

ON MANNHEIM PARTNER CURVES IN DUAL LORENTZIAN SPACE

Sıddıka Özkaldı^{*†‡}, Kazım İlarıslan[§] and Yusuf Yaylı^{*}

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Abstract

In this paper we define non-null Mannheim partner curves in three dimensional dual Lorentzian space \mathbb{D}_1^3 , and obtain necessary and sufficient conditions for the existence of non-null Mannheim partner curves in dual Lorentzian space \mathbb{D}_1^3 .

Keywords: Mannheim partner curve, Dual Lorentzian space, Dual Lorentzian space curve.

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1. Introduction

In the differential geometry of a regular curve in Euclidean 3-space \mathbb{E}^3 , it is well known that one of the important problems is the characterization of a regular curve. The curvature functions k_1 (curvature \varkappa) and k_2 (torsion τ) of a regular curve play an important role in determining the shape and size of the curve [4, 8]. For example: If $k_1 = k_2 = 0$, then the curve is a geodesic. If $k_1 \neq 0$ (constant) $k_2 = 0$, then the curve is a circle with radius $1/k_1$. If $k_1 \neq 0$ (constant) and $k_2 \neq 0$ (constant), then the curve is a helix in the space, etc.

Another route to the classification and characterization of curves is the relationship between the Frenet vectors of the curves. For example Saint Venant, in 1845, proposed the question whether upon the surface generated by the principal normal of a curve, a second curve can exist which has for its principal normal the principal normal of the given curve. This question was answered by Bertrand in 1850, when he showed that a necessary and sufficient condition for the existence of such a second curve is that a linear

^{*}Department of Mathematics, Faculty of Science, Ankara University, Tandođan, Ankara, Turkey. E-mail:

(S. Özkaldı) sozkaldi@science.ankara.edu.tr (Y. Yaylı) yayli@science.ankara.edu.tr

[†]Current Address: Department of Mathematics, Faculty of Science and Arts, Bilecik University, Bilecik, Turkey. E-mail: sozkaldi98@yahoo.com

[‡]Corresponding Author.

[§]Department of Mathematics, Faculty of Science and Arts, Kırıkkale University, Kırıkkale, Turkey. E-mail: kilarıslan@yahoo.com

relationship with constant coefficients shall exist between the first and second curvatures of the given original curve. The pairs of curves of this kind have been called Conjugate Bertrand Curves, or more commonly Bertrand Curves [4, 8, 13]. There are many works related with Bertrand curves in Euclidean space and Minkowski space. Associated curves of another kind, called Mannheim curves and Mannheim partner curves occur if there exists a relationship between the space curves α and β such that, at the corresponding points of the curves, the principal normal lines of α coincide with the binormal lines of β . Then, α is called a *Mannheim curve*, and β the *Mannheim partner curve* of α . Mannheim partner curves were studied by Liu and Wang [9] in Euclidean 3-space and in Minkowski 3-space.

Dual numbers had been introduced by *William Kingdon Clifford* (1845–1879) as a tool for his geometrical investigations. After him *Eduard Study* (1862–1930) used dual numbers and dual vectors in his research on line geometry and kinematics. He devoted special attention to the representation of oriented lines by dual unit vectors and defined the famous mapping: The set of oriented lines in a Euclidean three-dimensional space \mathbb{E}^3 is one to one correspondence with the points of a dual space \mathbb{D}^3 of triples of dual numbers [5].

Differential Geometric properties of regular curves in a dual space \mathbb{D}^3 , as well as in a dual Lorentzian space \mathbb{D}_1^3 have been studied in many papers. For example see [16, 3, 7, 17, 18, 1, 2, 11, 15, 14]. Mannheim curves in a dual space were studied by the authors in [12]

In this paper we study non-null Mannheim partner curves in a dual Lorentzian space \mathbb{D}_1^3 .

2. Preliminary

Dual numbers were introduced by W.K. Clifford (1845–1879) as a tool for his geometrical investigations. After him, E. Study used dual numbers and dual vectors in his research on the geometry of lines and kinematics. He devoted special attention to the representation of directed lines by dual unit vectors, and defined the mapping that is known by his name. Namely, there exists one-to-one correspondence between the points of dual unit sphere \mathbb{S}^2 and the directed lines in \mathbb{R}^3 [5].

If we take the Minkowski 3-space \mathbb{R}_1^3 instead of \mathbb{R}^3 the E. Study (1862–1930) mapping can be stated as follows: The dual timelike and spacelike unit vectors of the dual hyperbolic and Lorentzian unit spheres \mathbb{H}_0^2 and \mathbb{S}_1^2 in the dual Lorentzian space \mathbb{D}_1^3 are in one-to-one correspondence with the directed timelike and spacelike lines in \mathbb{R}_1^3 , respectively. Then a differentiable curve on \mathbb{H}_0^2 corresponds to a timelike ruled surface at \mathbb{R}_1^3 . Similarly, the timelike (resp. spacelike) curve on \mathbb{S}_1^2 corresponds to any spacelike (resp. timelike) ruled surface in \mathbb{R}_1^3 .

We will survey briefly various fundamental concepts and properties in Lorentzian space. We refer mainly to O’Neill [10].

Let \mathbb{R}_1^3 be the 3-dimensional Lorentzian space with Lorentzian metric $\langle \cdot, \cdot \rangle : -dx_1^2 + dx_2^2 + dx_3^2$. It is known that in \mathbb{R}_1^3 there are three categories of curves and vectors, namely, spacelike, timelike and null, depending on their causal character. Let \vec{x} be a tangent vector of Lorentzian space. Then \vec{x} is said to be spacelike if $\langle \vec{x}, \vec{x} \rangle > 0$ or $\vec{x} = \vec{0}$, timelike if $\langle \vec{x}, \vec{x} \rangle < 0$, null (lightlike) if $\langle \vec{x}, \vec{x} \rangle = 0$ and $\vec{x} \neq \vec{0}$.

Let $\alpha : I \subset \mathbb{R} \rightarrow \mathbb{R}_1^3$ be a regular curve in \mathbb{R}_1^3 . Then, the curve α is spacelike if all its velocity vectors are spacelike. Similarly, it is called timelike and a null curve if all its velocity vectors are timelike and null vectors, respectively.

A dual number \widehat{x} has the form $x + \varepsilon x^*$ with the properties

$$\varepsilon \neq 0, \quad 0\varepsilon = \varepsilon 0 = 0, \quad 1\varepsilon = \varepsilon 1 = \varepsilon, \quad \varepsilon^2 = 0,$$

where x and x^* are real numbers and ε is the dual unit (for the properties of dual numbers, see [16]). An ordered triple of dual numbers $(\widehat{x}_1, \widehat{x}_2, \widehat{x}_3)$ is called a *dual vector* and the set of dual vectors is denoted by

$$\begin{aligned} \mathbb{D}^3 &= \mathbb{D} \times \mathbb{D} \times \mathbb{D} \\ &= \{ \vec{\widehat{x}} \mid \vec{\widehat{x}} = (x_1 + \varepsilon x_1^*, x_2 + \varepsilon x_2^*, x_3 + \varepsilon x_3^*) \} \\ &= \{ \vec{\widehat{x}} \mid \vec{\widehat{x}} = (x_1, x_2, x_3) + \varepsilon (x_1^*, x_2^*, x_3^*) \} \\ &= \{ \vec{\widehat{x}} \mid \vec{\widehat{x}} = \vec{x} + \varepsilon \vec{x}^*, \vec{x}, \vec{x}^* \in \mathbb{R}^3 \} \end{aligned}$$

For any $\vec{\widehat{x}} = \vec{x} + \varepsilon \vec{x}^*, \vec{\widehat{y}} = \vec{y} + \varepsilon \vec{y}^* \in \mathbb{D}^3$, if the Lorentzian inner product of the dual vectors $\vec{\widehat{x}}$ and $\vec{\widehat{y}}$ is defined by

$$\langle \vec{\widehat{x}}, \vec{\widehat{y}} \rangle := \langle \vec{x}, \vec{y} \rangle + \varepsilon (\langle \vec{x}, \vec{y}^* \rangle + \langle \vec{x}^*, \vec{y} \rangle),$$

then the dual space \mathbb{D}^3 together with this Lorentzian inner product is called the *dual Lorentzian space*, and it is shown by \mathbb{D}_1^3 . A dual vector $\vec{\widehat{x}}$ in \mathbb{D}^3 is said to be spacelike, timelike and lightlike (null) if the vector \vec{x} is spacelike, timelike and lightlike (null), respectively. The Lorentzian vector product of dual vectors $\vec{\widehat{x}} = (\widehat{x}_1, \widehat{x}_2, \widehat{x}_3)$ and $\vec{\widehat{y}} = (\widehat{y}_1, \widehat{y}_2, \widehat{y}_3)$ in \mathbb{D}_1^3 is defined by

$$\vec{\widehat{x}} \wedge \vec{\widehat{y}} := (\widehat{x}_3 \widehat{y}_2 - \widehat{x}_2 \widehat{y}_3, \widehat{x}_3 \widehat{y}_1 - \widehat{x}_1 \widehat{y}_3, \widehat{x}_1 \widehat{y}_2 - \widehat{x}_2 \widehat{y}_1).$$

If $\vec{x} \neq 0$, the norm $\|\vec{\widehat{x}}\|$ of $\vec{\widehat{x}} = \vec{x} + \varepsilon \vec{x}^*$ is defined by

$$\|\vec{\widehat{x}}\| := \sqrt{|\langle \vec{\widehat{x}}, \vec{\widehat{x}} \rangle|}.$$

A dual vector $\vec{\widehat{x}}$ with norm 1 is called a *dual unit vector*. Let $\vec{\widehat{x}} = (\widehat{x}_1, \widehat{x}_2, \widehat{x}_3) \in \mathbb{D}_1^3$. Then,

i) The set

$$\mathbb{S}_1^2 = \{ \vec{\widehat{x}} = \vec{x} + \varepsilon \vec{x}^* \mid \|\vec{\widehat{x}}\| = (1, 0); \vec{x}, \vec{x}^* \in \mathbb{R}_1^3 \text{ and the vector } \vec{x} \text{ is spacelike} \}$$

is called the *pseudo dual sphere* with center $\vec{0}$ in \mathbb{D}_1^3 .

ii) The set

$$\mathbb{H}_0^2 = \{ \vec{\widehat{x}} = \vec{x} + \varepsilon \vec{x}^* \mid \|\vec{\widehat{x}}\| = (1, 0); \vec{x}, \vec{x}^* \in \mathbb{R}_1^3 \text{ and the vector } \vec{x} \text{ is timelike} \}$$

is called the *pseudo dual hyperbolic space* in \mathbb{D}_1^3 [15]

If all the real valued functions $x_i(t)$ and $x_i^*(t)$, $1 \leq i \leq 3$, are differentiable, the dual Lorentzian curve

$$\begin{aligned} \widehat{x} : I \subset \mathbb{R} &\rightarrow \mathbb{D}_1^3 \\ t &\mapsto \vec{\widehat{x}}(t) = (x_1(t) + \varepsilon x_1^*(t), x_2(t) + \varepsilon x_2^*(t), x_3(t) + \varepsilon x_3^*(t)) \\ &= \vec{x}(t) + \varepsilon \vec{x}^*(t) \end{aligned}$$

in \mathbb{D}_1^3 is differentiable. We call the real part $\vec{x}(t)$ the *indicatrix* of $\vec{x}(t)$. The *dual arc length* of the curve $\vec{x}(t)$ from t_1 to t is defined as

$$\widehat{s} := \int_{t_1}^t \|\vec{x}(t)'\| dt = \int_{t_1}^t \|\vec{x}(t)'\| dt + \varepsilon \int_{t_1}^t \langle \vec{t}, (\vec{x}^*)' \rangle = s + \varepsilon s^*,$$

where \vec{t} is a unit tangent vector of $\vec{x}(t)$. From now on we will take the arc length s of $\vec{x}(t)$ as the parameter instead of t .

The equalities relative to the derivatives of the dual Frenet vectors $\vec{t}, \vec{n}, \vec{b}$ throughout the dual space curve are written in the matrix form as

$$(2.1) \quad \frac{d}{d\widehat{s}} \begin{bmatrix} \vec{t} \\ \vec{n} \\ \vec{b} \end{bmatrix} = \begin{bmatrix} 0 & \widehat{\kappa} & 0 \\ -\varepsilon_1 \varepsilon_2 \widehat{\kappa} & 0 & \widehat{\tau} \\ 0 & -\varepsilon_2 \varepsilon_3 \widehat{\tau} & 0 \end{bmatrix} \begin{bmatrix} \vec{t} \\ \vec{n} \\ \vec{b} \end{bmatrix},$$

where $\widehat{\kappa} = \kappa + \varepsilon \kappa^*$ is the nowhere pure dual curvature, $\widehat{\tau} = \tau + \varepsilon \tau^*$ is the nowhere pure dual torsion and

$$\langle \vec{t}, \vec{t} \rangle = \varepsilon_1, \quad \langle \vec{n}, \vec{n} \rangle = \varepsilon_2, \quad \langle \vec{b}, \vec{b} \rangle = \varepsilon_3, \quad \langle \vec{t}, \vec{n} \rangle = \langle \vec{t}, \vec{b} \rangle = \langle \vec{n}, \vec{b} \rangle = 0.$$

The formulae (2.1) are called the *Frenet formulae* of the dual curve in dual Lorentzian space. The planes spanned by $\{\vec{t}, \vec{b}\}$, $\{\vec{t}, \vec{n}\}$ and $\{\vec{n}, \vec{b}\}$ at each point of the dual Lorentzian curve are called the *rectifying plane*, the *osculating plane*, and the *normal plane*, respectively [1, 2, 11, 14].

3. Mannheim partner curves in \mathbb{D}_1^3

In this section, we define Mannheim partner curves in the dual space \mathbb{D}_1^3 and we give characterizations for Mannheim partner curves in the same space.

3.1. Definition. Let \mathbb{D}_1^3 be dual Lorentzian space with the Lorentzian inner product $\langle \cdot, \cdot \rangle$. If there exists a correspondence between the dual Lorentzian space curves $\widehat{\alpha}$ and $\widehat{\beta}$ such that, at the corresponding points of the dual Lorentzian curves, the principal normal lines of $\widehat{\alpha}$ coincides with the binormal lines of $\widehat{\beta}$, then $\widehat{\alpha}$ is called a *dual Lorentzian Mannheim curve*, and $\widehat{\beta}$ a *dual Lorentzian Mannheim partner curve* of $\widehat{\alpha}$. The pair $\{\widehat{\alpha}, \widehat{\beta}\}$ is said to be a *dual Lorentzian Mannheim pair*.

Let $\widehat{\alpha} : \widehat{x}(\widehat{s})$ be a dual Lorentzian Mannheim curve in \mathbb{D}_1^3 parameterized by its arc length \widehat{s} and $\widehat{\beta} : \widehat{x}_1(\widehat{s}_1)$ the dual Lorentzian Mannheim partner curve with an arc length parameter \widehat{s}_1 . Denote by $\left\{ \vec{t}(\widehat{s}), \vec{n}(\widehat{s}), \vec{b}(\widehat{s}) \right\}$ the dual Lorentzian Frenet frame field along $\widehat{\alpha} : \widehat{x}(\widehat{s})$.

In the following theorems, we give necessary and sufficient conditions for a dual space curve to be a dual Lorentzian Mannheim curve.

3.2. Theorem. Let $\widehat{\alpha} : \widehat{x}(\widehat{s})$ be a Lorentzian curve in \mathbb{D}_1^3 and $\vec{t}(\widehat{s}), \vec{n}(\widehat{s})$ and $\vec{b}(\widehat{s})$ the tangent, principal normal and binormal vector fields of $\widehat{\alpha} : \widehat{x}(\widehat{s})$, respectively.

- i) In the case when $\vec{t}(\widehat{s})$ is a timelike vector, $\vec{n}(\widehat{s})$ and $\vec{b}(\widehat{s})$ are spacelike vectors, then $\widehat{\alpha} : \widehat{x}(\widehat{s})$ is a dual Lorentzian Mannheim curve if and only if its curvature $\widehat{\kappa}$ and torsion $\widehat{\tau}$ satisfy the formula $\widehat{\kappa} = \widehat{\lambda}(\widehat{\tau}^2 - \widehat{\kappa}^2)$, where $\widehat{\lambda}$ is a never pure dual constant.

- ii) In the case when $\vec{t}(\hat{s})$ and $\vec{n}(\hat{s})$ are spacelike vectors, $\vec{b}(\hat{s})$ a timelike vector, then $\hat{\alpha} : \hat{x}(\hat{s})$ is a dual Lorentzian Mannheim curve if and only if its curvature $\hat{\kappa}$ and torsion $\hat{\tau}$ satisfy the formula $\hat{\kappa} = \hat{\lambda}(\hat{\kappa}^2 - \hat{\tau}^2)$, where $\hat{\lambda}$ is a never pure dual constant.
- iii) In the case when $\vec{t}(\hat{s})$ and $\vec{b}(\hat{s})$ are spacelike vectors, $\vec{n}(\hat{s})$ a timelike vector, then $\hat{\alpha} : \hat{x}(\hat{s})$ is a dual Lorentzian Mannheim curve if and only if its curvature $\hat{\kappa}$ and torsion $\hat{\tau}$ satisfy the formula $\hat{\kappa} = -\hat{\lambda}(\hat{\kappa}^2 + \hat{\tau}^2)$, where $\hat{\lambda}$ is a never pure dual constant.

Proof. i) Let $\hat{\alpha} : \hat{x}(\hat{s})$ be a dual Lorentzian Mannheim curve in \mathbb{D}_1^3 with arc length parameter \hat{s} , and $\hat{\beta} : \hat{x}_1(\hat{s}_1)$ the dual Lorentzian Mannheim partner curve with arc length parameter \hat{s}_1 . Then by the definition we can assume that

$$(3.1) \quad \hat{x}_1(\hat{s}) = \hat{x}(\hat{s}) + \hat{\lambda}(\hat{s}) \vec{n}(\hat{s})$$

for some never pure dual constant $\hat{\lambda}(\hat{s})$. By taking the derivative of (3.1) with respect to \hat{s} and applying the Frenet formulas we have

$$\frac{d\hat{x}_1(\hat{s})}{d\hat{s}} = (1 + \hat{\lambda}\hat{\kappa}) \vec{t} + \frac{d\hat{\lambda}}{d\hat{s}} \vec{n} + \hat{\lambda}\hat{\tau} \vec{b}.$$

Since \vec{t}_1 is coincident with \vec{b}_1 in direction, we get

$$\frac{d\hat{\lambda}(\hat{s})}{d\hat{s}} = 0.$$

This means that $\hat{\lambda}$ is a never pure dual constant. Thus we have

$$\frac{d\hat{x}_1(\hat{s})}{d\hat{s}} = (1 + \hat{\lambda}\hat{\kappa}) \vec{t} + \hat{\lambda}\hat{\tau} \vec{b}.$$

On the other hand, we have

$$\vec{t}_1 = \frac{d\hat{x}_1}{d\hat{s}} \frac{d\hat{s}}{d\hat{s}_1} = \left((1 + \hat{\lambda}\hat{\kappa}) \vec{t} + \hat{\lambda}\hat{\tau} \vec{b} \right) \frac{d\hat{s}}{d\hat{s}_1}.$$

By taking the derivative of this equation with respect to \hat{s}_1 and applying the Frenet formulas we obtain

$$\begin{aligned} \frac{d\vec{t}_1}{d\hat{s}} \frac{d\hat{s}}{d\hat{s}_1} &= \left(\hat{\lambda} \frac{d\hat{\kappa}}{d\hat{s}} \vec{t} + (\hat{\kappa} + \hat{\lambda}\hat{\kappa}^2 - \hat{\lambda}\hat{\tau}^2) \vec{n} + \hat{\lambda} \frac{d\hat{\tau}}{d\hat{s}} \vec{b} \right) \frac{d\hat{s}}{d\hat{s}_1} \\ &\quad + \left((1 + \hat{\lambda}\hat{\kappa}) \vec{t} + \hat{\lambda}\hat{\tau} \vec{b} \right) \left(\frac{d\hat{s}}{d\hat{s}_1} \right)^2. \end{aligned}$$

From this equation we get

$$(\hat{\kappa} + \hat{\lambda}\hat{\kappa}^2 - \hat{\lambda}\hat{\tau}^2) \frac{d\hat{s}}{d\hat{s}_1} = 0,$$

$$\hat{\kappa} = \hat{\lambda}(\hat{\tau}^2 - \hat{\kappa}^2).$$

This completes the proof.

ii) Let $\hat{\alpha} : \hat{x}(\hat{s})$ be a dual Lorentzian Mannheim curve in \mathbb{D}_1^3 with arc length parameter \hat{s} and $\hat{\beta} : \hat{x}_1(\hat{s}_1)$ the dual Lorentzian Mannheim partner curve with arc length parameter \hat{s}_1 . Then by the definition we can assume that

$$(3.2) \quad \hat{x}_1(\hat{s}) = \hat{x}(\hat{s}) + \hat{\lambda}(\hat{s}) \vec{n}(\hat{s})$$

for some never pure dual constant $\widehat{\lambda}(\widehat{s})$. By taking the derivative of (3.2) with respect to \widehat{s} and applying the Frenet formulas we have

$$\frac{d\widehat{x}_1(\widehat{s})}{d\widehat{s}} = (1 - \widehat{\lambda}\widehat{\kappa}) \vec{t} + \frac{d\widehat{\lambda}}{d\widehat{s}} \vec{n} + \widehat{\lambda}\widehat{\tau} \vec{b}.$$

Since \vec{t}_1 is coincident with \vec{b}_1 in direction, we get

$$\frac{d\widehat{\lambda}(\widehat{s})}{d\widehat{s}} = 0.$$

This means that $\widehat{\lambda}$ is a never pure dual constant. Thus we have

$$\frac{d\widehat{x}_1(\widehat{s})}{d\widehat{s}} = (1 - \widehat{\lambda}\widehat{\kappa}) \vec{t} + \widehat{\lambda}\widehat{\tau} \vec{b}.$$

On the other hand, we have

$$\vec{t}_1 = \frac{d\widehat{x}_1}{d\widehat{s}} \frac{d\widehat{s}}{d\widehat{s}_1} = \left((1 - \widehat{\lambda}\widehat{\kappa}) \vec{t} + \widehat{\lambda}\widehat{\tau} \vec{b} \right) \frac{d\widehat{s}}{d\widehat{s}_1}.$$

By taking the derivative of this equation with respect to \widehat{s}_1 and applying the Frenet formulas we obtain

$$\begin{aligned} \frac{d\vec{t}_1}{d\widehat{s}} \frac{d\widehat{s}}{d\widehat{s}_1} &= \left(-\widehat{\lambda} \frac{d\widehat{\kappa}}{d\widehat{s}} \vec{t} + (\widehat{\kappa} - \widehat{\lambda}\widehat{\kappa}^2 + \widehat{\lambda}\widehat{\tau}^2) \vec{n} + \widehat{\lambda} \frac{d\widehat{\tau}}{d\widehat{s}} \vec{b} \right) \frac{d\widehat{s}}{d\widehat{s}_1} \\ &\quad + \left((1 - \widehat{\lambda}\widehat{\kappa}) \vec{t} + \widehat{\lambda}\widehat{\tau} \vec{b} \right) \left(\frac{d\widehat{s}}{d\widehat{s}_1} \right)^2. \end{aligned}$$

From this equation we get

$$(\widehat{\kappa} - \widehat{\lambda}\widehat{\kappa}^2 + \widehat{\lambda}\widehat{\tau}^2) \frac{d\widehat{s}}{d\widehat{s}_1} = 0,$$

$$\widehat{\kappa} = \widehat{\lambda}(\widehat{\kappa}^2 - \widehat{\tau}^2).$$

This completes the proof.

iii) Let $\widehat{\alpha} : \widehat{x}(\widehat{s})$ be a dual Lorentzian Mannheim curve in \mathbb{D}_1^3 with arc length parameter \widehat{s} , and $\widehat{\beta} : \widehat{x}_1(\widehat{s}_1)$ the dual Lorentzian Mannheim partner curve with arc length parameter \widehat{s}_1 . Then by the definition we can assume that

$$(3.3) \quad \widehat{x}_1(\widehat{s}) = \widehat{x}(\widehat{s}) + \widehat{\lambda}(\widehat{s}) \vec{n}(\widehat{s})$$

for some never pure dual constant $\widehat{\lambda}(\widehat{s})$. By taking the derivative of (3.3) with respect to \widehat{s} and applying the Frenet formulas we have

$$\frac{d\widehat{x}_1(\widehat{s})}{d\widehat{s}} = (1 + \widehat{\lambda}\widehat{\kappa}) \vec{t} + \frac{d\widehat{\lambda}}{d\widehat{s}} \vec{n} + \widehat{\lambda}\widehat{\tau} \vec{b}.$$

Since \vec{t}_1 is coincident with \vec{b}_1 in direction, we get

$$\frac{d\widehat{\lambda}(\widehat{s})}{d\widehat{s}} = 0.$$

This means that $\widehat{\lambda}$ is a never pure dual constant. Thus we have

$$\frac{d\widehat{x}_1(\widehat{s})}{d\widehat{s}} = (1 + \widehat{\lambda}\widehat{\kappa}) \vec{t} + \widehat{\lambda}\widehat{\tau} \vec{b}.$$

On the other hand, we have

$$\vec{t}_1 = \frac{d\widehat{x}_1}{d\widehat{s}} \frac{d\widehat{s}}{d\widehat{s}_1} = \left((1 + \widehat{\lambda}\widehat{\kappa}) \vec{t} + \widehat{\lambda}\widehat{\tau} \vec{b} \right) \frac{d\widehat{s}}{d\widehat{s}_1}.$$

By taking the derivative of this equation with respect to \widehat{s}_1 and applying the Frenet formulas, we obtain

$$\begin{aligned} \frac{d\vec{t}_1}{d\widehat{s}} \frac{d\widehat{s}}{d\widehat{s}_1} &= \left(\widehat{\lambda} \frac{d\widehat{\kappa}}{d\widehat{s}} \vec{t} + (\widehat{\kappa} + \widehat{\lambda}\widehat{\kappa}^2 + \widehat{\lambda}\widehat{\tau}^2) \vec{n} + \widehat{\lambda} \frac{d\widehat{\tau}}{d\widehat{s}} \vec{b} \right) \frac{d\widehat{s}}{d\widehat{s}_1} \\ &\quad + \left((1 + \widehat{\lambda}\widehat{\kappa}) \vec{t} + \widehat{\lambda}\widehat{\tau} \vec{b} \right) \left(\frac{d\widehat{s}}{d\widehat{s}_1} \right)^2. \end{aligned}$$

From this equation we get

$$(\widehat{\kappa} + \widehat{\lambda}\widehat{\kappa}^2 + \widehat{\lambda}\widehat{\tau}^2) \frac{d\widehat{s}}{d\widehat{s}_1} = 0,$$

$$\widehat{\kappa} = -\widehat{\lambda}(\widehat{\tau}^2 + \widehat{\kappa}^2).$$

This completes the proof. □

3.3. Theorem. Let $\widehat{\alpha} : \widehat{x}(\widehat{s})$ be a dual Lorentzian Mannheim curve in \mathbb{D}_1^3 with arc length parameter \widehat{s} .

- i) In the case where $\vec{t}(\widehat{s})$ is a timelike vector, $\vec{n}(\widehat{s})$ and $\vec{b}(\widehat{s})$ are spacelike vectors, then $\widehat{\beta} : \widehat{x}_1(\widehat{s}_1)$ is the dual Mannheim partner curve of $\widehat{\alpha}$ if and only if the curvature $\widehat{\kappa}_1$ and the torsion $\widehat{\tau}_1$ of $\widehat{\beta}$ satisfy the following equation

$$\frac{d\widehat{\tau}_1}{d\widehat{s}_1} = \frac{\widehat{\kappa}_1}{\widehat{\mu}} (1 - \widehat{\mu}^2 \widehat{\tau}_1^2)$$

for some never pure dual constant $\widehat{\mu}$.

- ii) In the case where $\vec{t}(\widehat{s})$ and $\vec{n}(\widehat{s})$ are spacelike vectors, $\vec{b}(\widehat{s})$ a timelike vector, then $\widehat{\beta} : \widehat{x}_1(\widehat{s}_1)$ is the dual Mannheim partner curve of $\widehat{\alpha}$ if and only if the curvature $\widehat{\kappa}_1$ and the torsion $\widehat{\tau}_1$ of $\widehat{\beta}$ satisfy the following equation

$$\frac{d\widehat{\tau}_1}{d\widehat{s}_1} = -\frac{\widehat{\kappa}_1}{\widehat{\mu}} (1 + \widehat{\mu}^2 \widehat{\tau}_1^2)$$

for some never pure dual constant $\widehat{\mu}$.

- iii) In the case where $\vec{t}(\widehat{s})$ and $\vec{b}(\widehat{s})$ are spacelike vectors, $\vec{n}(\widehat{s})$ a timelike vector, then $\widehat{\beta} : \widehat{x}_1(\widehat{s}_1)$ is the dual Mannheim partner curve of $\widehat{\alpha}$ if and only if the curvature $\widehat{\kappa}_1$ and the torsion $\widehat{\tau}_1$ of $\widehat{\beta}$ satisfy the following equation

$$\frac{d\widehat{\tau}_1}{d\widehat{s}_1} = -\frac{\widehat{\kappa}_1}{\widehat{\mu}} (1 - \widehat{\mu}^2 \widehat{\tau}_1^2)$$

for some never pure dual constant $\widehat{\mu}$.

Proof. i) Suppose that $\widehat{\alpha} : \widehat{x}(\widehat{s})$ is a dual Lorentzian Mannheim curve. Then by the definition we can assume that

$$(3.4) \quad \widehat{x}(\widehat{s}_1) = \widehat{x}_1(\widehat{s}_1) + \widehat{\mu}(\widehat{s}_1) \vec{b}_1(\widehat{s}_1)$$

for some function $\widehat{\mu}(\widehat{s}_1)$. By taking the derivative of (3.4) with respect to \widehat{s}_1 and applying the Frenet formulas, we have

$$(3.5) \quad \vec{t} \frac{d\widehat{s}}{d\widehat{s}_1} = \vec{t}_1 + \frac{d\widehat{\mu}}{d\widehat{s}_1} \vec{b} - \widehat{\mu}\widehat{\tau}_1 \vec{n}_1.$$

Since \vec{b} is coincident with \vec{n} in direction, we get

$$\frac{d\widehat{\mu}}{d\widehat{s}_1} = 0.$$

This means that $\widehat{\mu}$ is a never pure dual constant. Thus we have

$$(3.6) \quad \vec{t} \frac{d\widehat{s}}{d\widehat{s}_1} = \vec{t}_1 - \widehat{\mu}\widehat{\tau}_1 \vec{n}_1.$$

On the other hand, we have

$$(3.7) \quad \vec{t} = \vec{t}_1 \cosh \widehat{\theta} + \vec{n}_1 \sinh \widehat{\theta},$$

where $\widehat{\theta}$ is the dual hyperbolic angle between \vec{t} and \vec{t}_1 at the corresponding points of $\widehat{\alpha}$ and $\widehat{\beta}$. By taking the derivative of this equation with respect to \widehat{s}_1 , we obtain

$$\widehat{\kappa} \vec{n} \frac{d\widehat{s}}{d\widehat{s}_1} = \left(\widehat{\kappa}_1 + \frac{d\widehat{\theta}}{d\widehat{s}_1} \right) \sinh \widehat{\theta} \vec{t}_1 + \left(\widehat{\kappa}_1 + \frac{d\widehat{\theta}}{d\widehat{s}_1} \right) \cosh \widehat{\theta} \vec{n}_1 + \widehat{\tau}_1 \sinh \widehat{\theta} \vec{b}_1.$$

From this equation and the fact that the direction of \vec{n} is coincident with that of \vec{b}_1 , we get

$$\left(\widehat{\kappa}_1 + \frac{d\widehat{\theta}}{d\widehat{s}_1} \right) \cosh \widehat{\theta} = 0.$$

Therefore we have

$$(3.8) \quad \frac{d\widehat{\theta}}{d\widehat{s}_1} = -\widehat{\kappa}_1.$$

From (3.6) and (3.7) and noting that \vec{t}_1 is orthogonal to \vec{b}_1 , we find that

$$\frac{d\widehat{s}}{d\widehat{s}_1} = \frac{1}{\cosh \widehat{\theta}} = -\frac{\widehat{\mu}\widehat{\tau}_1}{\sinh \widehat{\theta}}.$$

Then we have

$$\widehat{\mu}\widehat{\tau}_1 = -\tanh \widehat{\theta}.$$

By taking the derivative of this equation and applying (3.8), we get

$$\widehat{\mu} \frac{d\widehat{\tau}_1}{d\widehat{s}_1} = \widehat{\kappa}_1 (1 - \widehat{\mu}^2 \widehat{\tau}_1^2),$$

that is

$$\frac{d\widehat{\tau}_1}{d\widehat{s}_1} = \frac{\widehat{\kappa}_1}{\widehat{\mu}} (1 - \widehat{\mu}^2 \widehat{\tau}_1^2).$$

Conversely, if the curvature $\widehat{\kappa}_1$ and torsion $\widehat{\tau}_1$ of the dual Lorentzian curve $\widehat{\beta}$ satisfy

$$\frac{d\widehat{\tau}_1}{d\widehat{s}_1} = \frac{\widehat{\kappa}_1}{\widehat{\mu}} (1 - \widehat{\mu}^2 \widehat{\tau}_1^2)$$

for some never pure dual constant $\widehat{\mu}(\widehat{s})$, then we define a dual curve by

$$(3.9) \quad \widehat{x}(\widehat{s}_1) = \widehat{x}_1(\widehat{s}_1) + \widehat{\mu} \vec{b}_1(\widehat{s}_1),$$

and we will prove that $\widehat{\alpha}$ is a dual Lorentzian Mannheim curve and that $\widehat{\beta}$ is the dual Lorentzian partner curve of $\widehat{\alpha}$.

By taking the derivative of (3.9) with respect to \widehat{s}_1 twice, we get

$$(3.10) \quad \vec{t} \frac{d\widehat{s}}{d\widehat{s}_1} = \vec{t}_1 - \widehat{\mu}\widehat{\tau}_1 \vec{n}_1,$$

$$(3.11) \quad \widehat{\kappa} \vec{n} \left(\frac{d\widehat{s}}{d\widehat{s}_1} \right)^2 + \vec{t} \frac{d^2\widehat{s}}{d\widehat{s}_1^2} = -\widehat{\mu}\widehat{\kappa}_1\widehat{\tau}_1 \vec{t}_1 + \left(\widehat{\kappa}_1 - \widehat{\mu} \frac{d\widehat{\tau}_1}{d\widehat{s}_1} \right) \vec{n}_1 - \widehat{\mu}\widehat{\tau}_1^2 \vec{b}_1,$$

respectively. Taking the cross product of (3.10) with (3.11) and noticing that

$$\widehat{\kappa}_1 - \widehat{\mu} \frac{d\widehat{\tau}_1}{d\widehat{s}_1} - \widehat{\mu}^2 \widehat{\kappa}_1 \widehat{\tau}_1^2 = 0,$$

we have

$$(3.12) \quad \widehat{\kappa} \vec{b} \left(\frac{d\widehat{s}}{d\widehat{s}_1} \right)^3 = -\widehat{\mu}^2 \widehat{\tau}_1^3 \vec{t}_1 + \widehat{\mu} \widehat{\tau}_1^2 \vec{n}_1$$

By taking the cross product of (3.12) with (3.10), we obtain also

$$\widehat{\kappa} \vec{n} \left(\frac{d\widehat{s}}{d\widehat{s}_1} \right)^4 = -\widehat{\mu} \widehat{\tau}_1^2 (1 - \widehat{\mu}^2 \widehat{\tau}_1^2) \vec{b}_1.$$

This means that the principal normal direction \vec{n} of $\widehat{\alpha} : \widehat{x}(\widehat{s})$ coincides with the binormal direction \vec{b}_1 of $\widehat{\beta} : \widehat{x}_1(\widehat{s}_1)$. Hence $\widehat{\alpha} : \widehat{x}(\widehat{s})$ is a dual Lorentzian Mannheim curve and $\widehat{\beta} : \widehat{x}_1(\widehat{s}_1)$ is its dual Lorentzian Mannheim partner curve.

ii) Suppose that $\widehat{\alpha} : \widehat{x}(\widehat{s})$ is a dual Lorentzian Mannheim curve. Then by the definition we can assume that

$$(3.13) \quad \widehat{x}(\widehat{s}_1) = \widehat{x}_1(\widehat{s}_1) + \widehat{\mu}(\widehat{s}_1) \vec{b}_1(\widehat{s}_1)$$

for some function $\widehat{\mu}(\widehat{s}_1)$. By taking the derivative of (3.13) with respect to \widehat{s}_1 and applying the Frenet formulas, we have

$$(3.14) \quad \vec{t} \frac{d\widehat{s}}{d\widehat{s}_1} = \vec{t}_1 + \frac{d\widehat{\mu}}{d\widehat{s}_1} \vec{b} + \widehat{\mu} \widehat{\tau}_1 \vec{n}_1.$$

Since \vec{b} is coincident with \vec{n} in direction, we get

$$\frac{d\widehat{\mu}}{d\widehat{s}_1} = 0.$$

This means that $\widehat{\mu}$ is a never pure dual constant. Thus we have

$$(3.15) \quad \vec{t} \frac{d\widehat{s}}{d\widehat{s}_1} = \vec{t}_1 + \widehat{\mu} \widehat{\tau}_1 \vec{n}_1.$$

On the other hand, we have

$$(3.16) \quad \vec{t} = \vec{t}_1 \cos \widehat{\theta} + \vec{n}_1 \sin \widehat{\theta},$$

where $\widehat{\theta}$ is the dual angle between \vec{t} and \vec{t}_1 at the corresponding points of $\widehat{\alpha}$ and $\widehat{\beta}$. By taking the derivative of this equation with respect to \widehat{s}_1 , we obtain

$$\widehat{\kappa} \vec{n} \frac{d\widehat{s}}{d\widehat{s}_1} = - \left(\widehat{\kappa}_1 + \frac{d\widehat{\theta}}{d\widehat{s}_1} \right) \sin \widehat{\theta} \vec{t}_1 + \left(\widehat{\kappa}_1 + \frac{d\widehat{\theta}}{d\widehat{s}_1} \right) \cos \widehat{\theta} \vec{n}_1 + \widehat{\tau}_1 \sin \widehat{\theta} \vec{b}_1.$$

From this equation and the fact that the direction of \vec{n} is coincident with that of \vec{b}_1 , we get

$$\begin{cases} \left(\widehat{\kappa}_1 + \frac{d\widehat{\theta}}{d\widehat{s}_1} \right) \sin \widehat{\theta} = 0 \\ \left(\widehat{\kappa}_1 + \frac{d\widehat{\theta}}{d\widehat{s}_1} \right) \cos \widehat{\theta} = 0. \end{cases}$$

Therefore we have

$$(3.17) \quad \frac{d\widehat{\theta}}{d\widehat{s}_1} = -\widehat{\kappa}_1.$$

From (3.15), (3.16) and noticing that \vec{t}_1 is orthogonal to \vec{b}_1 , we find that

$$\frac{d\hat{s}}{d\hat{s}_1} = \frac{1}{\cos \hat{\theta}} = \frac{\hat{\mu}\hat{\tau}_1}{\sin \hat{\theta}}.$$

Then we have

$$\hat{\mu}\hat{\tau}_1 = \tan \hat{\theta}.$$

By taking the derivative of this equation and applying (3.17), we get

$$\hat{\mu} \frac{d\hat{\tau}_1}{d\hat{s}_1} = -\hat{\kappa}_1(1 + \hat{\mu}^2\hat{\tau}_1^2),$$

that is

$$\frac{d\hat{\tau}_1}{d\hat{s}_1} = -\frac{\hat{\kappa}_1}{\hat{\mu}}(1 + \hat{\mu}^2\hat{\tau}_1^2).$$

Conversely, if the curvature $\hat{\kappa}_1$ and torsion $\hat{\tau}_1$ of the dual Lorentzian curve $\hat{\beta}$ satisfy

$$\frac{d\hat{\tau}_1}{d\hat{s}_1} = -\frac{\hat{\kappa}_1}{\hat{\mu}}(1 + \hat{\mu}^2\hat{\tau}_1^2)$$

for some never pure dual constant $\hat{\mu}(\hat{s})$, then we define a dual curve by

$$(3.18) \quad \hat{x}(\hat{s}_1) = \hat{x}_1(\hat{s}_1) + \hat{\mu}\vec{b}_1(\hat{s}_1),$$

and we will prove that $\hat{\alpha}$ is a dual Lorentzian Mannheim curve and that $\hat{\beta}$ is the dual Lorentzian partner curve of $\hat{\alpha}$.

By taking the derivative of (3.18) with respect to \hat{s}_1 twice, we get

$$(3.19) \quad \vec{t} \frac{d\hat{s}}{d\hat{s}_1} = \vec{t}_1 + \hat{\mu}\hat{\tau}_1 \vec{n}_1,$$

$$(3.20) \quad \hat{\kappa} \vec{n} \left(\frac{d\hat{s}}{d\hat{s}_1} \right)^2 + \vec{t} \frac{d^2\hat{s}}{d\hat{s}_1^2} = -\hat{\mu}\hat{\kappa}_1\hat{\tau}_1 \vec{t}_1 + \left(\hat{\kappa}_1 + \hat{\mu} \frac{d\hat{\tau}_1}{d\hat{s}_1} \right) \vec{n}_1 + \hat{\mu}\hat{\tau}_1^2 \vec{b}_1,$$

respectively. Taking the cross product of (3.19) with (3.20) and noticing that

$$-\hat{\kappa}_1 - \hat{\mu} \frac{d\hat{\tau}_1}{d\hat{s}_1} - \hat{\mu}^2\hat{\kappa}_1\hat{\tau}_1^2 = 0,$$

we have

$$(3.21) \quad \hat{\kappa} \vec{b} \left(\frac{d\hat{s}}{d\hat{s}_1} \right)^3 = -\hat{\mu}^2\hat{\tau}_1^3 \vec{t}_1 + \hat{\mu}\hat{\tau}_1^2 \vec{n}_1.$$

By taking the cross product of (3.21) with (3.19), we obtain also

$$\hat{\kappa} \vec{n} \left(\frac{d\hat{s}}{d\hat{s}_1} \right)^4 = \hat{\mu}\hat{\tau}_1^2(1 + \hat{\mu}^2\hat{\tau}_1^2) \vec{b}_1.$$

This means that the principal normal direction \vec{n} of $\hat{\alpha} : \hat{x}(\hat{s})$ coincides with the binormal direction \vec{b}_1 of $\hat{\beta} : \hat{x}_1(\hat{s}_1)$. Hence, $\hat{\alpha} : \hat{x}(\hat{s})$ is a dual Lorentzian Mannheim curve and $\hat{\beta} : \hat{x}_1(\hat{s}_1)$ is its dual Lorentzian Mannheim partner curve.

iii) Suppose that $\hat{\alpha} : \hat{x}(\hat{s})$ is a dual Lorentzian Mannheim curve. Then by the definition we can assume that

$$(3.22) \quad \hat{x}(\hat{s}_1) = \hat{x}_1(\hat{s}_1) + \hat{\mu}(\hat{s}_1)\vec{b}_1(\hat{s}_1)$$

for some function $\hat{\mu}(\hat{s}_1)$. By taking the derivative of (3.22) with respect to \hat{s}_1 and applying the Frenet formulas, we have

$$(3.23) \quad \vec{t} \frac{d\hat{s}}{d\hat{s}_1} = \vec{t}_1 + \frac{d\hat{\mu}}{d\hat{s}_1} \vec{b} + \hat{\mu}\hat{\tau}_1 \vec{n}_1.$$

Since \vec{b} is coincident with \vec{n} in direction, we get

$$\frac{d\hat{\mu}}{d\hat{s}_1} = 0.$$

This means that $\hat{\mu}$ is a never pure dual constant. Thus we have

$$(3.24) \quad \vec{t} \frac{d\hat{s}}{d\hat{s}_1} = \vec{t}_1 + \hat{\mu}\hat{\tau}_1 \vec{n}_1.$$

On the other hand, we have

$$(3.25) \quad \vec{t} = \vec{t}_1 \cosh \hat{\theta} + \vec{n}_1 \sinh \hat{\theta},$$

where $\hat{\theta}$ is the dual hyperbolic angle between \vec{t} and \vec{t}_1 at the corresponding points of $\hat{\alpha}$ and $\hat{\beta}$. By taking the derivative of this equation with respect to \hat{s}_1 , we obtain

$$\hat{\kappa} \vec{n} \frac{d\hat{s}}{d\hat{s}_1} = \left(\hat{\kappa}_1 + \frac{d\hat{\theta}}{d\hat{s}_1} \right) \sinh \hat{\theta} \vec{t}_1 + \left(\hat{\kappa}_1 + \frac{d\hat{\theta}}{d\hat{s}_1} \right) \cosh \hat{\theta} \vec{n}_1 + \hat{\tau}_1 \sinh \hat{\theta} \vec{b}_1.$$

From this equation and the fact that the direction of \vec{n} is coincident with that of \vec{b}_1 , we get

$$\left(\hat{\kappa}_1 + \frac{d\hat{\theta}}{d\hat{s}_1} \right) \cosh \hat{\theta} = 0.$$

Therefore we have

$$(3.26) \quad \frac{d\hat{\theta}}{d\hat{s}_1} = -\hat{\kappa}_1.$$

From (3.24), (3.25) and noticing that \vec{t}_1 is orthogonal to \vec{b}_1 , we find that

$$\frac{d\hat{s}}{d\hat{s}_1} = \frac{1}{\cosh \hat{\theta}} = \frac{\hat{\mu}\hat{\tau}_1}{\sinh \hat{\theta}}.$$

Then we have

$$\hat{\mu}\hat{\tau}_1 = \tanh \hat{\theta}.$$

By taking the derivative of this equation and applying (3.26), we get

$$\hat{\mu} \frac{d\hat{\tau}_1}{d\hat{s}_1} = -\hat{\kappa}_1 (1 - \hat{\mu}^2 \hat{\tau}_1^2),$$

that is

$$\frac{d\hat{\tau}_1}{d\hat{s}_1} = -\frac{\hat{\kappa}_1}{\hat{\mu}} (1 - \hat{\mu}^2 \hat{\tau}_1^2).$$

Conversely, if the curvature $\hat{\kappa}_1$ and torsion $\hat{\tau}_1$ of the dual curve $\hat{\beta}$ satisfy

$$\frac{d\hat{\tau}_1}{d\hat{s}_1} = -\frac{\hat{\kappa}_1}{\hat{\mu}} (1 - \hat{\mu}^2 \hat{\tau}_1^2)$$

for some never pure dual constant $\hat{\mu}(\hat{s})$, then we define a dual Lorentzian curve by

$$(3.27) \quad \hat{x}(\hat{s}_1) = \hat{x}_1(\hat{s}_1) + \hat{\mu} \vec{b}_1(\hat{s}_1),$$

and we will prove that $\hat{\alpha}$ is a dual Lorentzian Mannheim curve and that $\hat{\beta}$ is the dual Lorentzian partner curve of $\hat{\alpha}$.

By taking the derivative of (3.27) with respect to \widehat{s}_1 twice, we get

$$(3.28) \quad \vec{t} \frac{d\widehat{s}}{d\widehat{s}_1} = \vec{t}_1 + \widehat{\mu}\widehat{\tau}_1 \vec{n}_1,$$

$$(3.29) \quad \widehat{\kappa} \vec{n} \left(\frac{d\widehat{s}}{d\widehat{s}_1} \right)^2 + \vec{t} \frac{d^2\widehat{s}}{d\widehat{s}_1^2} = \widehat{\mu}\widehat{\kappa}_1\widehat{\tau}_1 \vec{t}_1 + \left(\widehat{\kappa}_1 + \widehat{\mu} \frac{d\widehat{\tau}_1}{d\widehat{s}_1} \right) \vec{n}_1 + \widehat{\mu}\widehat{\tau}_1^2 \vec{b}_1,$$

respectively. Taking the cross product of (3.28) with (3.29) and noticing that

$$\widehat{\kappa}_1 + \widehat{\mu} \frac{d\widehat{\tau}_1}{d\widehat{s}_1} - \widehat{\mu}^2 \widehat{\kappa}_1 \widehat{\tau}_1^2 = 0,$$

we have

$$(3.30) \quad \widehat{\kappa} \vec{b} \left(\frac{d\widehat{s}}{d\widehat{s}_1} \right)^3 = \widehat{\mu}^2 \widehat{\tau}_1^3 \vec{t}_1 + \widehat{\mu}\widehat{\tau}_1^2 \vec{n}_1$$

By taking the cross product of (3.30) with (3.28), we obtain also

$$\widehat{\kappa} \vec{n} \left(\frac{d\widehat{s}}{d\widehat{s}_1} \right)^4 = -\widehat{\mu}\widehat{\tau}_1^2 (1 - \widehat{\mu}^2 \widehat{\tau}_1^2) \vec{b}_1.$$

This means that the principal normal direction \vec{n} of $\widehat{\alpha} : \widehat{x}(\widehat{s})$ coincides with the binormal direction \vec{b}_1 of $\widehat{\beta} : \widehat{x}_1(\widehat{s}_1)$. Hence, $\widehat{\alpha} : \widehat{x}(\widehat{s})$ is a dual Lorentzian Mannheim curve and $\widehat{\beta} : \widehat{x}_1(\widehat{s}_1)$ is its dual Lorentzian Mannheim partner curve. \square

3.4. Remark. Let $\widehat{\alpha} = \alpha + \varepsilon\alpha^*$ and $\widehat{\beta} = \beta + \varepsilon\beta^*$ be non-null dual curves in the dual Lorentzian space \mathbb{D}_1^3 . As a special case, if we consider the dual part of the given curves as $\alpha^* = 0$ and $\beta^* = 0$, then $\widehat{\alpha}$ and $\widehat{\beta}$ are non-null Lorentzian curves in \mathbb{R}_1^3 . In this case all the results of this paper are similar to the results given in [9].

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References

- [1] Ayyıldız, N., Çöken, A. C. and Yücesan, A. *A characterization of dual Lorentzian spherical curves in the dual Lorentzian space*, Taiwanese J. Math. **11** (4), 999–1018, 2007.
- [2] Ayyıldız, N., Çöken, A. C. and Yücesan, A. *On the dual Darboux rotation axis of the space-like dual space curve*. Demonstratio Math. **37** (1), 197–202, 2004.
- [3] Çöken, A. C. and Görgülü, A. *On the dual Darboux rotation axis of the dual space curve*, Demonstratio Math. **35** (2), 385–390, 2002.
- [4] do Carmo, M. P. *Differential Geometry of Curves and Surfaces* (Prentice-Hall, Inc., Englewood Cliffs, N. J., 1976).
- [5] Guggenheimer, W. *Differential Geometry* (McGraw-Hill, New York, 1963).
- [6] Hacısalıhoğlu, H. H. *On the pitch of a closed ruled surface*, Mech. Mach. Theory **7** (3), 291–305, 1972.
- [7] Köse, Ö., Nizamoglu, Ş. and Sezer, M. *An explicit characterization of dual spherical curves*, Doğa Mat. **12** (3), 105–113, 1988.
- [8] Kuhnel, W. *Differential Geometry: Curves-Surfaces-Manifolds* (Braunschweig, Wiesbaden, 1999).
- [9] Liu, H. and Wang, F. *Mannheim partner curves in 3-space*, J. Geom. **88**, 120–126, 2008.
- [10] O'Neill, B. *Semi-Riemannian Geometry with Applications to Relativity* (Academic Press, London, 1983).
- [11] Özbey, E. and Oral, M. *A Study on rectifying curves in the dual Lorentzian space*, Bull. Korean Math. Soc. **46** (5), 967–978, 2009.

- [12] Özkaldı, S., İlarıslan, K. and Yaylı, Y. *On Mannheim partner curves in dual space*, An. Şt. Univ. Ovidius Constanta **17** (2), 131–142, 2009.
- [13] Struik, D. J. *Differential Geometry, Second Ed.* (Addison-Wesley, Reading, Massachusetts, 1961).
- [14] Turgut, M. *On the invariants of time-like dual curves*, Hacet. J. Math. Stat. **37** (2), 129–133, 2008.
- [15] Uğurlu H. H., and Çalışkan, A. *The Study mapping for directed spacelike and timelike lines in Minkowski 3-space \mathbb{R}_1^3* , Mathematical and Computational Applications **1** (2), 142–148, 1996.
- [16] Veldkamp, G. R. *On the use of dual numbers, vectors and matrices in instantaneous, spatial kinematics*, Mech. Mach. Theory **11** (2), 141–156, 1976.
- [17] Yücesan, A., Ayyıldız, N. and Çöken, A. C. *On rectifying dual space curves*, Rev. Mat. Complut. **20** (2), 497–506, 2007.
- [18] Yücesan, A., Çöken, A. C. and Ayyıldız, N. *On the dual Darboux rotation axis of the timelike dual space curve*, Balkan J. Geom. Appl. **7** (2), 137–142, 2002.