A new characterization of symmetric groups for some $n$

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Abstract
Let $G$ be a finite group and let $\pi_e(G)$ be the set of element orders of $G$. Let $k \in \pi_e(G)$ and let $m_k$ be the number of elements of order $k$ in $G$. Set $\text{nse}(G) := \{ m_k | k \in \pi_e(G) \}$. In this paper, we prove that if $G$ is a group such that $\text{nse}(G) = \text{nse}(S_n)$ where $n \in \{3, 4, 5, 6, 7\}$, then $G \cong S_n$.

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1. Introduction

If $n$ is an integer, then we denote by $\pi(n)$ the set of all prime divisors of $n$. Let $G$ be a finite group. Denote by $\pi(G)$ the set of primes $p$ such that $G$ contains an element of order $p$. Also the set of element orders of $G$ is denoted by $\pi_e(G)$. A finite group $G$ is called a simple $K_n$-group, if $G$ is a simple group with $|\pi(G)| = n$. Set $m_i = m_i(G) = |\{ g \in G | \text{the order of } g \text{ is } i \}|$ and $\text{nse}(G) := \{ m_i | i \in \pi_e(G) \}$.

Let $L_t(G) := \{ g \in G | g^t = 1 \}$. Then $G_1$ and $G_2$ are of the same order type if and only if $|L_t(G_1)| = |L_t(G_2)|$, $t = 1, 2, \ldots$. The idea of this paper springs from Thompson’s Problem as follows:

Thompson’s Problem. Suppose that $G_1$ and $G_2$ are of the same order type. If $G_1$ is solvable, is it true that $G_2$ is also necessarily solvable?

Unfortunately, as so far, no one can prove it completely, or even give a counterexample. However, if groups $G_1$ and $G_2$ are of the same order type, we see clearly that $|G_1| = |G_2|$

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Lemma 2.5. In [13], it is proved that all simple $K_4$-groups can be uniquely determined by $nse(G)$ and $|G|$. Further, it is claimed that some simple groups could be characterized by exactly the set $nse(G)$ without considering group order. For instance, in [3, 12], it is proved that the alternating groups $A_n$ where $n \in \{4, 5, 6, 7, 8\}$ are uniquely determined by $nse(G)$. Also in [10], it is proved that $L_2(q)$ where $q \in \{7, 8, 11, 13\}$ are uniquely determined by $nse(A)$.

Analogously, for infinite simple groups, there are also some interesting results: In [1], the author prove that $G \cong PGL_2(p)$ if and only the two conditions hold: (1) $p \in \pi(G)$ but $p^2 \nmid |G|$; (2) $nse(G) = nse(PGL_2(p))$, where $p > 3$ is a prime. In [2], the authors proved that all sporadic groups characterizable by $nse(G)$ and $|G|$. In this paper, we show that the symmetric group $S_n$ is characterizable by $nse(G)$ for $n \in \{3, 4, 5, 6, 7\}$. In fact the main theorem of our paper is as follows:

**Main Theorem:** Let $G$ be a group such that $nse(G) = nse(S_n)$ where $n \in \{3, 4, 5, 6, 7\}$. Then $G \cong S_n$.

Note that not all groups can be characterized by $nse(G)$ and $|G|$. For instance, in 1987, Thompson gave an example as follows: Let $G_1 = (C_2 \times C_2 \times C_2 \times C_2) \rtimes A_7$ and $G_2 = L_3(4) \rtimes C_2$ be maximal subgroups of $M_{23}$, where $M_{23}$ is the Mathieu group of degree 23. Although $nse(G_1) = nse(G_2)$ and $|G_1| = |G_2|$, we still have $G_1 \not\cong G_2$.

Throughout this paper, we denote by $\phi$ the Euler totient function. If $G$ is a finite group, then we denote by $P_q$ a Sylow $q$-subgroup of $G$ and $n_q(G)$ is the number of Sylow $q$-subgroup of $G$, that is, $n_q(G) = [Syl_q(G)]$. All other notations are standard and we refer to [5], for example.

**2. Some lemmas**

In this section we collect some preliminary lemmas used in the proof of the main theorem.

**Lemma 2.1.** [6] Let $G$ be a finite group and $m$ be a positive integer dividing $|G|$. If $L_m(G) = \{g \in G \mid g^m = 1\}$, then $m \mid |L_m(G)|$.

**Lemma 2.2.** [12] Let $G$ be a group containing more than two elements. Let $k \in \pi_e(G)$ and $m_k$ be the number of elements of order $k$ in $G$. If $s = sup\{m_k \mid k \in \pi_e(G)\}$ is finite, then $G$ is finite and $|G| \leq s(s^2 - 1)$.

**Lemma 2.3.** [11] Let $G$ be a finite group and $p \in \pi(G)$ be odd. Suppose that $P$ is a Sylow $p$-subgroup of $G$ and $n = p^a$ where $p \mid m$, $a > 1$. If $P$ is not cyclic and $s > 1$, then the number of elements of order $n$ is always a multiple of $p^s$.

**Lemma 2.4.** [8] Let $G$ be a finite solvable group and $|G| = mn$, where $m = p_1^{a_1} \cdots p_r^{a_r}$, $(m, n) = 1$. Let $\pi = \{p_1, \ldots, p_r\}$ and let $h_m$ be the number of $\pi$-Hall subgroups of $G$. Then $h_m = q_1^{b_1} \cdots q_r^{b_r}$ satisfies the following conditions for all $i \in \{1, 2, \ldots, s\}$:

1. $q_i^{b_i} \equiv 1 \pmod{p_j}$, for some $p_j$.
2. The order of some chief factor of $G$ is divisible by $q_i^{b_i}$.

**Lemma 2.5.** [13] Let $G$ be a finite group, $P \in Syl_p(G)$ where $p \in \pi(G)$. Let $G$ have a normal series $K \subseteq L \subseteq G$. If $P \subseteq L$ and $p \nmid |K|$, then the following hold:

1. $N_G(K)|PK/K = N_G(P)K/K$;
2. $|G : N_G(P)| = |L : N_L(P)|$, that is, $n_p(G) = n_p(L)$;
3. $|L/K : N_L(K)|t = |G : N_G(P)| = |L : N_L(P)|$, that is, $n_p(L/K)t =$
Lemma 2.6. [9] If $G$ is a simple $K_3$-group, then $G$ is isomorphic to one of the following groups: $A_5, A_6, L_2(7), L_2(8), L_2(17), L_3(3), U_3(3)$ or $U_3(2)$.

Lemma 2.7. [14] Let $G$ be a simple group of order $2^a \cdot 3^b \cdot 5^c$ where $p \not\in \{2, 3, 5\}$ is a prime and $abc \neq 0$. Then $G$ is isomorphic to one of the following groups: $A_7, A_8, A_9; M_{11}, M_{12}; L_2(q), q = 11, 16, 19, 31, 81; L_3(4), L_4(3), S_6(2), U_4(3)$ or $U_5(2)$. In particular, if $p = 11$, then $G \cong M_{11}, M_{12}, L_2(11)$ or $U_5(2)$; if $p = 7$, then $G \cong A_7, A_8, A_9, A_{10}, L_2(49), L_3(4), S_4(7), S_6(2), U_3(5), U_4(3), J_2$, or $O_{10}^+(2)$.

Let $G$ be a group such that $\text{nse}(G) = \text{nse}(S_n)$ where $n \in \{3, 4, 5, 6, 7\}$. By Lemma 2.2, we can assume that $G$ is finite. Let $m_n$ be the number of elements of order $n$. We note that $m_n = k\phi(n)$, where $k$ is the number of cyclic subgroups of order $n$ in $G$. Also we note that if $n > 2$, then $\phi(n)$ is even. If $n \mid |G|$, then by Lemma 2.1 and the above notation, we have

\[
\begin{align*}
\phi(n) & \mid m_n \\
|n| & \mid \sum_d|n| m_d
\end{align*}
\]

In the proof of the main theorem, we often apply $(\ast)$ and the above comments.

3. Proof of the Main Theorem.

Case 1. Let $G$ be a group such that $\text{nse}(G) = \text{nse}(S_3) = \{1, 2, 3\}$. First we prove that $\pi(G) \subseteq \{2, 3\}$. Since $3 \in \text{nse}(G)$, it follows that by $(\ast)$, $2 \in \pi(G)$ and $m_2 = 3$. Let $2 \neq p \in \pi(G)$. By $(\ast)$, $p | (1 + m_p)$ and $(p - 1) | m_p$, which implies that $p = 3$. Thus $\pi(G) \subseteq \{2, 3\}$. If $3 \in \pi(G)$, then $m_3 = 2$. If $6 \in \pi(G)$, then by $(\ast)$, $m_6 = 2$ and $6 | (1 + m_2 + m_3 + m_6) = 8$, a contradiction. If $2 \in \pi(G)$ for some $i \geq 2$, then $2^{i-1} = \phi(2^i) | m_{2^i} = 2$. So $i = 2$. If $3^j \in \pi(G)$ for some $j \geq 2$, then $2 \times 3^{j-1} = \phi(3^j) | m_{3^j} = 2$, a contradiction. Therefore, $\pi(G) \subseteq \{1, 2, 3, 4\}$ and $|G| = 6 + 2k = 2^m \times 3^n$ where $k, m$ and $n$ are non-negative integers. Now we consider the following subcases:

Subcase (a). If $\pi(G) = \{2\}$, then $\pi_e(G) \subseteq \{1, 2, 4\}$. Since $|\pi_e(G)| \leq 3$, $|G| = 2^m = 6 + 2k$ where $k = 0$, a contradiction.

Subcase (b). If $\pi(G) = \{2, 3\}$, then since $|G| = 6 + 2k = 2^m \times 3^n$ and $|\pi_e(G)| \leq 4$, $0 \leq k \leq 1$. It easy to check that the only solution of the equation is $(k, m, n) = (0, 1, 1)$. Thus $|G| = 6$, $\pi_e(G) = \{1, 2, 3\}$. Therefore, $G \cong S_3$.

Case 2. Let $G$ be a group such that $\text{nse}(G) = \text{nse}(S_4) = \{1, 6, 8, 9\}$. First we prove that $\pi(G) \subseteq \{2, 3\}$. Since $9 \in \text{nse}(G)$, it follows that by $(\ast)$, $2 \in \pi(G)$ and $m_9 = 2$. Let $2 \neq p \in \pi(G)$. By $(\ast)$, $p \in \{3, 7\}$. Thus $\pi(G) \subseteq \{2, 3, 7\}$. If $7 \in \pi(G)$, then $m_7 = 6$.

We prove that $14 \not\in \pi_e(G)$. If $14 \in \pi_e(G)$, then $m_{14} = 6$. On the other hand, $14 | (1 + m_2 + m_7 + m_{14}) = 22$, a contradiction. Therefore, the group $P_2$ acts fixed point freely on the set of elements of order $2$. Hence $|P_2| | m_2 = 9$, a contradiction. Hence $\pi(G) \subseteq \{2, 3\}$. If $3 \in \pi(G)$, then by $(\ast)$, $m_3 = 8$. It is clear that by $(\ast)$, $G$ does not contain any elements of order 6 and 9.

If $2^i \in \pi(G)$ for some $i \geq 2$, then $2^{i-1} \mid m_{2^i}$ where $m_{2^i} \in \{6, 8\}$. So $2 \leq i \leq 4$. Also by $(\ast)$, $m_4 = 6$, $m_8 = 8$ and $m_{16} = 8$. Therefore $\pi_e(G) \subseteq \{1, 2, 3, 4, 8, 16\}$ and
\[ G = 24 + 6k_1 + 8k_2 = 2^n \times 3^m \] where \( k_1, k_2, m \) and \( n \) are non-negative integers. Now we consider the following subcases:

**Subcase (a).** If \( \pi(G) = \{2\} \), then since \( |G| = 24 + 6k_1 + 8k_2 = 2^n \times 3^m \) and \( |\pi_e(G)| \leq 6 \), \( 0 \leq k_1 + k_2 \leq 1 \). It easy to check that the only solution is \((k_1, k_2, m) = (0, 1, 5)\). Thus \( |G| = 2^5 \) and \( \pi_e(G) = \{1, 2, 4, 8, 16\} \). But all groups of order 32 with element of order 16 are known, in particular, there are only four such non-Abelian groups by [7] (see, Chapter 5, Theorem 4.4). Since nse of such non-Abelian groups are not equal to nse\((G)\), it is impossible.

**Subcase (b).** Suppose that \( \pi(G) = \{2, 3\} \). By assumption \( |G| = 24 + 6k_1 + 8k_2 = 2^n \times 3^m \).

Since \( |\pi_e(G)| \leq 6 \), \( 0 \leq k_1 + k_2 \leq 2 \). Hence \( 12 + 3k_1 + 4k_2 = 2^{n-1} \times 3^m \). Since \( 3 \mid k_2 \), it follows that \( k_2 = 0 \). It easy to check that the only solutions of equation are \((k_1, k_2, m, n) = (0, 0, 3, 1) \) or \((2, 0, 2, 2)\).

If \((k_1, k_2, m, n) = (2, 0, 2, 2)\), then \(|G| = 36 \) and \( |\pi_e(G)| = 6 \). On the other hand, \(|P_2| = 4 \) so \( \pi_e(G) = \{1, 2, 3, 4\} \), a contradiction.

Therefore \((k_1, k_2, m, n) = (0, 0, 3, 1)\). \(|G| = 24 \) and \( \pi_e(G) = \{1, 2, 3, 4\} \), which implies that \( G \cong S_4 \).

**Case 3.** Let \( G \) be a group such that \( \text{nse}(G) = \text{nse}(S_3) = \{1, 20, 24, 25, 30\} \). First we prove that \( \pi(G) \subseteq \{2, 3, 5\} \). Since \( 25 \in \text{nse}(G) \), it follows that \( 2 \in \pi(G) \) and \( m_2 = 25 \).

Let \( 2 \neq p \in \pi(G) \). By \((*) \), \( p \in \{3, 5, 31\} \).

If \( p = 31 \), then \( m_{31} = 30 \). On the other hand, if \( 62 \in \pi_e(G) \), then \( m_{62} = 30 \) and \( |G| = 62 | 1 + m_2 + m_{31} + m_{62} = 86 \), a contradiction. So \( 62 \notin \pi_e(G) \). Then the group \( P_{31} \) acts fixed point freely on the set of elements of order 2. Thus \( |P_{31}| \mid m_2 \), a contradiction.

Therefore, \( \pi(G) \subseteq \{2, 3, 5\} \). If \( 3, 5 \in \pi(G) \), then \( m_3 = 20 \) and \( m_5 = 24 \). It is clear that by \((*) \), \( G \) does not contain any elements of order 15, 16, 18 or 25. If \( 4 \in \pi_e(G) \), then \( m_4 = 30 \) and \( m_8 = 24 \). If \( 2^i \in \pi_e(G) \) for some \( i \geq 2 \), then \( 2^{i-1} \mid m_{2^i} \) where \( m_{2i} \in \{20, 24, 30\} \). Hence \( 2 \leq i \leq 3 \). If \( 3^j \in \pi_e(G) \) for some \( j \geq 2 \), then \( 2 \times 3^{j-1} \mid m_{3^j} \) where \( m_{3^j} \in \{20, 24, 30\} \). Thus \( j \geq 2 \). If \( 5^k \in \pi_e(G) \) for some \( k \geq 2 \), then \( 4 \times 5^{k-1} \mid m_{5^k} \) where \( m_{5^k} \in \{20, 24, 25\} \). Thus \( k = 2 \). Since \( 25 \notin \pi_e(G) \), we get a contradiction.

Therefore \( \pi_e(G) \subseteq \{1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 24\} \) and \( |G| = 100 + 20k_1 + 24k_2 + 30k_3 = 2^m \times 3^n \times 5^r \) where \( k_1, k_2, k_3, m, n \) and \( r \) are non-negative integers. Now we consider the following subcases:

**Subcase (a).** Suppose that \( \pi(G) = \{2\} \). Then \( |\pi_e(G)| \leq 4 \). Since \( \text{nse}(G) \) have five elements and \( |\pi_e(G)| \leq 4 \), we get a contradiction.

**Subcase (b).** Suppose that \( \pi(G) = \{2, 5\} \). Then \( |G| = 100 + 20k_1 + 24k_2 + 30k_3 = 2^m \times 5^n \) and by \( |\pi_e(G)| \leq 6 \), we have \( 0 \leq k_1 + k_2 + k_3 \leq 1 \). Hence \( 5 \mid k_2 \), which implies that \( k_2 = 0 \), and so \( 50 + 10k_1 + 15k_3 = 2^{m-1} \times 5^n \). Hence \( 2 \mid k_3 \), which implies that \( k_3 = 0 \). It is easy to check that the only solution of the equation is \((k_1, k_2, k_3, m, n) = (0, 0, 0, 2, 2) \). Thus \( |G| = 2^2 \times 5^2 \). It is clear that \( \pi_e(G) = \{1, 2, 4, 5, 10\} \), and so \( \exp(P_2) = 4 \). Then \( P_2 \) is cyclic. Thus \( m_2 = m_4 / \phi(4) = 15 \). Since every cyclic Sylow 2-subgroup has one element of order 2, \( m_2 \leq 15 \), a contradiction.

**Subcase (c).** Suppose that \( \pi(G) = \{2, 3\} \). Since \( 27 \notin \pi_e(G) \), \( \exp(P_3) = 3 \) or 9. If \( \exp(P_3) = 3 \), then \( \pi_e(G) \subseteq \{1, 2, 3, 4, 6, 8, 12, 24\} \). By Lemma 2.1, \(|P_3| \mid (1 + m_3) = 21 \). Hence \( |P_3| = 3 \). Therefore, \( 100 + 20k_1 + 24k_2 + 30k_3 = 2^m \times 3 = |G| \) and \( 0 \leq k_1 + k_2 + k_3 \leq 10 \), which is a contradiction.
3. It is clear that $100 \leq 2^m \times 3 \leq 190$. Hence $m = 6$ and $20k_1 + 24k_2 + 30k_3 = 92$. It is easy to check that the equation has no solution.

If $\exp(P_3) = 9$, then since $m_9 = 24$, $|P_3| | (1 + m_3 + m_9) = 45$. Hence $|P_3| = 9$ and $n_3 = m_9/\phi(9) = 4$. Since a cyclic group of order 9 have two elements of order 3, $m_3 \leq 4 \times 2 = 8$, a contradiction.

Subcase (d). Suppose that $\pi(G) = \{2, 3, 5\}$. Since $G$ has no element of order 15, the group $P_3$ acts fixed point freely on the set of elements of order 3. Thus $|P_3| | m_3 = 20$, which implies that $r = 1$. Similarly, the group $P_3$ acts fixed point freely on the set of elements of order 5. Thus $|P_3| | m_5 = 24$, which implies that $n = 1$.

We will show $10 \not\in \pi_e(G)$. Suppose that $10 \in \pi_e(G)$. We know that if $P$ and $Q$ are Sylow 5-subgroups of $G$, then $P$ and $Q$ are conjugate, which implies that $C_G(P)$ and $C_G(Q)$ are conjugate. Therefore $m_{110} = \phi(10) \cdot n_5 \cdot k$, where $k$ is the number of cyclic subgroups of order 2 in $C_G(P)$. Since $n_5 = m_5/\phi(5) = 6, 24 | m_{110}$. Hence $m_{110} = 24$.

By Lemma 2.1, $10 | (1 + m_2 + m_5 + m_{110}) = 74$, a contradiction.

Therefore the group $P_2$ acts fixed point freely on the set of elements of order 5. Then $|P_2| | m_5 = 24$, which implies that $|P_2| | 8$. Thus $100 + 20k_1 + 24k_2 + 30k_3 = 2^m \times 3 \times 5$, where $0 \leq k_1 + k_2 + k_3 \leq 4$ and $m \leq 3$. It is clear that $100 \leq 2^m \times 3 \times 5 \leq 190$, hence $m = 3$. It is easy to check that the only solution of the equation is $(k_1, k_2, k_3) = (1, 0, 0)$. Thus $|G| = 2^3 \times 3 \times 5$, $\pi_e(G) = \{1, 2, 3, 4, 5, 6\}$, and by the main result of [4], $G \cong S_5$.

**Case 4.** Let $G$ be a group such that $\mathrm{nse}(G) = \mathrm{nse}(S_6) = \{1, 75, 80, 180, 144, 240\}$. First we prove that $\pi(G) \subseteq \{2, 3, 5\}$. Since $75 \in \mathrm{nse}(G)$, it follows that $2 \in \pi(G)$ and $m_2 = 75$. Let $2 \not\in \pi(G)$. By $(\ast)$, $p \in \{3, 5, 181, 241\}$. If $181 \in \pi(G)$, then by $(\ast)$, $m_{181} = 180$. If $282 \in \pi_3(G)$, then we conclude that $m_{282} = 180$, but by $(\ast)$, we get a contradiction. Therefore $282 \not\in \pi_e(G)$.

Since $282 \not\in \pi_e(G)$, the group $P_{281}$ acts fixed point freely on the set of elements of order 2. Then $|P_{281}| | m_2$, a contradiction. Similarly, if $241 \in \pi(G)$, we get a contradiction. Hence $\pi(G) \subseteq \{2, 3, 5\}$.

If $3, 5 \in \pi_e(G)$, then $m_3 = 80$ and $m_5 = 144$, by $(\ast)$. If $2^i \in \pi_e(G)$ for some $i \geq 2$, then $2^{i-1} = \phi(2^i) | m_{2^i}$. Thus $2 \leq i \leq 5$. If $3^j \in \pi_e(G)$ for some $j \geq 2$, then $2 \times 3^{j-1} = \phi(3^j) | m_{3^j}$. Then $2 \leq j \leq 3$. If $5^k \in \pi_e(G)$ for some $k \geq 2$, then $4 \times 5^{k-1} | m_{5^k}$ and so $k = 2$. If $2^i \times 3^j \in \pi_e(G)$ for some $a, b > 0$, then $1 \leq a \leq 4$ and $1 \leq b \leq 3$. If $2^i \times 5^k \in \pi_e(G)$ for some $a, b > 0$, then $1 \leq a \leq 5$ and $1 \leq b \leq 2$. If $3^j \times 5^k \in \pi_e(G)$ for some $a, b > 0$, then $1 \leq a \leq 2$ and $1 \leq b \leq 2$. If $2^i \times 3^j \times 5^k \in \pi_e(G)$ for some $a, b, c > 0$, then $1 \leq a \leq 3$, $1 \leq b \leq 2$ and $1 \leq c \leq 2$.

Therefore $\pi_e(G) \subseteq \{1, 2, 2^2, 2^3, 2^4, 2^5, 3, 3^2, 3^3, 3^4, 3^5, 5, 5^2\} \cup \{2^a \times 3^b \mid 1 \leq a \leq 4, 1 \leq b \leq 3\} \cup \{2^a \times 5^b \mid 1 \leq a \leq 3, 1 \leq b \leq 2\} \cup \{3^a \times 5^b \mid 1 \leq a \leq 2, 1 \leq b \leq 2\} \cup \{2^a \times 3^b \times 5^c \mid 1 \leq a \leq 3, 1 \leq b \leq 2, 1 \leq c \leq 2\}$.

Hence $|G| = 2^m \times 3^n \times 5^r = 720 + 80k_1 + 144k_2 + 180k_3 + 240k_4$ where $k_1, k_2, k_3, n, r$ are non-negative integers. Now we consider the following subcases:

Subcase (a). Suppose that $\pi_e(G) = \{2\}$. Then $360 + 40k_1 + 72k_2 + 90k_3 + 120k_4 = 2^{m-1}$. Since $|\pi_e(G)| \leq 6$, $k_1 + k_2 + k_3 + k_4 = 0$. It is easy to see that this equation has no solution.

Subcase (b). Suppose that $\pi_e(G) = \{2, 5\}$. Since $5^3 \not\in \pi_e(G), \exp(P_5) = 5$ or 25. Let $\exp(P_5) = 5$, then by Lemma 2.1, $|P_5| | (1 + m_3) = 45$. Hence $|P_5| = 5$. On the other hand, $|\pi_e(G)| \leq 10$. Therefore $720 + 80k_1 + 144k_2 + 180k_3 + 240k_4 = 2^m \times 5$, where $0 \leq k_1 + k_2 + k_3 + k_4 \leq 4$. Hence $5 \mid k_2$, then $k_2 = 0$. It is easy to see that this equation has no solution.
If \(\exp(P_3) = 25\), then by Lemma 2.1, \(|P_3| \mid (1 + m_5 + m_{25})\). Hence \(|P_3| = 25\) and \(P_3\) is cyclic. Thus \(n_5 = m_{25}/\phi(25)\). Since \(m_{25} \in \{80, 180\}\), \(n_5 = 4\) or 9, a contradiction.

Subcase (c). Suppose that \(\pi(G) = \{2, 3\}\). Since \(3^4 \notin \pi_e(G)\), \(\exp(P_3) = 3\), 9 or 27. Let \(\exp(P_3) = 3\). Then \(|P_3| \mid (1 + m_3) = 81\), by Lemma 2.1. If \(|P_3| = 3\), then \(n_3 = m_3/\phi(3) = 40\) \(\mid |G|\), we get a contradiction by \(5 \notin \pi(G)\).

If \(|P_3| = 9\), then \(|G| = 720 + 80k_1 + 144k_2 + 180k_3 + 240k_4 = 2^n \times 9\). Since \(\pi_e(G) \subseteq \{1, 2, 2^2, 2^3, 2^4, 2^5, 3, 3 \times 2, 3 \times 2^2, 3 \times 2^3, 3 \times 2^4\}\), \(0 \leq k_1 + k_2 + k_3 + k_4 \leq 5\). As \(720 \leq 2^n \times 9 \leq 1920\), \(m = 7\). Therefore, \(432 = 80k_1 + 144k_2 + 180k_3 + 240k_4\). The only solution of this equation is \((k_1, k_2, k_3, k_4) = (0, 3, 0, 0)\). Then \(|\pi_e(G)| = 9\), it is clear that \(\exp(P_3) = 16\) or 32.

If \(\exp(P_3) = 16\), then \(\pi_e(G) = \{1, 2, 3, 4, 6, 8, 12, 16, 24\}\). Since \(\exp(P_3) = 16\), \(\pi_e(G)\), the group \(P_3\) acts fixed point freely on the set of elements of order 16. Hence \(|P_3| \mid m_{16}\). We have \(m_{16} \in \{144, 240\}\). If \(m_{16} = 240\), we get a contradiction by \(|P_3| \mid m_{16}\). If \(m_{16} = 144\), then \((*)\), \(m_{24} = 240\). If \(m_8 = 144\), then \(m_4 = 180\) and if \(m_8 = 180\), then \(m_4 = 144\).

By Lemma 2.1, \(|P_2| \mid (1 + m_2 + m_8 + m_{16}) = 544\). Because \(|P_2| = 2^7\), we get a contradiction.

If \(\exp(P_2) = 32\), then \(\pi_e(G) = \{1, 2, 3, 4, 6, 8, 12, 16, 32\}\). Since \(48 \notin \pi_e(G)\), then the group \(P_3\) acts fixed point freely on the set of elements of order 8, 16 or 32. Hence \(|P_3| \mid m_8, m_{16} or m_{32}\). We know that \(m_8 \in \{144, 180, 240\}\), if \(m_8 = 144\), then \((*)\), \(m_4 = 180\). Therefore \(m_{16} or m_{32} \neq 144\). Since \(|P_3| \mid m_{16} or m_{32}\), we get a contradiction. If \(m_8 = 180\), then \(m_4 = 144\). Thus \(m_{16} or m_{32} \neq 144\). Since \(|P_3| \mid m_{16} or m_{32}\), a contradiction. Similarly, if \(m_8 = 240\), we get a contradiction by \(|P_3| \mid m_8\).

Suppose that \(\exp(P_3) = 27\). Then \(|G| = 720 + 80k_1 + 144k_2 + 180k_3 + 240k_4 = 2^n \times 27\) where \(0 \leq k_1 + k_2 + k_3 + k_4 \leq 5\). Hence 720 \(\leq 2^n \times 27 \leq 1920\). Thus \(m = 5\) or 6.

If \(m = 5\), then 144 = 80k_1 + 144k_2 + 180k_3 + 240k_4. The only solution of this equation is \((k_1, k_2, k_3, k_4, k_4) = (0, 1, 0, 0)\). Thus \(|\pi_e(G)| = 7\), it is clear that \(\exp(P_3) = 8\), 16 or 32.

If \(\exp(P_2) = 8\), then \(\pi_e(G) = \{1, 2, 3, 4, 6, 8, 12\}\). Since \(24 \notin \pi_e(G)\), the group \(P_3\) acts fixed point freely on the set of elements of order 8. Hence \(|P_3| \mid m_8\), by \(m_8 \in \{144, 180, 240\}\), we get a contradiction.

If \(\exp(P_2) = 16\), then \(\pi_e(G) = \{1, 2, 3, 4, 6, 8, 16\}\). Since \(12 \notin \pi_e(G)\), the group \(P_3\) acts fixed point freely on the set of elements of order 4. Hence \(|P_3| \mid m_4\). By \(m_4 \in \{144, 80, 240\}\), we get a contradiction.

If \(\exp(P_2) = 32\), then \(\pi_e(G) = \{1, 2, 3, 4, 6, 8, 16, 32\}\). Since \(6 \notin \pi_e(G)\), the group \(P_3\) acts fixed point freely on the set of elements of order 2. Hence \(|P_3| \mid m_2\). This is a contradiction because \(|P_3| = 9\).

If \(m = 6\), then by arguing as above we can rule out this case. Also by arguing as above we can rule out the case \(|P_3| = 81\).

Suppose that \(\exp(P_3) = 9\). By \((*)\), \(m_9 \in \{144, 180\}\). Then \(|P_3| \mid (1 + m_3 + m_9) = 225\) or 261. Hence \(|P_3| = 9\) and \(n_9 = m_9/\phi(9) = (24, 30)\), a contradiction.

If \(\exp(P_3) = 27\), then \((*)\), \(m_{27} \in \{144, 180\}\). If \(P_3\) be a cyclic group, then since \(\exp(P_3) = 27\), \(n_3 = m_{27}/\phi(27) \in \{8, 10\}\). If \(n_3 = 8\), then we get a contradiction by Sylow theorem and if \(n_3 = 10\), then since a cyclic group of order 27 have two elements of order 3, \(m_3 \leq 10 \times 2 = 20\), a contradiction. Therefore \(P_3\) is not cyclic. By Lemma 2.3, \(27 \mid m_{27}\), a contradiction.

Subcase (d). Suppose that \(\pi(G) = \{2, 3, 5\}\). We know that \(\exp(P_3) = 5\) and \(|P_3| = 5\).

Suppose that 15 \(\in \pi_e(G)\), then \(m_{15} = \phi(15) \cdot n_5 \cdot k\), where \(k\) is the number of cyclic subgroups of order 3 in \(C_G(P_3)\). Since \(n_5 = m_5/\phi(5) = 36, 288 \mid m_{15}\), a contradiction. Thus 15 \(\notin \pi_e(G)\). Similarly, 10 \(\notin \pi_e(G)\).
Suppose that $5 \notin \pi_e(G)$, the group $P_3$ acts fixed point freely on the set of elements of order 5. Hence $|P_3| \mid m_5 = 144$. Then $|P_3| = 3$ or 9. Since $10 \notin \pi_e(G)$, the group $P_3$ acts fixed point freely on the set of elements of order 5. Hence $|P_3| \mid m_5 = 144$. Then $|P_3| = 2^m$, where $1 \leq m \leq 4$. Therefore $|G| = 720 + 80k_1 + 144k_2 + 180k_3 + 240k_4 = 2^m \times 3^n \times 5$ where $1 \leq m \leq 4$ and $1 \leq n \leq 2$. The only solution of this equation is $(k_1, k_2, k_3, k_4, m, n) = (0, 0, 0, 4, 2)$. Thus $\pi_e(G) = \{1, 2, 3, 4, 5, 6\}, \{1, 2, 3, 4, 5, 8\}, \{1, 2, 3, 4, 5, 9\}$ or $\{1, 2, 3, 5, 6, 9\}$.

If $\pi_e(G) = \{1, 2, 3, 4, 5, 9\}$ or $\{1, 2, 3, 5, 6, 9\}$, then $\exp(P_3) = 9$ and $|P_3| = 9$. Because $m_9 \in \{144, 180\}$, $n_9 = m_9/\phi(9) \in \{24, 30\}$, we get a contradiction.

If $\pi_e(G) = \{1, 2, 3, 4, 5, 8\}$, then $6 \notin \pi_e(G)$. Thus the group $P_3$ acts fixed point freely on the set of elements of order 2. So, $|P_3| \mid m_2 = 75$. Since $|P_3| = 9$, we get a contradiction.

Therefore $\pi_e(G) = \{1, 2, 3, 4, 5, 6\}$. Now by the main result of [4], $G \cong S_6$.

**Case 5.** Let $G$ be a group such that $\text{nse}(G) = \text{nse}(S_7) = \{1, 213, 350, 420, 504, 720, 840, 1470\}$. First we prove that $\pi(G) \subseteq \{2, 3, 5, 7\}$. Since $231 \notin \text{nse}(G)$, it follows that $2 \notin \pi(G)$ and $m_2 = 231$. Let $2 \neq p \in \pi(G)$. By $(*)$, $p \in \{3, 5, 29, 421, 1471\}$. If $29 \notin \pi(G)$, then $m_{29} = 840$. If $58 \notin \pi_e(G)$, then $m_{58} \in \{504, 420, 840\}$, but by $(*)$, we get a contradiction. Thus $58 \notin \pi_e(G)$.

Since $58 \notin \pi_e(G)$, the group $P_{29}$ acts fixed point freely on the set of elements of order 2. Then $|P_{29}| \mid m_2$ and this is a contradiction. Similarly, if 421 and 1471 $\in \pi(G)$, we get a contradiction. Hence $\pi(G) \subseteq \{2, 3, 5, 7\}$. If $3, 5, 7 \in \pi(G)$, then $m_3 = 350, m_5 = 504$ and $m_7 = 720$. It is clear that $G$ does not contain any elements of order 64, 81, 125 and 343.

Let $25 \in \pi_e(G)$. Then $m_{25} = 420$ or 720 by $(*)$. By Lemma 2.1, $|P_5| \mid (1 + m_5 + m_{25}) = 920$ or 1225. Hence $|P_5| = 25$ and $n_5 = m_{25}/\phi(25) = 21$ or 36. Since in a cyclic group of order 25, there are four elements of order 5, so $n_5 \leq 21 \times 4 = 84$ or $n_5 \leq 36 \times 4 = 144$, a contradiction. Therefore $25 \notin \pi_e(G)$.

Let $49 \in \pi_e(G)$. Then $m_{49} = 504$. By Lemma 2.1, $|P_7| \mid (1 + m_7 + m_{49}) = 1225$. Then $|P_7| = 49$, and so $n_7 = m_{49}/\phi(49) = 12$. By Sylow’s theorem $n_7 = 1 + 7k$ for some $k$, as $n_7 = 12$, we get a contradiction. So $49 \notin \pi_e(G)$.

Therefore if $5, 7 \in \pi(G)$, then $\exp(P_5) = 5$ and $\exp(P_7) = 7$, and by Lemma 2.1, $|P_5| = 5$ and $|P_7| = 7$. Hence $n_5 = m_5/\phi(5) = 2 \times 9 \times 7$ and $n_7 = m_7/\phi(7) = 8 \times 3 \times 5$. We conclude that if $5 \notin \pi(G)$, then $3, 7 \in \pi(G)$, and if $7 \notin \pi_e(G)$, then $3, 5 \notin \pi_e(G)$. In follows, we show that $\pi(G)$ could not be the sets $\{2\}, \{2, 3\}$, and so $\pi(G)$ must be equal to $\{2, 3, 5, 7\}$. Now we consider the following subcases:

**Subcase (a).** Suppose that $\pi(G) = \{2\}$. Since $64 \notin \pi_e(G)$, $\pi_e(G) \subseteq \{1, 2, 4, 8, 16, 32\}$. Therefore $|G| = 2^m \times 4536 + 350k_1 + 504k_2 + 420k_3 + 720k_4 + 840k_5 + 1470k_6$ where $k_1 + k_2 + k_3 + k_4 + k_5 + k_6 = 0$. It is easy to see that this equation has no solution.

**Subcase (b).** Suppose that $\pi(G) = \{2, 3\}$. Since $81 \notin \pi_e(G)$, $\exp(P_3) = 3$ or 9. By the calculation of the exponent of $P_3$, we get $\exp(P_3) = 3$. Then $|P_3| \mid (1 + m_3) = 351$, by Lemma 2.1. Hence $|P_3| \mid 27$. If $|P_3| = 3$, then $n_3 = m_3/\phi(3) = 175 \mid |G|$, because $5 \notin \pi(G)$, we get a contradiction.

If $|P_3| = 9$, then $\exp(P_3) = 3$ and 64, 96 $\notin \pi_e(G)$, $\pi_e(G) \subseteq \{1, 2, 3, 4, 5, 6\}$. Therefore $|G| = 2^m \times 9 = 4536 + 350k_1 + 504k_2 + 420k_3 + 720k_4 + 840k_5 + 1470k_6$ where $k_1 + k_2 + k_3 + k_4 + k_5 + k_6 = 0$. Then $4536 \leq 2^m \times 9 \leq 4536 + 1470 \times 3$, so $m = 10$. Then $4680 = 4536 + 350k_1 + 504k_2 + 420k_3 + 720k_4 + 840k_5 + 1470k_6$ where $0 \leq k_1 + k_2 + k_3 + k_4 + k_5 + k_6 \leq 3$. By an easy computer calculation, it is easy to see this equation has no solution.

Similarly, we can rule out the case $|P_3| = 27$. 

Let \( \exp(P_3) = 9 \). By (\( * \)), \( m_9 \in \{ 504, 720 \} \). Hence by Lemma 2.1, \( |P_3| = 9 \). Therefore \( n_3 = m_9/\phi(9) \in \{ 84, 120 \} \). Because 5, 7 \( \notin \pi(G) \), we get a contradiction.

If \( \exp(P_3) = 27 \), then \( m_{27} \in \{ 504, 720 \} \). If \( P_3 \) be a cyclic group, then since \( \exp(P_3) = 27 \), \( n_3 = m_{27}/\phi(27) \in \{ 28, 40 \} \). Because 5, 7 \( \notin \pi(G) \), we get a contradiction. Thus \( P_3 \) is not cyclic. By Lemma 2.3, \( 27 \mid m_{27} \), a contradiction.

Therefore \( \pi(G) = \{ 2, 3, 5, 7 \} \). We prove that \( 21 \notin \pi_n(G) \). Suppose that \( 21 \in \pi_n(G) \). Then \( m_{21} = \phi(21) \cdot n \), where \( n \) is the number of cyclic subgroups of order 3 in \( C_G(P_7) \). Since \( n_7 = m_7/\phi(7) = 120, 720 \mid m_{21}, \) a contradiction. Thus \( 21 \notin \pi_n(G) \). Similarly, \( 14 \notin \pi_n(G) \).

Since \( 21 \notin \pi_n(G) \), the group \( P_3 \) acts fixed point freely on the set of elements of order 7. Hence \( |P_3| \mid m_7 = 720 \). Then \( |P_3| = 3 \) or 9. Also since \( 14 \notin \pi_n(G) \), the group \( P_2 \) acts fixed point freely on the set of elements of order 7. Hence \( |P_2| \mid m_7 = 720 \). Then \( |P_2| = 16 \). On the other hand, \( 4536 \leq |G| \), thus \( |G| = 2^4 \times 3^2 \times 5 \times 7 = |S_7| \).

Now we claim that \( G \) is non-solvable group. Suppose that \( G \) is solvable. Since \( n_7 = 120 \) by Lemma 2.4, \( 3 \equiv 1 \pmod{7} \), a contradiction. Hence \( G \) is non-solvable group and \( p \parallel |G| \), where \( p \in \{ 5, 7 \} \). Therefore \( G \) has a normal series

\[
1 \leq N \leq H \leq G
\]

such that \( N \) is a maximal solvable normal subgroup of \( G \) and \( H/N \) is an non-solvable minimal normal subgroup of \( G/N \). Then \( H/N \) is a non-Abelian simple \( K_3 \)-group or simple \( K_3 \)-group. If \( H/N \) be simple \( K_3 \)-group, then by Lemma 2.6, \( H/N \) is isomorphic to one of the groups: \( A_5, A_6, L_2(7) \) or \( L_2(8) \).

Suppose that \( H/N \cong A_5 \). If \( P_5 \in \text{Syl}_5(G) \), then \( P_5 N/N \in \text{Syl}_5(H/N) \), \( n_5(H/N) t = n_5(G) \) for some positive integer \( t \) and 5 \( \nmid t \), by Lemma 2.5. Since \( n_5(A_5) = 6, n_5(G) = 6t \). Therefore \( n_5 = n_5(G) \times 4 \times 21 = 504 \) and so \( t = 21 \). By Lemma 2.5, \( 21 \times |N_5| = |N| \). Since \( |N| = 2^3 \times 3 \times 7 \), then \( n_7(N) = 108 \). So \( m_7 = 6 \) or 48, a contradiction.

Suppose that \( H/N \cong A_6 \). If \( P_6 \in \text{Syl}_5(G) \), then \( P_6 N/N \in \text{Syl}_5(H/N) \), \( n_6(H/N) t = n_6(G) \) for some positive integer \( t \) and 5 \( \nmid t \), by Lemma 2.5. Since \( n_6(A_6) = 36, n_6(G) = 36t \) and \( m_6 = n_6(G) \times 4 = 144t = 504 \), a contradiction.

Suppose that \( H/N \cong L_2(7) \). If \( P_7 \in \text{Syl}_7(G) \), then \( P_7 N/N \in \text{Syl}_7(H/N) \), \( n_7(H/N) t = n_7(G) \) for some positive integer \( t \) and 7 \( \nmid t \), by Lemma 2.5. Since \( n_7(L_2(7)) = 8, n_7(G) = 8t \) and \( m_7 = n_7(G) \times 6 = 48t = 720 \). Hence \( t = 15 \). By Lemma 2.5, \( 15 \times |N_7| = |N| \). Since \( |N| = 2 \times 3^2 \times 5, n_5(N) = 1 \) or 6. So \( m_5 = 4 \) or 24, a contradiction.

Similarly, if \( H/N \cong L_4(8) \), we get a contradiction. Hence \( H/N \) is simple \( K_4 \)-group. Then by Lemma 2.7, \( H/N \) is isomorphic to \( A_7 \). Now set \( \mathcal{T} := H/N \cong A_7 \) and \( \mathcal{G} := G/N \). We have

\[
A_7 \cong \mathcal{T} \cong \mathcal{T}C_{\mathcal{T}}(\mathcal{T})/C_{\mathcal{T}}(\mathcal{T}) \leq \mathcal{G}/C_{\mathcal{G}}(\mathcal{G}) = N_{\mathcal{G}}(\mathcal{T})/C_{\mathcal{G}}(\mathcal{T}) \leq \text{Aut}(\mathcal{T}).
\]

Let \( K = \{ x \in G \mid xN \in C_{\mathcal{G}}(\mathcal{T}) \} \), then \( G/K \cong \mathcal{G}/C_{\mathcal{G}}(\mathcal{T}) \). Hence \( A_7 \leq G/K \leq \text{Aut}(A_7) \).

Then \( A_7 \cong A_7 \) or \( G/K \cong S_7 \). If \( G/K \cong A_7 \), then \( |K| = 2 \). We have \( N \leq K \) and \( N \) is a maximal solvable normal subgroup of \( G \), then \( N = K \). Now we know that \( G/N \cong A_7 \) where \( |N| = 2 \), so \( G \) has a normal subgroup \( G \) of order 2, generated by a central involution \( z \). Let \( x \) be an element of order 7 in \( G \). Since \( xz = zx \) and \( (o(x), o(z)) = 1, o(xz) = 14 \).

Hence \( 14 \notin \pi_s(G) \). We know \( 14 \notin \pi_s(G) \), a contradiction.

If \( G/K \cong S_7 \), then \( |K| = 1 \) and \( G \cong S_7 \). Now the proof of the main theorem is complete.
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