right)^{\xi(n)} q_{n-1} \\ &= \frac{a^{1-\xi(n)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \left(\frac{\alpha^n - \beta^n}{\alpha - \beta} \right) + 2\left(\frac{b}{a}\right)^{\xi(n)} \frac{a^{1-\xi(n-1)}}{(ab)^{\lfloor \frac{n-1}{2} \rfloor}} \left(\frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta} \right) \\ &= \frac{a^{1-\xi(n)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \left(\frac{\alpha^n - \beta^n}{\alpha - \beta} \right) + 2b\frac{a^{1-\xi(n)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \left(\frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta} \right) \end{aligned}$$

$$= \frac{a^{1-\xi(n)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \left(\frac{\alpha^n \left(1 + 2b\alpha^{-1}\right) - \beta^n \left(1 + 2b\beta^{-1}\right)}{\alpha - \beta} \right)$$
$$= \frac{a^{1-\xi(n)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \left(\frac{\alpha^n \left(\frac{\alpha-\beta}{a} + (1-b)\right) - \beta^n \left(-\frac{\alpha-\beta}{a} + (1-b)\right)}{\alpha - \beta} \right)$$
$$= \frac{a^{1-\xi(n)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \left(\frac{\alpha^n + \beta^n}{a} + (1-b) \frac{\alpha^n - \beta^n}{\alpha - \beta} \right),$$

as claimed. □

When we take $a = b = 1$, we obtain Binet formula for Lucas sequences.

3. Several Identities Involving the Generalized Fibonacci And Lucas Numbers

In this section, we derive several identities involving the generalized Fibonacci and Lucas numbers. We start with the following result:

3.1. Theorem. *For $n \geq 0$*

$$p_n q_n = \left(\frac{b}{a}\right)^{\xi(n)} q_{2n} + (1-b)\, q_n^2.$$

Proof. By using the Binet formulas of $\{q_n\}$ and $\{p_n\}$, we have

$$p_n q_n = \left(\frac{a^{1-\xi(n)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \right)^2 \left(\frac{\alpha^n + \beta^n}{a} + (1-b) \frac{\alpha^n - \beta^n}{\alpha - \beta} \right) \left(\frac{\alpha^n - \beta^n}{\alpha - \beta} \right)$$
$$= \left(\frac{a^{1-\xi(n)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \right)^2 \left(\frac{\alpha^{2n} - \beta^{2n}}{a(\alpha - \beta)} + (1-b) \left(\frac{\alpha^n - \beta^n}{\alpha - \beta} \right)^2 \right)$$
$$= \left(\frac{b}{a}\right)^{\xi(n)} \frac{a^{1-\xi(2n)}}{(ab)^{\lfloor \frac{2n}{2} \rfloor}} \left(\frac{\alpha^{2n} - \beta^{2n}}{\alpha - \beta} \right) + (1-b) \left(\frac{a^{1-\xi(n)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \frac{\alpha^n - \beta^n}{\alpha - \beta} \right)^2$$
$$= \left(\frac{b}{a}\right)^{\xi(n)} q_{2n} + (1-b)\, q_n^2.$$

Thus the proof is complete. □

When $a = b = 1$, we obtain the well known result for the usual Fibonacci and Lucas numbers :

$$L_n F_n = F_{2n}.$$

3.2. Theorem. *For $n \geq 0$*

$$q_{n+1} + q_{n-1} = ab^{-\xi(n)} \left(p_n - (1-b)\, q_n\right).$$

Proof. In order to prove the claim, we again use the extended Binet formulas of the sequences $\{q_n\}$ and $\{p_n\}$:

$$\begin{aligned}
q_{n+1}+q_{n-1} &= \frac{a^{1-\xi(n+1)}}{(ab)^{\lfloor\frac{n+1}{2}\rfloor}}\left(\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}\right)+\frac{a^{1-\xi(n-1)}}{(ab)^{\lfloor\frac{n-1}{2}\rfloor}}\left(\frac{\alpha^{n-1}-\beta^{n-1}}{\alpha-\beta}\right)\\
&= \frac{a^{1-\xi(n-1)}}{(ab)^{\lfloor\frac{n-1}{2}\rfloor}(\alpha-\beta)}\left(\frac{\alpha^{n+1}-\beta^{n+1}}{ab}+\alpha^{n-1}-\beta^{n-1}\right)\\
&= \frac{a^{1-\xi(n-1)}}{(ab)^{\lfloor\frac{n-1}{2}\rfloor}(\alpha-\beta)}\left(\alpha^{n}\left(\frac{\alpha}{ab}-\frac{\beta}{ab}\right)+\beta^{n}\left(\frac{\alpha}{ab}-\frac{\beta}{ab}\right)\right)\\
&= \frac{a^{1-\xi(n-1)}}{(ab)^{\lfloor\frac{n-1}{2}\rfloor}b}\left(\frac{\alpha^{n}+\beta^{n}}{a}+(1-b)\,\frac{\alpha^{n}-\beta^{n}}{\alpha-\beta}\right)\\
&- \frac{a^{1-\xi(n-1)}}{(ab)^{\lfloor\frac{n-1}{2}\rfloor}}\frac{(1-b)}{b}\frac{\alpha^{n}-\beta^{n}}{\alpha-\beta}\\
&= \left(\frac{a}{b}\right)^{\xi(n)}\left(p_n-(1-b)\,q_n\right),
\end{aligned}$$

as claimed. □

When $a=b=1$ in Theorem 3, we obtain the well known the formula:

$$F_{n+1}+F_{n-1}=L_n.$$

3.3. Theorem. *For* $n\geq 0$,

$$p_{n+1}+p_{n-1}=\left(\frac{a}{b}\right)^{\xi(n)}\left(\left(\frac{\alpha-\beta}{a}\right)^{2}q_n+(1-b)\,p_n-(1-b)^{2}\,q_n\right).$$

Proof. Consider

$$\begin{aligned}
p_{n+1}+p_{n-1} &= \frac{a^{1-\xi(n+1)}}{(ab)^{\lfloor\frac{n+1}{2}\rfloor}}\left(\frac{\alpha^{n+1}+\beta^{n+1}}{a}+(1-b)\,\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}\right)\\
&+ \frac{a^{1-\xi(n-1)}}{(ab)^{\lfloor\frac{n-1}{2}\rfloor}}\left(\frac{\alpha^{n-1}+\beta^{n-1}}{a}+(1-b)\,\frac{\alpha^{n-1}-\beta^{n-1}}{\alpha-\beta}\right)\\
&= \frac{a^{1-\xi(n-1)}}{(ab)^{\lfloor\frac{n-1}{2}\rfloor}}\left(\frac{1}{a}\alpha^{n}\left(\frac{\alpha}{ab}-\frac{\beta}{ab}\right)+\frac{1}{a}\beta^{n}\left(\frac{\beta}{ab}-\frac{\alpha}{ab}\right)\right.\\
&\left.+\frac{1-b}{\alpha-\beta}\alpha^{n}\left(\frac{\alpha}{ab}-\frac{\beta}{ab}\right)-\frac{(1-b)}{\alpha-\beta}\beta^{n}\left(\frac{\beta}{ab}-\frac{\alpha}{ab}\right)\right)\\
&= \frac{a^{1-\xi(n-1)}}{(ab)^{\lfloor\frac{n-1}{2}\rfloor}}\left(\frac{(\alpha-\beta)^{2}}{a^{2}b}\frac{\alpha^{n}-\beta^{n}}{\alpha-\beta}+(1-b)\,\frac{\alpha^{n}+\beta^{n}}{ab}\right)\\
&= \frac{(\alpha-\beta)^{2}}{a^{2}b}a^{\xi(n)}b^{\xi(n-1)}\frac{a^{1-\xi(n)}}{(ab)^{\lfloor\frac{n}{2}\rfloor}}\frac{\alpha^{n}-\beta^{n}}{\alpha-\beta}\\
&+ \frac{1-b}{b}a^{\xi(n)}b^{\xi(n-1)}\frac{a^{1-\xi(n)}}{(ab)^{\lfloor\frac{n}{2}\rfloor}}\left(\frac{\alpha^{n}+\beta^{n}}{a}+(1-b)\,\frac{\alpha^{n}-\beta^{n}}{\alpha-\beta}\right)\\
&- a^{\xi(n)}b^{\xi(n-1)}\frac{(1-b)^{2}}{b}\frac{a^{1-\xi(n)}}{(ab)^{\lfloor\frac{n}{2}\rfloor}}\frac{\alpha^{n}-\beta^{n}}{\alpha-\beta}
\end{aligned}$$

$$= \left(\frac{a}{b}\right)^{\xi(n)} \left(\left(\frac{\alpha-\beta}{a}\right)^2 q_n + (1-b)\,p_n - (1-b)^2\,q_n\right).$$

□

When we take $a = b = 1$, we obtain $L_{n+1} + L_{n-1} = 5F_n$.

3.4. Theorem. *For $m, n \geq 0$*

$$q_m p_n + q_n p_m = 2\left(\frac{b}{a}\right)^{\xi(mn)} q_{m+n} + 2\,(1-b)\,q_n q_m.$$

Proof. Using the Binet formulas, and the identity follows easily from definition $\xi\,(m+n) = \xi\,(m) + \xi\,(n) - 2\xi\,(m)\,\xi\,(n)$. Then consider

$$\begin{aligned} q_m p_n + q_n p_m &= \left(\frac{a^{1-\xi(m)}}{(ab)^{\lfloor \frac{m}{2} \rfloor}} \frac{\alpha^m - \beta^m}{\alpha-\beta}\right)\left(\frac{a^{1-\xi(n)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}}\left(\frac{\alpha^n+\beta^n}{a} + (1-b)\,\frac{\alpha^n-\beta^n}{\alpha-\beta}\right)\right) \\ &+ \left(\frac{a^{1-\xi(n)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \frac{\alpha^n - \beta^n}{\alpha-\beta}\right)\left(\frac{a^{1-\xi(m)}}{(ab)^{\lfloor \frac{m}{2} \rfloor}}\left(\frac{\alpha^m+\beta^m}{a} + (1-b)\,\frac{\alpha^m-\beta^m}{\alpha-\beta}\right)\right) \\ &= 2\frac{a^{2-\xi(m)-\xi(n)}}{(ab)^{\lfloor \frac{n}{2} \rfloor + \lfloor \frac{m}{2} \rfloor}}\left(\frac{\alpha^{m+n}-\beta^{m+n}}{\alpha-\beta}\right) \\ &+ 2\,(1-b)\left(\frac{a^{1-\xi(n)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \frac{\alpha^n - \beta^n}{\alpha-\beta}\right)\left(\frac{a^{1-\xi(m)}}{(ab)^{\lfloor \frac{m}{2} \rfloor}} \frac{\alpha^m - \beta^m}{\alpha-\beta}\right) \\ &= 2\left(\frac{b}{a}\right)^{\xi(mn)} \frac{a^{1-\xi(m+n)}}{(ab)^{\lfloor \frac{m+n}{2} \rfloor}}\left(\frac{\alpha^{m+n}-\beta^{m+n}}{\alpha-\beta}\right) \\ &+ 2\,(1-b)\left(\frac{a^{1-\xi(n)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \frac{\alpha^n - \beta^n}{\alpha-\beta}\right)\left(\frac{a^{1-\xi(m)}}{(ab)^{\lfloor \frac{m}{2} \rfloor}} \frac{\alpha^m - \beta^m}{\alpha-\beta}\right) \\ &= 2\left(\frac{b}{a}\right)^{\xi(mn)} q_{m+n} - (1-b)\,q_n q_m. \end{aligned}$$

□

When we take $a = b = 1$, we obtain $2F_{m+n} = F_m L_n + F_n L_m$.

3.5. Theorem. *For $n, m \geq 0$,*

$$\left(\frac{b}{a}\right)^{\xi(n)} a^{2\xi(mn)} p_m q_{n-m+1} + \left(\frac{b}{a}\right)^{\xi(mn)} p_{m-1} q_{n-m} = p_n.$$

Proof. If we use the Binet formulas of generalized Fibonacci and Lucas sequences and by using the identity $\xi\,(m+n) = \xi\,(m) + \xi\,(n) - 2\xi\,(m)\,\xi\,(n)$, we obtain that

$$\begin{aligned} &\left(\frac{b}{a}\right)^{\xi(n)} a^{2\xi(mn)} p_m q_{n-m+1} + \left(\frac{b}{a}\right)^{\xi(mn)} p_{m-1} q_{n-m} \\ =&\frac{a^{1-\xi(n)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \frac{\alpha^{n-m+1}-\beta^{n-m+1}}{\alpha-\beta}\left(\frac{\alpha^m+\beta^m}{a} + (1-b)\,\frac{\alpha^m-\beta^m}{\alpha-\beta}\right) \\ &+ \frac{a^{1-\xi(n)}}{(ab)^{\lfloor \frac{n}{2} \rfloor - 1}} \frac{\alpha^{n-m}-\beta^{n-m}}{\alpha-\beta}\left(\frac{\alpha^{m-1}+\beta^{m-1}}{a} + (1-b)\,\frac{\alpha^{m-1}-\beta^{m-1}}{\alpha-\beta}\right) \end{aligned}$$

$$= \frac{a^{1-\xi(n)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \left(\frac{\alpha^n \left(\alpha + \frac{ab}{\alpha}\right) - \beta^n \left(\beta + \frac{ab}{\alpha}\right)}{a(\alpha - \beta)} + (1-b) \frac{\alpha^n \left(\alpha + \frac{ab}{\alpha}\right) + \beta^n \left(\beta + \frac{ab}{\beta}\right)}{(\alpha - \beta)^2} \right)$$

$$= \frac{a^{1-\xi(n)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \left(\frac{\alpha^n + \beta^n}{a} + (1-b) \frac{\alpha^n - \beta^n}{\alpha - \beta} \right) = p_n,$$

as claimed. □

When $a = b = 1$ in Theorem above, we deduce the following well known formula:

$$L_n = L_m F_{n-m+1} + L_{m-1} F_{n-m}.$$

3.6. Theorem. *For $n \geq 0$*

$$\frac{1}{a} p_{2n+1} + b(-1)^n + \frac{(1-b)}{a} q_{2n+1} + (1-b)^2 q_n q_{n+1} = p_n p_{n+1}.$$

Proof. Consider

$$\begin{aligned} p_n p_{n+1} =& \frac{a^{1-\xi(n)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \left(\frac{\alpha^n + \beta^n}{a} + (1-b) \frac{\alpha^n - \beta^n}{\alpha - \beta} \right) \\ & \times \frac{a^{1-\xi(n+1)}}{(ab)^{\lfloor \frac{n+1}{2} \rfloor}} \left(\frac{\alpha^{n+1} + \beta^{n+1}}{a} + (1-b) \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} \right) \\ & = \frac{a}{(ab)^{\lfloor \frac{2n+1}{2} \rfloor}} \left(\frac{\alpha^{2n+1} + \beta^{2n+1}}{a^2} + \frac{1-b}{a} \frac{\alpha^{2n+1} - \beta^{2n+1}}{\alpha - \beta} \right) \\ & + b(-1)^n + (1-b) \frac{a^{1-\xi(2n+1)}}{(ab)^{\lfloor \frac{2n+1}{2} \rfloor}} \frac{\alpha^{2n+1} - \beta^{2n+1}}{\alpha - \beta} \\ & + (1-b)^2 \frac{a^{2-(\xi(n)+\xi(n+1))}}{(ab)^{\lfloor \frac{n}{2} \rfloor + \lfloor \frac{n+1}{2} \rfloor}} \frac{\alpha^n - \beta^n}{\alpha - \beta} \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} \\ & = \frac{1}{a} p_{2n+1} + b(-1)^n + \frac{(1-b)}{a} q_{2n+1} + (1-b)^2 q_n q_{n+1}. \end{aligned}$$

So the proof is complete. □

When $a = b = 1$ in Theorem above, we obtain well known identity:

$$L_n L_{n+1} = L_{2n+1} + (-1)^n$$

In the following theorem, we list Binomial sums with $\{p_n\}$ sequence. And we proved one of them. The other one can be prove in the same way.

3.7. Theorem. *The sequence $\{p_n\}$ satisfies the following identities.*

(*a*) $\sum_{k=0}^{n} \binom{n}{k} a^{\xi(k)} (ab)^{\lfloor \frac{k}{2} \rfloor} p_k = p_{2n}$

(*b*) $\sum_{k=0}^{n} \binom{n}{k} a^{\xi(k+r)} (ab)^{\lfloor \frac{k}{2} \rfloor + \xi(r)\xi(k)} p_{k+r} = p_{2n+r}$.

Proof. (*b*) Using Binet formula of $\{p_n\}$,

$$\begin{aligned} \sum_{k=0}^{n} \binom{n}{k} a^{\xi(k+r)} (ab)^{\lfloor \frac{k}{2} \rfloor + \xi(r)\xi(k)} p_{k+r} =& \sum_{k=0}^{n} \binom{n}{k} a^{\xi(k+r)} (ab)^{\lfloor \frac{k}{2} \rfloor + \xi(r)\xi(k)} \\ & \times \left(\frac{a^{1-\xi(k+r)}}{(ab)^{\lfloor \frac{k+r}{2} \rfloor}} \left(\frac{\alpha^{k+r} + \beta^{k+r}}{a} - (1-b) \frac{\alpha^{k+r} - \beta^{k+r}}{\alpha - \beta} \right) \right) \end{aligned}$$

$$= \sum_{k=0}^{n} \binom{n}{k} \frac{a}{(ab)^{\lfloor \frac{r}{2} \rfloor}} \left(\frac{\alpha^{k+r} + \beta^{k+r}}{a} + (1-b)\, \frac{\alpha^{k+r} - \beta^{k+r}}{\alpha - \beta} \right)$$
$$= \frac{a^{1-\xi(2n)}}{(ab)^{\lfloor \frac{2n+r}{2} \rfloor}} \left(\frac{\alpha^{2n+r} + \beta^{2n+r}}{a} + (1-b)\, \frac{\alpha^{2n+r} - \beta^{2n+r}}{\alpha - \beta} \right)$$
$$= p_{2n+r}.$$

□

When we take $a = b = 1$, we obtain $\sum_{k=0}^{n} \binom{n}{k} L_k = L_{2n}$ and $\sum_{k=0}^{n} \binom{n}{k} L_{k+r} = L_{2n+r}$.

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