A classification of biharmonic hypersurfaces in the Minkowski spaces of arbitrary dimension

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Abstract
In this paper we study hypersurfaces with the mean curvature function $H$ satisfying $\langle \nabla H, \nabla H \rangle = 0$ in a Minkowski space of arbitrary dimension. First, we obtain some conditions satisfied by connection forms of biconservative hypersurfaces with the mean curvature function whose gradient is light-like. Then, we use these results to get a classification of biharmonic hypersurfaces. In particular, we prove that if a hypersurface is biharmonic, then it must have at least 6 distinct principal curvatures under the hypothesis of having mean curvature function satisfying the condition above.

Keywords: biharmonic submanifolds, Lorentzian hypersurfaces, biconservative hypersurfaces, finite type submanifolds.

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1. Introduction
After Bang-Yen Chen conjectured that every biharmonic submanifold of a Euclidean space is minimal, biharmonic and biconservative submanifolds in semi-Euclidean spaces have been studied by many geometers (cf. [4, 5, 7, 8]). In particular, many results on biharmonic submanifolds in the Minkowski 4-space $\mathbb{E}^4_1$ and the semi-Euclidean space $\mathbb{E}^4_2$ have appeared since the middle of 1990s, [1, 2, 6, 9, 18].

On the other hand, several geometrical properties of biconservative submanifolds in Euclidean spaces have been obtained and some classification results of biconservative hypersurfaces have been given so far, [3, 12, 15, 17]. For example in [12], Hasanis and Vlachos obtained the complete classification of biconservative hypersurfaces in $\mathbb{E}^3_1$ and $\mathbb{E}^4$. Furthermore, Yu Fu have recently proved that the only biconservative surfaces in $\mathbb{E}^4_1$ are surfaces of revolution and null scrolls, [10]. Most recently, the complete classification of biconservative surfaces in 4-dimensional Lorentzian space forms is obtained in [11].
Let $M$ be a hypersurface in $\mathbb{E}^{n+1}_s$, $s = 0, 1$ with the shape operator $S$, mean curvature $H$ and $x: M \rightarrow \mathbb{E}^m$ an isometric immersion. $M$ is said to be biharmonic if the equation $\Delta^2 x = 0$ is satisfied or, equivalently, the system of differential equations

\begin{align*}
\text{(BC)} & \quad S(\nabla H) + \varepsilon \frac{nH}{2}\nabla H = 0,
\text{(BH1)} & \quad \Delta H + H\text{tr}S^2 = 0
\end{align*}

is satisfied, where $N$ is the unit normal vector field (see [6, 13]) and $\varepsilon = \langle N, N \rangle$.

On the other hand, a hypersurface satisfying (BC) is said to be biconservative. From (BC), one can see that if a hypersurface $M$ with non-constant mean curvature is biconservative, then $\nabla H$ is an eigenvector of its shape operator. Note that along with the increase of index, the difference between Euclidean space and Minkowski space is the appearance of light-like vectors. Thus, the hypersurfaces with light-like $\nabla H$ has no counterparts in Euclidean spaces and they are worth to be studied separately in terms of being biconservative or biharmonic.

1.1. Remark. For ease of elaboration, we want to abbreviate a hypersurface with mean curvature whose gradient is light-like to a MCGL-hypersurface.

In this work we study MCGL-hypersurfaces in the Minkowski space of arbitrary dimension. In Sect. 2, after we describe our notations, we give a summary of the basic facts and formulas that we will use. In Sect. 3, we focus on biconservative MCGL-hypersurfaces and obtain some necessary conditions. In Sect. 4, we prove the non-existence of biharmonic MCGL-hypersurfaces under some conditions.

2. Preliminaries

Let $\mathbb{E}_s^n$ denote the pseudo-Euclidean $n$-space with the canonical pseudo-Euclidean metric tensor $g$ of index $s$ given by

\[ g = -\sum_{i=1}^{s} dx^2_i + \sum_{j=s+1}^{m} dx^2_j, \]

where $(x_1, x_2, \ldots, x_m)$ is a rectangular coordinate system in $\mathbb{E}_s^n$. A non-zero vector $v \in \mathbb{E}_s^n$ is called space-like, time-like or light-like if $\langle v, v \rangle > 0$, $\langle v, v \rangle < 0$ or $\langle v, v \rangle = 0$, respectively.

Consider an oriented hypersurface $M$ of the Minkowski space $\mathbb{E}_s^{n+1}$. We denote the Levi-Civita connections of $\mathbb{E}_s^{n+1}$ and $M$ by $\tilde{\nabla}$ and $\nabla$, respectively. Then, the Gauss and Weingarten formulas are given, respectively, by

\begin{align*}
\tilde{\nabla}_XY &= \nabla_XY + h(X,Y)N, \\
\tilde{\nabla}_XN &= -S(X)
\end{align*}

for all tangent vectors fields $X, Y$, where $h, \nabla^\perp$ and $S$ are the second fundamental form, the normal connection and the shape operator of $M$, respectively, and $N$ is the unit normal vector field associated with the orientation of $M$.

The Gauss and Codazzi equations are given, respectively, by

\begin{align*}
R(X, Y, Z, W) &= \langle h(Y, Z), h(X, W) \rangle - \langle h(X, Z), h(Y, W) \rangle, \\
(\nabla_Xh)(Y, Z) &= (\nabla_Yh)(X, Z),
\end{align*}

where $R$ is the curvature tensor associated with the connection $\nabla$ and $\nabla h$ is defined by

\[ (\nabla_Xh)(Y, Z) = \nabla^\perp_X h(Y, Z) - h(\nabla_XY, Z) - h(Y, \nabla_XZ). \]
$M$ is said to be Lorentzian if its tangent space $T_m M$ at every point $m \in M$ has two linearly independent null vectors. In this case, there exists a pseudo-orthonormal frame field $\{e_1, e_2, \ldots, e_n\}$ of the tangent bundle of $M$ satisfying

$$\langle e_A, e_B \rangle = 1 - \delta_{AB}, \quad \langle e_A, e_a \rangle = 0, \quad \langle e_a, e_b \rangle = \delta_{ab}$$

for all $A, B = 1, 2, a, b = 3, 4, \ldots, n$. Then, the Levi-Civita connection $\nabla$ of $M$ becomes

$$\nabla_{e_i} e_1 = \phi_i e_1 + \sum_{b=3}^{n} \omega_{1b}(e_i) e_b,$$

$$\nabla_{e_i} e_2 = -\phi_i e_2 + \sum_{b=3}^{n} \omega_{2b}(e_i) e_b,$$

$$\nabla_{e_i} e_a = \omega_{a1}(e_i) e_1 + \omega_{a2}(e_i) e_2 + \sum_{b=3}^{n} \omega_{ab}(e_i) e_b,$$

where $\phi_i = \phi(e_i) = \langle \nabla_{e_i} e_2, e_1 \rangle$ and $\omega_{jk}(e_i) = \langle \nabla_{e_j} e_k, e_i \rangle$, i.e., $\phi = -\omega_{12}$.

On the other hand, the shape operator $S$ of an oriented Lorentzian hypersurface in $E^{n+1}_1$ can be non-diagonalizable. If $S$ is non-diagonalizable, then its characteristic polynomial may also have complex roots. However, in this case all eigenvectors of $S$ are space-like.

Now, assume that $M$ has non-diagonalizable shape operator $S$ and consider the case that all of the eigenvalues of $S$ are real at any point of $M$. In this case, locally, we may assume that the multiplicities of eigenvalues are constant at every point of $M$. Therefore, [14, Lemma 2.3 and Lemma 2.5] imply that there exists an appropriate pseudo-orthonormal frame field $\{e_1, e_2, \ldots, e_n\}$ of smooth vector fields such that the matrix representation of $S$ is in one of the following two forms.

Case I. $S = \begin{pmatrix}
    k_1 & 1 & 0 & 0 & \ldots & 0 \\
    0 & k_1 & 0 & 0 & \ldots & 0 \\
    0 & 0 & k_3 & 0 & \ldots & 0 \\
    0 & 0 & 0 & k_4 & \ldots & 0 \\
    \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & 0 & 0 & \ldots & k_n
  \end{pmatrix}$

Case II. $S = \begin{pmatrix}
    k_1 & 0 & 1 & 0 & \ldots & 0 \\
    0 & k_1 & 0 & 0 & \ldots & 0 \\
    0 & 0 & 1 & 0 & \ldots & 0 \\
    0 & 0 & 0 & k_3 & \ldots & 0 \\
    \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & 0 & 0 & \ldots & k_n
  \end{pmatrix}$

for some smooth functions $k_1, k_3, k_4, \ldots, k_n$, where the eigenvector $e_1$ of $S$ is light-like, (see also [13, 16]). With the abuse of terminology, we call these vector fields $e_1, e_2, \ldots, e_n$ as principal directions and the functions $k_1, k_3, k_4, \ldots, k_n$ as principal curvatures. Moreover, we put

$$s_1 = 2k_1 + k_3 + \cdots + k_n = nH,$$

where $H$ is the mean curvature of $M$. 

3. Biconservative MCGL-hypersurfaces

In this section we focus on biconservative MCGL-hypersurfaces in the Minkowski space $\mathbb{E}^{n+1}_1$. As we described in the previous section, the shape operator $S$ of a MCGL-hypersurfaces in the Minkowski space $\mathbb{E}^{n+1}_1$ is one of two forms given in (2.6). We study these two cases separately.

3.1. Case I. Consider a hypersurface $M$ in the Minkowski space $\mathbb{E}^{n+1}_1$ with the shape operator $S$ given by case I of (2.6). Then, we have

$$h(e_1, e_2) = -k_1, \quad h(e_2, e_2) = -1,$$

(3.1)  

$$h(e_A, e_B) = \delta_{AB} k_A,$$

$$h(e_1, e_1) = h(e_1, e_A) = h(e_2, e_A) = 0, \quad A, B = 3, 4, \ldots, n.$$  

Now, assume that $M$ is a biconservative MCGL-hypersurface, i.e., $\nabla s_1$ is light-like and (BC) is satisfied. Then, $e_1$ is proportional to $\nabla s_1$ and we have

$$(3.2a) \quad k_1 = -\frac{81}{2}, \quad k_3 + k_4 + \cdots + k_n = 2s_1,$$

(3.2b)  

$$e_1(k_1) = e_3(k_1) = e_4(k_1) = \cdots = e_n(k_1) = 0, \quad e_2(k_1) \neq 0.$$  

Let the distinct principal curvatures of $M$ be $K_1, K_2, \ldots, K_p$ with the multiplicities $\nu_1, \nu_2, \ldots, \nu_p$, respectively, i.e., the characteristic polynomial of $S$ is

$$p_S(t) = (t - K_1)^{\nu_1} (t - K_2)^{\nu_2} \cdots (t - K_p)^{\nu_p}$$  

with $K_1 = k_1$ and $\nu_1 \geq 2$. We also suppose that the functions $K_\alpha - K_\beta$ does not vanish on $M$, for all $\alpha \neq \beta \in \{1, 2, \ldots, p\}$. Then, (3.2a) becomes

$$(3.3) \quad K_1 = -\frac{81}{2}, \quad \nu_2 K_2 + \nu_3 K_3 + \cdots + \nu_p K_p = (-2 - \nu_1) K_1.$$  

On the other hand, from the Codazzi equation (2.4) for $X = e_1$, $Y = Z = e_A$ we get

$$(3.5) \quad \psi_0 = \omega_A(e_A) = \frac{e_1(K_A)}{K_1 - K_A} \quad \text{if} \quad k_A = K_\alpha, \quad \alpha = 2, 3, \ldots, p.$$  

By rearranging the indices if necessary, we may assume that $\psi_2, \psi_3, \ldots, \psi_r \neq 0$ and $\psi_{r+1} = \psi_{r+2} = \cdots = \psi_p = 0$ for some $r \leq p$. Thus, from (3.5) we have

$$(3.6) \quad e_1(K_A) = 0 \quad \text{if} \quad k_A = K_\alpha, \quad \alpha > r.$$  

From Codazzi equation (2.4) for $X = e_1$, $Y = e_A$, $Z = e_B$ and $X = e_A$, $Y = e_B$, $Z = e_1$ we obtain

$$(3.7) \quad \omega_A(e_B)(k_1 - k_A) = \omega_B(e_A)(k_1 - k_B) = \omega_A(e)(k_A - k_B), \quad A, B = 2, 3, \ldots, n.$$  

Moreover, from the equation $\{e_A, e_B\}(k_1) = 0$ we have

$$\omega_A(e_B) = \omega_B(e_A).$$  

By combining the above equation with (3.7) one may obtain

$$(3.8) \quad \omega_A(e_A) = 0 \quad \text{if} \quad k_A, k_B \neq K_1.$$  

On the other hand, from the Codazzi equation $X = e_1, Y = e_1, Z = e_j$ and $X = e_2, Y = e_1, Z = e_j$ we have

$$(3.9) \quad \omega_{ij}(e_1) = 0, \quad j = 3, 4, \ldots, n.$$  

In addition, by combining the Codazzi equation (2.4) for $X = e_A, Y = e_1, Z = e_a$ and $[e_a, e_A](k_1) = 0$, we obtain

$$(3.10) \quad \omega_{aA}(e_1) = \omega_{1A}(e_a) = \omega_{1A}(e_A) = 0.$$
for all \( a, A = 3, 4, \ldots, n \) such that \( k_a = K_1 \neq k_A \). By summing up (3.8)-(3.10) we obtain
\[
\nabla e_1 e_1 = \phi_1 e_1, \quad \nabla e_A e_1 = \phi_A e_1 + \omega_1 A(e_A)e_A, \\
\omega_1 A(e_a) = 0, \quad x \neq 2, x \neq A
\]
for all \( A = 3, 4, \ldots, n \) such that \( K_1 \neq k_A \).

Hence, by combining (3.11) and the Gauss equation \( R(e_A, e_1, e_1, e_A) = 0 \) we obtain
\[
e_1(\omega_1 A(e_A)) = \omega_1 A(e_A)(\phi_1 - \omega_1 A(e_A)) \quad \text{if} \quad k_A \neq K_1
\]
from which we get
\[
e_1(\psi_3) = \psi_3(\phi_1 - \psi_3), \quad \psi_3 = 2, 3, \ldots, r.
\]

Next, we obtain the following lemma which we will use later.

**3.1. Lemma.** Let \( M \) be a biconservative MCGL-hypersurface in the Minkowski space \( E_{n+1}^1 \) with the shape operator given by (3.1). Then the functions \( \psi_3, \psi_4, \ldots, \psi_r \) defined above satisfy
\[
W(\psi_2, \psi_3, \ldots, \psi_r) = 0,
\]
where \( W(\psi_2, \psi_3, \ldots, \psi_r) \) is an \( r \times r \) matrix given by
\[
W(\psi_2, \psi_3, \ldots, \psi_r) = \begin{pmatrix}
\psi_2 & \psi_3 & \cdots & \psi_r \\
\psi_2^2 & \psi_3^2 & \cdots & \psi_r^2 \\
\vdots & \vdots & \ddots & \vdots \\
\psi_2^r & \psi_3^r & \cdots & \psi_r^r
\end{pmatrix}.
\]

**Proof.** By applying \( e_1 \) to the second equation in (3.4) and using (3.2b), we obtain
\[
\nu_2 e_1(K_2) + \nu_3 e_1(K_3) + \cdots + \nu_p e_1(K_p) = 0.
\]

Now, by induction we would like to show
\[
\sum_{\alpha=2}^{r} (\psi_\alpha)^t \nu_\alpha(K_1 - K_\alpha) = 0, \quad t = 1, 2, \ldots
\]
\[
\sum_{\alpha=2}^{r} (\psi_\alpha)^{t-1} \nu_\alpha(K_1 - K_\alpha) = 0, \quad n = 1, 2, \ldots
\]
By applying \( e_1 \) to this equation and using (3.2b), (3.5) and (3.12) we obtain
\[
\sum_{\alpha=2}^{r} (l-1)(\psi_\alpha)^{t-1} \nu_\alpha(\phi_1 - \psi_\alpha)(K_1 - K_\alpha) = \sum_{\alpha=2}^{r} (\psi_\alpha)^t \nu_\alpha(K_1 - K_\alpha).
\]
By combining this equation and (3.16) we obtain (3.15) for \( t = 1 \). Thus, we have (3.15) for all \( t \) which implies (3.13). \( \square \)
3.2. Case II. In this subsection, we consider the hypersurfaces with the shape operator given by case II of (2.6) in the Minkowski space $\mathbb{E}^{n+1}$. Now, assume that $M$ is a biconservative MCGL-hypersurface. In this case, we have

$$h(e_1, e_2) = -k_1, \quad h(e_1, e_1) = h(e_1, e_3) = h(e_2, e_2) = 0,$$

(3.17) \hspace{1cm} h(e_3, e_3) = k_1, \quad h(e_A, e_B) = \delta_{AB}k_A,

$$h(e_1, e_1) = h(e_1, e_A) = h(e_2, e_A) = h(e_3, e_A) = 0, \quad A, B = 4, 5, \ldots, n.$$ Assume that the characteristic polynomial of $S$ is as given by (3.3) with $k_1 = k_1 = -s_1/2$ and $\nu_1 \geq 3$. Then, we have (3.4) and

$$e_1(K_1) = e_3(K_1) = e_4(K_1) = \cdots = e_n(K_1) = 0, \quad e_2(K_1) \neq 0.$$ We again suppose that the functions $K_\alpha - K_\beta$ does not vanish on $M$.

Note that the Codazzi equation (2.4) for $X = e_1, Y = e_A, Z = e_B$ gives $e_1(k_A) = \omega_1A(e_A)(k_1 - k_A)$ if $k_1 \neq k_A$. Let $\psi_2, \psi_3, \ldots, \psi_p$ be the functions defined by (3.5) such that $\psi_2, \psi_3, \ldots, \psi_p \neq 0$ and $\psi_{r+1} = \psi_{r+2} = \cdots = \psi_p = 0$ for some $r \leq p$.

(3.18) implies $[e_1, e_A](k_1) = 0$. By computing the left-hand side of this equation we get $\omega_1A(e_1) = 0, A = 3, 4, \ldots, n$. In addition, the Codazzi equation (2.4) for $X = e_1, Y = e_2, Z = e_3$ gives $\psi_1 = 0$. Thus, we have $\nabla_{e_1}e_1 = 0$. Next, similar to previous subsection, we apply the Codazzi equation (2.4) for $X = e_i, Y = e_j, Z = e_k$ for each triplet $(i, j, k)$ in the set \{(1, 2, a), (1, 3, A), (3, A, 1), (1, A, B), (A, B, 1), (1, a, A)\} and combine equations obtained with $[e_A, e_B](k_1) = [e_A, e_B](k_1) = 0$ to get $\nabla_{e_1}(e_1) \in \text{span} \{e_1, e_A\}$ and $\omega_1A(e_1) = 0, x = 2, A$, where $A, B, a = 4, 5, \ldots, n$ with $A \neq B, k_A, k_B \neq k_1, k_0 = K_1$. By combining these equations with the Gauss equation $R(e_3, e_1, e_1, e_3) = 0$ we obtain

$$e_1(\psi_\alpha) = -\psi_\alpha^2, \quad \alpha = 1, 2, \ldots, r.$$ Therefore, similar to Lemma 3.1 we have

3.2. Lemma. Let $M$ be a biconservative MCGL-hypersurface in the Minkowski space $\mathbb{E}^{n+1}$ with the shape operator given by (3.17). Then the functions $\psi_3, \psi_4, \ldots, \psi_r$ defined above satisfy (3.13).

3.3. Biconservative hypersurfaces. In this subsection, we would like to obtain conditions satisfied by connection forms of biconservative MCGL-hypersurfaces (See [17, 10, 11] for implicit examples of biconservative hypersurfaces that have recently obtained).

Now we would like to obtain some necessary conditions for being biconservative of an MCGL-hypersurface by using Lemma 3.1 and Lemma 3.2.

3.3. Proposition. Let $M$ be an MCGL-hypersurface in the Minkowski space $\mathbb{E}^{n+1}$ and $e_1, e_2, \ldots, e_n$ its principal directions with corresponding principal curvatures $k_1, k_1, k_3, k_4, \ldots, k_n$ such that $e_1$ is proportional to gradient of its mean curvature. If $M$ is biconservative, then

(i) For any $3 \leq i \leq n$ such that $k_i \neq k_1, \omega_1i(e_i) \neq 0$, there exists a $j \neq i$ such that $\omega_1j(e_j) = \omega_1i(e_i), k_j \neq k_i$.

(ii) Let $I_i = \{3 \leq j \leq n | \omega_1j(e_j) = \omega_1i(e_i)\}$. Then,

$$\sum_{j \in I_i} (k_1 - k_j) = 0.$$ (3.19)

(iii) There exists a $j \in \{3, 4, \ldots, n\}$ such that $e_1(k_j) = \omega_1i(e_i) = 0, k_1 \neq k_i$.

Proof. Let $K_1, \ldots, K_n$ and $\psi_2, \ldots, \psi_r$ be the functions defined on the beginning of this section.
Assume that $\psi_2 \neq 0$ and $\psi_2 \neq \psi_j$, $2 < j \leq r$. Then, we have $\det W(\psi_2, \psi_3, \ldots, \psi_r) = 0$ from (3.13) since the functions $K_1 - K_2$ is non-vanishing by the assumptions. Therefore, $\psi_3, \ldots, \psi_r$ are not distinct and we may assume $\psi_{r-1} = \psi_r$. Thus (3.13) gives

$$W(\psi_2, \psi_3, \ldots, \psi_{r-1}) \begin{pmatrix} \nu_2(K_1 - K_2) \\ \nu_3(K_1 - K_3) \\ \vdots \\ \nu_r(K_1 - K_r) + \nu_{r-1}(K_1 - K_{r-1}) \end{pmatrix} = 0.$$ 

Since $(K_1 - K_2)$ is non-vanishing, the above equation implies that $\psi_3, \ldots, \psi_{r-1}$ are not distinct and we may assume either $\psi_{r-2} = \psi_{r-1}$ or $\psi_3 = \psi_4$. By repeating this procedure, one can get $\psi_3 = \cdots = \psi_{r-1}$ and

$$\psi_2(K_1 - K_2) + \psi_3 \left( \sum_{\alpha=3}^r \nu_\alpha(K_1 - K_\alpha) \right) = 0,$$

$$\psi_2^2(K_1 - K_2) + \psi_3^2 \left( \sum_{\alpha=3}^r \nu_\alpha(K_1 - K_\alpha) \right) = 0$$

which gives $\psi_2 = \psi_3$ or $K_1 - K_2 = 0$ which yields a contradiction. Hence we have (i) of the proposition.

Let $l - 1$ of $\psi_2, \psi_3, \ldots, \psi_r$ be distinct and by rearranging indices if necessary, assume that they are $\psi_2, \psi_3, \ldots, \psi_l$. Note that we have $l - 1 \leq (r - 1)/2$ because of (i) of the proposition. Moreover, we have $\det W(\psi_2, \psi_3, \ldots, \psi_l) \neq 0$. Thus, (3.13) implies

$$W(\psi_2, \psi_3, \ldots, \psi_l) \begin{pmatrix} \sum_{j \in I_2} \nu_j(K_1 - K_j) \\ \sum_{j \in I_3} \nu_j(K_1 - K_j) \\ \vdots \\ \sum_{j \in I_l} \nu_j(K_1 - K_j) \end{pmatrix} = 0$$

which gives (ii) of the proposition.

Now, assume that all of the functions $\omega_j(e_j)$ are non-zero, i.e., $r = p$ and $\psi_2, \psi_3, \ldots, \psi_l$ are distinct. Note that we have $\bigcup_{j=2}^l I_j = \{2, 3, \ldots, p\}$ and (ii) of the proposition implies

$$\sum_{j \in I_\alpha} \nu_j(K_1 - K_j) = 0$$

or, equivalently,

$$\sum_{j \in I_\alpha} \nu_j K_j = \left( \sum_{j \in I_\alpha} \nu_j \right) K_1, \quad \alpha = 2, 3, \ldots, l.$$ 

By summing these equations over $\alpha$ we get

$$\nu_2 K_2 + \nu_3 K_3 + \cdots + \nu_p K_p = (\nu_2 + \nu_3 + \cdots + \nu_p) K_1.$$ 

However, this equation and (3.4) give $K_1 \equiv 0$ on $M$ which implies $\nabla s_1 = 0$. This is a contradiction because we have assumed that $\nabla s_1$ is light-like. Hence, we have (iii) of the proposition.

4. Biharmonic MCGL-Hypersurfaces

In this section we study biharmonic MCGL-hypersurfaces with the shape operator given by (3.1) in the Minkowski space $\mathbb{E}^{n+1}_1$ and obtain some classification results.
Let $M$ be a biharmonic MCGL-hypersurface with the shape operator given by (3.1). Then, we have (3.2a)-(3.13) obtained in the Sect. 3.1. In addition, from the Codazzi equation $X = e_2, Y = e_1, Z = e_2$ and $X = e_A, Y = e_2, Z = e_A$ we have

\[(4.1) \quad e_2(k_1) = 2\phi_1 = \omega_{1A}(e_A), \quad \text{if} \; k_A = K_1, \; A > 2.\]

Moreover, since $e_1e_2(k_1) = [e_1, e_2](k_1)$, by using (3.2b) we get

\[(4.2) \quad e_1e_2(k_1) = -\phi_1 e_2(k_1).\]

This equation and (4.1) imply using (3.2b) and (4.2) we get

\[(4.3) \quad e_1(\phi_1) = -\phi_1^2.\]

Now we would like to consider the biharmonic equation (BHI). By a direct calculation using (3.2b) and (4.2) we get

\[
\left( e_1e_2 + e_2e_1 - \sum_{j=3}^n e_je_j - \nabla e_1e_2 - \nabla e_2e_1 \right)(k_1) = 0
\]

which gives

\[
\Delta k_1 = \sum_{j=3}^n \omega_{1j}(e_j) e_2(k_1) = \sum_{\alpha=1}^r \left( \sum_{k_A=K_\alpha} \omega_{1A}(e_A) e_2(k_1) \right)
\]

\[
= (2\nu_1 \phi_1 + 2\nu_3 \phi_2 + 2\nu_4 \phi_3 + \cdots + 2\nu_r \phi_r) e_2(k_1).
\]

By combining the above equation and (4.1), we see that the biharmonic equation (BHI) becomes

\[(BH2) \quad (4\nu_1 \phi_1 + 2\nu_2 \phi_2 + 2\nu_3 \phi_3 + \cdots + 2\nu_r \phi_r) \phi_1 = -k_1(\nu_1 K_1^2 + \nu_2 K_2^2 + \cdots + \nu_r K_r^2).\]

### 4.1. Theorem. There exists no biharmonic MCGL-hypersurface with at most 5 distinct principal curvatures and the shape operator given by (3.1) in the Minkowski space $E_4^{n+1}$.

**Proof.** Let the distinct principal curvatures of $M$ be $K_1, K_2, K_3, K_4, K_5$ with the multiplicities $\nu_1, \nu_2, \nu_3, \nu_4, \nu_5$, respectively, and consider the functions $\psi_2, \psi_3, \psi_4, \psi_5$ defined by (3.5). Now, toward contradiction we assume that $M$ is a biharmonic MCGL-hypersurface, i.e., (BC) and (BHI) are satisfied.

**Case I.** $p < 4$. If the number of distinct principal curvatures is less than 4, the proof directly follows from Proposition 3.3.

**Case II.** $p = 4$. Next, we consider the case that $M$ has exactly 4 distinct principal curvatures, i.e., $K_4 = K_5$. Then, because of (iii) of Proposition 3.3, we may assume $\psi_2 = 0$. Note that if $\psi_3 = 0$, then (i) of Proposition 3.3 implies $\psi_4 = 0$. In this subcase, we have $r = 1$ and (3.6) implies $e_1(K_\alpha) = 0, \; \alpha = 1, 2, 3, 4$. Thus (BH2) becomes

\[(4.4) \quad 4\nu_1 \phi_1^2 = -k_1(\nu_1 K_1^2 + \nu_2 K_2^2 + \nu_3 K_3^2 + \nu_4 K_4^2).\]

By applying $e_1$ to this equation and using (4.3) one can find $\nu_1 \phi_1^3 = 0$. However, this equation and (4.4) implies $k_1 \equiv 0$. Thus, we have $\nabla \psi_1 = 0$ which contradicts with being light-like of $\nabla \psi_1$. Hence, $\psi_3$ and $\psi_4$ are non-zero.

Therefore, (i) and (ii) of Proposition 3.3 imply

\[(4.5) \quad \psi_3 = \psi_4, \quad \nu_3(K_1 - K_4) + \nu_4(K_1 - K_4) = 0.\]

Thus, (BH2) becomes

\[(4.6) \quad (a\phi_1 + b\psi_3) \phi_1 = -k_1(\nu_1 K_1^2 + \nu_2 K_2^2 + \nu_3 K_3^2 + \nu_4 K_4^2),\]

where $a = 4\nu_2$ and $b = 2(\nu_3 + \nu_4)$ are some non-negative constants. Note that $\psi_2 = 0$ and (3.5) imply $e_1(K_2) = 0$. 

\[
\sum_{j=3}^n \omega_{1j}(e_j) e_2(k_1) = \sum_{\alpha=1}^r \left( \sum_{k_A=K_\alpha} \omega_{1A}(e_A) e_2(k_1) \right)
\]

\[
= (2\nu_1 \phi_1 + 2\nu_3 \phi_2 + 2\nu_4 \phi_3 + \cdots + 2\nu_r \phi_r) e_2(k_1).
\]
Next, we apply \( e_1 \) to (4.6) and use (3.2b), (4.3), (3.12) to get

\[
(4.7) \quad -(2a\psi_1^2 + b\psi_3^3)\phi_1 = -k_1 e_1 (\nu_3 K_3^2 + \nu_4 K_4^2).
\]

Then we use, (3.5) and (4.5) to compute the right-hand side of (4.7) and get

\[
(4.8) \quad -(2a\psi_1^2 + b\psi_3^3)\phi_1 = -2k_1 \psi_3 (bK_1^2 - \nu_3 K_3^2 - \nu_4 K_4^2).
\]

By applying \( e_1 \) to (4.8) again and using (3.2b), (4.3), (3.12) we get

\[
(4.9) \quad (6a\phi_1^3 - b\phi_1^3 \psi_3^3 + 2b\psi_3^3)\phi_3 = -2k_1 \psi_3 (\phi_1 - \psi_3) (bK_1^2 - \nu_3 K_3^2 - \nu_4 K_4^2)
\]

\[
+ 2k_1 \psi_3 e_1 (\nu_3 K_3^2 + \nu_4 K_4^2)
\]

By combining (4.7), (4.8) and (4.9) we get

\[
(4.10) \quad (6a\phi_1^3 - b\phi_1^3 \psi_3^3 + 2b\psi_3^3 + (\phi_1 - 3\psi_3)(2a\phi_1^2 + b\psi_3^2) \phi_1 = 0.
\]

Thus, we have \( \psi_3 = c\phi_1 \) for a constant \( c \). However, in this case, from (4.3) and (3.12) we get \( c = 2 \), i.e., \( \psi_3 = 2\phi_1 \). However, this equation and (4.10) give \( (a + 2b)\phi_1^3 = 0 \) which is impossible to be satisfied because \( a, b \) are non-negative constants. Thus, the proof for this case is completed.

**Case III.** \( p = 5 \). Then, because of (iii) of Proposition 3.3, we may assume \( \psi_2 = 0 \). Note that, if \( \psi_3 = 0 \), then we have either \( \psi_4 = \psi_5 \neq 0 \) or \( \psi_3 = \psi_4 = \psi_5 = 0 \). However, these subcases and the other possible subcase \( \psi_3 = \psi_4 = \psi_5 \) are similar to case II. \( \square \)

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**References**


