

INEQUALITIES FOR ONE SIDED APPROXIMATION IN ORLICZ SPACES

Ramazan Akgün*

Received 16:06:2010 : Accepted 10:01:2011

Abstract

In the present article some inequalities of trigonometric approximation are proved in Orlicz spaces generated by a quasiconvex Young function. Also, the main one-sided approximation problems are investigated.

Keywords: Fractional moduli of smoothness, Direct theorem, Converse theorem, Fractional derivative, One-sided approximation.

2010 AMS Classification: 26 A 33, 41 A 17, 41 A 20, 41 A 25, 41 A 27, 42 A 10.

Communicated by Cihan Orhan

1. Introduction

A function Φ is called a *Young function* if Φ is even, continuous, nonnegative in $\mathbb{R} := (-\infty, +\infty)$, increasing on $\mathbb{R}^+ := (0, \infty)$ and such that

$$\Phi(0) = 0, \lim_{x \rightarrow \infty} \Phi(x) = \infty.$$

A function $\varphi : [0, \infty) \rightarrow [0, \infty)$ is said to be *quasiconvex* if there exist a convex Young function Φ and a constant $c_1 \geq 1$ such that

$$\Phi(x) \leq \varphi(x) \leq \Phi(c_1 x) \quad \forall x \geq 0.$$

Set $\mathbb{T} := [0, 2\pi]$ and let φ be a quasiconvex Young function. We denote by $\varphi(L)$ the class of complex valued Lebesgue measurable functions $f : \mathbb{T} \rightarrow \mathbb{C}$ satisfying the condition

$$\int_{\mathbb{T}} \varphi(|f(x)|) dx < \infty.$$

The class of functions $f : \mathbb{T} \rightarrow \mathbb{C}$ having the property

$$\int_{\mathbb{T}} \varphi(c_2 |f(x)|) dx < \infty$$

*Balıkesir University, Faculty of Arts and Sciences, Department of Mathematics, Balıkesir, Turkey. E-mail: rakgun@balikesir.edu.tr

for some $c_2 \in \mathbb{R}^+$ is denoted by $L_\varphi(\mathbb{T})$. The set $L_\varphi(\mathbb{T})$ becomes a normed space with the Orlicz norm

$$\|f\|_\varphi := \sup \left\{ \int_{\mathbb{T}} |f(x)g(x)| dx : \int_{\mathbb{T}} \tilde{\varphi}(|g|) dx \leq 1 \right\},$$

where $\tilde{\varphi}(y) := \sup_{x \geq 0} (xy - \varphi(x))$, $y \geq 0$, is the *complementary function* of φ .

For a quasiconvex function φ we define the index $p(\varphi)$ of φ as

$$\frac{1}{p(\varphi)} := \inf \{p : p > 0, \varphi^p \text{ is quasiconvex}\}$$

and the conjugate index of φ as

$$p'(\varphi) := \frac{p(\varphi)}{p(\varphi) - 1}.$$

It can be easily seen that the functions in $L_\varphi(\mathbb{T})$ are summable on \mathbb{T} , $L_\varphi(\mathbb{T}) \subset L^1(\mathbb{T})$ and $L_\varphi(\mathbb{T})$ becomes a Banach space with the Orlicz norm. The Banach space $L_\varphi(\mathbb{T})$ is called the *Orlicz space*.

A Young function Φ is said to be satisfy the Δ_2 condition if there is a constant $c_3 > 0$ such that

$$\Phi(2x) \leq c_3 \Phi(x)$$

for all $x \in \mathbb{R}$.

We will denote by $QC_2^\theta(0, 1)$ the class of functions g satisfying the condition Δ_2 such that g^θ is quasiconvex for some $\theta \in (0, 1)$.

In the present work we consider the trigonometric polynomial approximation problems for functions and their fractional derivatives in the spaces $L_\varphi(\mathbb{T})$, where $\varphi \in QC_2^\theta(0, 1)$. We prove a Jackson type direct theorem, and a converse theorem of trigonometric approximation with respect to the fractional order moduli of smoothness in Orlicz spaces. As a particular case, we obtain a constructive description of the Lipschitz class in Orlicz spaces. A direct theorem of one sided trigonometric approximation is also obtained.

Let

$$(1.1) \quad f(x) \sim \sum_{k=-\infty}^{\infty} c_k e^{ikx} \text{ and } \tilde{f}(x) \sim \sum_{k=-\infty}^{\infty} (-i \operatorname{sign} k) c_k e^{ikx}$$

be the *Fourier* and the *conjugate Fourier series* of $f \in L^1(\mathbb{T})$, respectively. We define

$$S_n(f) := S_n(x, f) := \sum_{k=-n}^n c_k e^{ikx}, \quad n = 0, 1, 2, \dots$$

For a given $f \in L^1(\mathbb{T})$, assuming $c_0 = 0$ in (1.1), we define the α^{th} fractional ($\alpha \in \mathbb{R}^+$) *integral* of f as in [7, v.2, p.134] by

$$I_\alpha(x, f) := \sum_{k \in \mathbb{Z}^*} c_k (ik)^{-\alpha} e^{ikx},$$

where \mathbb{Z} is the set of integers, $\mathbb{Z}^* := \{z \in \mathbb{Z} : z \neq 0\}$, and

$$(ik)^{-\alpha} := |k|^{-\alpha} e^{(-1/2)\pi i \alpha \operatorname{sign} k}$$

as principal value.

Let $\alpha \in \mathbb{R}^+$ be given. We define the *fractional derivative* of a function $f \in L^1(\mathbb{T})$, satisfying $c_0 = 0$ in (1.1), as

$$f^{(\alpha)}(x) := \frac{d^{[\alpha]+1}}{dx^{[\alpha]+1}} I_{1+\alpha-[\alpha]}(x, f),$$

provided the righthand side exists, where $[x]$ denotes the integer part of the real number x .

Setting $h \in \mathbb{T}$, $r \in \mathbb{R}^+$, $\varphi \in QC_2^\theta(0, 1)$ and $f \in L_\varphi(\mathbb{T})$, we define

$$\Delta_h^r f(\cdot) := (T_h - I)^r f(\cdot) = \sum_{k=0}^\infty (-1)^k \binom{r}{k} f(\cdot + (r - k)h),$$

where $\binom{r}{k} := \frac{r(r-1)\dots(r-k+1)}{k!}$ for $k > 1$, $\binom{r}{1} := r$ and $\binom{r}{0} := 1$ are the binomial coefficients, $T_h f(x) := f(x + h)$ is the translation operator and I the identity operator.

Since $\sum_{k=0}^\infty \left| \binom{r}{k} \right| < \infty$ we get

$$(1.2) \quad \|\Delta_h^r f\|_\varphi \leq c \|f\|_\varphi < \infty$$

under the condition $f \in L_\varphi(\mathbb{T})$, where $\varphi \in QC_2^\theta(0, 1)$.

Here and in the following we will denote by B a translation invariant Banach Function Space. Also, the notation $\|\cdot\|_B$ stands for the norm of B .

For $r \in \mathbb{R}^+$, we define the *fractional modulus of smoothness of order r for $f \in B$* , as

$$\omega_B^r(f, \delta) := \sup_{|h| \leq \delta} \|\Delta_h^r f\|_B, \quad \delta \geq 0.$$

If $\varphi \in QC_2^\theta(0, 1)$ and $B = L_\varphi(\mathbb{T})$, we will set $\omega_B^r(f, \cdot) =: \omega_\varphi^r(f, \cdot)$. Hence for $\varphi \in QC_2^\theta(0, 1)$ and $f \in L_\varphi(\mathbb{T})$, we have by (1.2) that

$$\omega_\varphi^r(f, \delta) \leq c \|f\|_\varphi,$$

where the constant $c > 0$ dependent only on r and φ .

Let \mathcal{T}_n be the class of trigonometric polynomials of degree not greater than n . We begin with the fractional Nikolski-Civin inequality:

1.1. Theorem. *Suppose that $\alpha \in \mathbb{R}^+$, $T_n \in \mathcal{T}_n$ and $0 < h < 2\pi/n$. Then*

$$\|T_n^{(\alpha)}\|_B \leq \left(\frac{n}{2 \sin(nh/2)} \right)^\alpha \|\Delta_h^\alpha T_n\|_B.$$

In particular, if $h = \pi/n$, then

$$(1.3) \quad \|T_n^{(\alpha)}\|_B \leq 2^{-\alpha} n^\alpha \|\Delta_{\pi/n}^\alpha T_n\|_B.$$

Proof. Let $T_n(x) = \frac{a_n}{2} + \sum_{\nu \in \mathbb{Z}_n^*} c_\nu e^{i\nu x}$, where $\mathbb{Z}_n^* := \{z \in \mathbb{Z} : z < n, z > -n, z \neq 0\}$. Then

$$T_n^{(\alpha)}(x) = \sum_{\nu \in \mathbb{Z}_n^*} (i\nu)^\alpha c_\nu e^{i\nu x}, \text{ and}$$

$$\Delta_h^\alpha T_n\left(x + \frac{\alpha}{2}h\right) = \sum_{\nu \in \mathbb{Z}_n^*} \left(2i \sin \frac{h}{2}\nu\right)^\alpha c_\nu e^{i\nu x}.$$

We set

$$\varphi(t) := \left(2i \sin \frac{h}{2}t\right)^\alpha, \quad g(t) := \left(\frac{t}{2 \sin \frac{h}{2}t}\right)^\alpha \text{ for } -n \leq t \leq n \text{ and } g(0) := h^{-\alpha}.$$

Then for $x \in \mathbb{R}$, $h \in (0, 2\pi/n)$, we obtain

$$\Delta_h^\alpha T_n\left(x + \frac{\alpha}{2}h\right) = \sum_{\nu \in \mathbb{Z}_n^*} \varphi(\nu) c_\nu e^{i\nu x}$$

and

$$T_n^{(\alpha)}(x) = \sum_{\nu \in \mathbb{Z}_n^*} \varphi(\nu) g(\nu) c_\nu e^{i\nu x}.$$

The convergence

$$g(t) = \sum_{k=-\infty}^{\infty} d_k e^{ik\pi t/n}$$

is uniform for $t \in [-n, n]$. Since $(-1)^k d_k \geq 0$, we find

$$\begin{aligned} T_n^{(\alpha)}(x) &= \sum_{\nu \in \mathbb{Z}_n^*} \varphi(\nu) \sum_{k=-\infty}^{\infty} d_k e^{\frac{ik\pi\nu}{n}} c_\nu e^{i\nu x} \\ &= \sum_{k=-\infty}^{\infty} d_k \sum_{\nu \in \mathbb{Z}_n^*} \varphi(\nu) c_\nu e^{i\nu(x + \frac{k\pi}{n})} \\ &= \sum_{k=-\infty}^{\infty} d_k \Delta_h^\alpha T_n \left(x + \frac{k\pi}{n} + \frac{\alpha}{2} h \right). \end{aligned}$$

Hence we conclude

$$\begin{aligned} \|T_n^{(\alpha)}\|_B &\leq \|\Delta_h^\alpha T_n\|_B \sum_{k=-\infty}^{\infty} |d_k e^{ik\pi}| \\ &= \|\Delta_h^\alpha T_n\|_B \sum_{k=-\infty}^{\infty} d_k e^{ik\pi} \\ &= \left(\frac{n}{2 \sin(nh/2)} \right)^\alpha \|\Delta_h^\alpha T_n\|_B, \end{aligned}$$

and Theorem 1.1 is proved. \square

We denote by B^α , $\alpha > 0$, the linear space of 2π -periodic complex valued functions $f \in B$ such that $f^{(\alpha-1)}$ is absolutely continuous (AC), and $f^{(\alpha)} \in B$. If $\varphi \in QC_2^\theta(0, 1)$ and $B = L_\varphi(\mathbb{T})$ we will let $B^\alpha =: W_\varphi^\alpha(\mathbb{T})$.

We set $L_0^\infty := \{f \in L^\infty : f \text{ is real valued and bounded on } \mathbb{T}\}$. If $f \in L_0^\infty$ we define

$$\begin{aligned} \mathcal{T}_n^-(f) &:= \{t \in \mathcal{T}_n : t \text{ is real valued } 2\pi \text{ periodic and } t(x) \leq f(x) \text{ for every } x \in \mathbb{R}\}, \\ \mathcal{T}_n^+(f) &:= \{T \in \mathcal{T}_n : T \text{ is real valued } 2\pi \text{ periodic and } f(x) \leq T(x) \text{ for every } x \in \mathbb{R}\}, \\ E_n^-(f)_\varphi &:= \inf_{t \in \mathcal{T}_n^-(f)} \|f - t\|_\varphi, \quad E_n^+(f)_\varphi := \inf_{T \in \mathcal{T}_n^+(f)} \|T - f\|_\varphi. \end{aligned}$$

The quantities $E_n^-(f)_\varphi$ and $E_n^+(f)_\varphi$ are, respectively, called the *best lower (upper) one sided approximation errors* for $f \in L_0^\infty$. Similarly, the best *trigonometric approximation error* of $f \in L_\varphi(\mathbb{T})$ is defined as $E_n(f)_\varphi := \inf_{S \in \mathcal{T}_n} \|f - S\|_\varphi$. We note that $E_n(f)_\varphi \leq E_n^\pm(f)_\varphi$.

If $\varphi \in QC_2^\theta(0, 1)$, $f \in L_\varphi(\mathbb{T})$, $g \in L^1(\mathbb{T})$, we introduce the convolution

$$(f * g)(x) = \frac{1}{2\pi} \int_{\mathbb{T}} f(x-u) g(u) du.$$

This convolution exists for every $x \in \mathbb{R}$ and is a measurable function. Furthermore

$$\|f * g\|_\varphi \leq \|f\|_\varphi \|g\|_{L^1(\mathbb{T})}.$$

If f is continuous (AC) then $f * g$ is continuous (AC).

1.2. Theorem. Let $\varphi \in QC_2^\theta(0, 1)$, $1 \leq \beta < \infty$ and $f \in W_\varphi^\beta(\mathbb{T})$. If $0 \leq \alpha \leq \beta$ and $n = 1, 2, 3, \dots$, then there exists a constant $c > 0$ depending only on α and β such that

$$(1.4) \quad E_n(f^{(\alpha)})_\varphi \leq \frac{c}{n^{\beta-\alpha}} E_n(f^{(\beta)})_\varphi$$

holds. If f is real valued, $0 \leq \alpha \leq \beta - 1$ and $n = 1, 2, 3, \dots$, then

$$(1.5) \quad E_n^\pm(f^{(\alpha)})_\varphi \leq \frac{c}{n^{\beta-\alpha}} E_n(f^{(\beta)})_\varphi$$

holds.

Proof. 1° First we prove that $f^{(\alpha)}$ is AC for $0 \leq \alpha \leq \beta - 1$ and $f^{(\alpha)} \in L_\varphi(\mathbb{T})$ for $\beta - 1 \leq \alpha \leq \beta$. It is well known that the function $\Psi_\alpha(u) := \lim_{n \rightarrow \infty} \sum_{\nu \in \mathbb{Z}_n^*} \frac{e^{i\nu u}}{(i\nu)^\alpha} = \sum_{\nu \in \mathbb{Z}^*} \frac{e^{i\nu u}}{(i\nu)^\alpha}$,

$\alpha \in \mathbb{R}^+$, is defined for every $u \in \mathbb{R}$ if $1 \leq \alpha < \infty$ (for $u \neq 2k\pi$, $k \in \mathbb{Z}$ if $0 < \alpha < 1$) and Ψ_α is of class $L^1(\mathbb{T})$. In this case

$$(1.6) \quad f(x) = (f^{(\beta)} * \Psi_\beta)(x) \text{ for every } x \in \mathbb{R}.$$

Furthermore,

$$(1.7) \quad f^{(\alpha)}(x) = (f^{(\beta)} * \Psi_{\beta-\alpha})(x)$$

is satisfied for every $x \in \mathbb{R}$ if $0 \leq \alpha < \beta - 1$ (for almost every $x \in \mathbb{R}$ if $\beta - 1 < \alpha < \beta$). Now (1.6) implies that if $\beta \geq 1$, then f is absolutely continuous, and (1.7) implies that $f^{(\alpha)}$ is AC for $0 \leq \alpha \leq \beta - 1$ and $f^{(\alpha)} \in L_\varphi(\mathbb{T})$ for $\beta - 1 \leq \alpha \leq \beta$.

2° If $\alpha = \beta$, then (1.4) is obvious. If $\alpha = 0$, then (1.4) was proved in [3]. Let $0 \leq \alpha < \beta$. We choose a $S_{\alpha,n} \in \mathcal{T}_n$ with $\|S_{\alpha,n} - \Psi_{\beta-\alpha}\|_{L^1(\mathbb{T})} = E_n(\Psi_{\beta-\alpha})_{L^1(\mathbb{T})}$. Let $U_{n,\alpha}[f] = f^{(\beta)} * S_{\alpha,n}$, $n = 1, 2, 3, \dots$. Then

$$f^{(\alpha)}(x) - U_{n,\alpha}[f](x) = \frac{1}{2\pi} \int_{\mathbb{T}} f^{(\beta)}(u) \{ \Psi_{\beta-\alpha}(x-u) - S_{\alpha,n}(x-u) \} du$$

holds a.e. Therefore,

$$\|f^{(\alpha)} - U_{n,\alpha}[f]\|_\varphi \leq \|\Psi_{\beta-\alpha} - S_{\alpha,n}\|_{L^1(\mathbb{T})} \|f^{(\beta)}\|_\varphi.$$

Since by [4]

$$\|\Psi_{\beta-\alpha} - S_{\alpha,n}\|_{L^1(\mathbb{T})} \leq cn^{\alpha-\beta}$$

we get (since $U_{n,\alpha}[f] \in \mathcal{T}_n$) that

$$E_n(f^{(\alpha)})_\varphi \leq cn^{\alpha-\beta} \|f^{(\beta)}\|_\varphi.$$

Let $Q_n \in \mathcal{T}_n$ be such that

$$\|f^{(\beta)} - Q_n\|_\varphi = E_n(f^{(\beta)})_\varphi, \quad n = 1, 2, 3, \dots$$

We suppose

$$\phi(x) = f(x) - I_\beta[Q_n](x), \quad x \in \mathbb{R}.$$

Then

$$\phi^{(\beta)}(x) = f^{(\beta)}(x) - Q_n(x),$$

and hence

$$\|\phi^{(\beta)}\|_\varphi = \|f^{(\beta)} - Q_n\|_\varphi = E_n(f^{(\beta)})_\varphi.$$

Therefore we find

$$E_n(\phi^{(\alpha)})_\varphi \leq cn^{\alpha-\beta} \|\phi^{(\beta)}\|_\varphi \leq cn^{\alpha-\beta} E_n(f^{(\beta)})_\varphi.$$

Since

$$E_n(\phi^{(\alpha)})_\varphi = E_n(f^{(\alpha)})_\varphi,$$

we conclude that (1.4) holds.

3° Let

$$f_+^{(\beta)}(u) = \frac{1}{2} \left\{ |f^{(\beta)}(u)| + f^{(\beta)}(u) \right\} \quad \text{and} \quad f_-^{(\beta)}(u) = \frac{1}{2} \left\{ |f^{(\beta)}(u)| - f^{(\beta)}(u) \right\}$$

for $u \in \mathbb{R}$. Then

$$\begin{aligned} f(x) &= (f_+^{(\beta)} * \Psi_\beta)(x) - (f_-^{(\beta)} * \Psi_\beta)(x), \\ f^{(\alpha)}(x) &= (f_+^{(\beta)} * \Psi_{\beta-\alpha})(x) - (f_-^{(\beta)} * \Psi_{\beta-\alpha})(x) \end{aligned}$$

for every $0 < \alpha \leq \beta - 1$. Let $t_{\alpha,n} \in \mathcal{T}_n^-(\Psi_{\beta-\alpha})$, $T_{\alpha,n} \in \mathcal{T}_n^+(\Psi_{\beta-\alpha})$ be such that

$$\|f - t_{\alpha,n}\|_\varphi = E_n^-(\Psi_{\beta-\alpha})_{L^1(\mathbb{T})} \quad \text{and} \quad \|T_{\alpha,n} - f\|_\varphi = E_n^+(\Psi_{\beta-\alpha})_{L^1(\mathbb{T})}$$

for $n = 1, 2, 3, \dots$. Let also

$$U_{0,n}^+[f] = (f_+^{(\beta)} * T_{0,n}) - (f_-^{(\beta)} * t_{0,n}), \quad U_{0,n}^-[f] = (f_+^{(\beta)} * t_{0,n}) - (f_-^{(\beta)} * T_{0,n}),$$

and for $0 < \alpha < \beta - 1$ we set

$$U_{\alpha,n}^+[f] = (f_+^{(\beta)} * T_{\alpha,n}) - (f_-^{(\beta)} * t_{\alpha,n}), \quad U_{\alpha,n}^-[f] = (f_+^{(\beta)} * t_{\alpha,n}) - (f_-^{(\beta)} * T_{\alpha,n}).$$

Hence,

$$\begin{aligned} U_{\alpha,n}^+[f](x) - f^{(\alpha)}(x) &= \frac{1}{2\pi} \int_{\mathbb{T}} f_+^{(\beta)}(u) \{T_{\alpha,n}(x-u) - \Psi_{\beta-\alpha}(x-u)\} du \\ &\quad + \frac{1}{2\pi} \int_{\mathbb{T}} f_-^{(\beta)}(u) \{\Psi_{\beta-\alpha}(x-u) - t_{\alpha,n}(x-u)\} du \end{aligned}$$

for every $x \in \mathbb{R}$. Then

$$U_{\alpha,n}^+[f](x) \geq f^{(\alpha)}(x)$$

for every $x \in \mathbb{R}$. This implies that

$$U_{\alpha,n}^+[f] \in \mathcal{T}_n^+(f^{(\alpha)}), \quad 0 \leq \alpha \leq \beta - 1.$$

Similarly,

$$U_{\alpha,n}^-[f] \in \mathcal{T}_n^-(f^{(\beta)}), \quad 0 \leq \alpha \leq \beta - 1.$$

We obtain

$$\|U_{n,\alpha}^\pm[f] - f^{(\alpha)}\|_\varphi \leq cn^{\alpha-\beta} \|f^{(\beta)}\|_\varphi,$$

and hence

$$\|U_{n,\alpha}^\pm[f] - f^{(\alpha)}\|_\varphi \leq cn^{\alpha-\beta} E_n(f^{(\beta)})_\varphi$$

for $0 \leq \alpha \leq \beta - 1$. Since

$$U_{\alpha,n}^-[f](x) \leq \phi^{(\alpha)}(x) \leq U_{\alpha,n}^+[f](x)$$

and

$$\phi^{(\alpha)}(x) = f^{(\alpha)}(x) - Q_n^{(\alpha-\beta)}(x)$$

we have

$$U_{\alpha,n}^-[f](x) + Q_n^{(\alpha-\beta)}(x) \leq f^{(\alpha)}(x) \leq U_{\alpha,n}^+[f](x) + Q_n^{(\alpha-\beta)}(x)$$

for every $x \in \mathbb{R}$. Therefore

$$\begin{aligned} E_n^\pm(f^{(\alpha)})_\varphi &\leq \|U_{n,\alpha}^\pm[\phi] - Q_n^{(\alpha-\beta)} - f^{(\alpha)}\|_\varphi \\ &= \|U_{n,\alpha}^\pm[\phi] - \phi^{(\alpha)}\|_\varphi \\ &\leq cn^{\alpha-\beta} E_n(f^{(\beta)})_\varphi, \end{aligned}$$

and the required result holds. \square

1.3. Theorem. *Let $\varphi \in QC_2^\theta(0, 1)$. If $1 \leq \beta < \infty$, $f \in W_\varphi^\beta(\mathbb{T})$ and $\beta \geq \alpha \geq 0$, then for $n = 1, 2, 3, \dots$ there is a constant $c > 0$ dependent only on α, β and φ such that*

$$(1.8) \quad \|f^{(\alpha)}(\cdot) - S_n^{(\alpha)}(\cdot, f)\|_\varphi \leq \frac{c}{n^{\beta-\alpha}} E_n(f^{(\beta)})_\varphi$$

holds. If f is real valued and there exist polynomials $t_n \in \mathcal{T}_n^-(f)$, $T_n \in \mathcal{T}_n^+(f)$ such that $\|f - t_n\|_\varphi \leq cE_n^-(f)_\varphi$, $\|T_n - f\|_\varphi \leq cE_n^+(f)_\varphi$, then for $0 \leq \alpha \leq \beta$ and $n = 1, 2, 3, \dots$,

$$(1.9) \quad \|f^{(\alpha)} - t_n^{(\alpha)}\|_\varphi \leq \frac{c}{n^{\beta-\alpha}} E_n(f^{(\beta)})_\varphi, \text{ and}$$

$$(1.10) \quad \|T_n^{(\alpha)} - f^{(\alpha)}\|_\varphi \leq \frac{c}{n^{\beta-\alpha}} E_n(f^{(\beta)})_\varphi$$

hold.

Proof. If $\alpha = 0$, then the results follows from Theorem 1.2. If $\alpha = \beta$, then it was proved in [2] that

$$(1.11) \quad \|f^{(\alpha)}(\cdot) - S_n^{(\alpha)}(\cdot, f)\|_\varphi \leq cE_n(f^{(\alpha)})_\varphi.$$

From Theorem 1.2 and last inequality, (1.8) follows.

Now we prove (1.9) and (1.10) for $0 \leq \alpha < \beta$. Let

$$W_n(f) := W_n(x, f) := \frac{1}{n+1} \sum_{\nu=n}^{2n} S_\nu(x, f), \quad n = 0, 1, 2, \dots$$

Suppose that $u := u(\cdot, f) \in \mathcal{T}_n$ satisfies $\|f - u\|_\varphi = E_n(f)_\varphi$. Since

$$W_n(\cdot, f^{(\alpha)}) = W_n^{(\alpha)}(\cdot, f)$$

we have

$$\begin{aligned} \|f^{(\alpha)} - t_n^{(\alpha)}\|_\varphi &\leq \|f^{(\alpha)} - W_n(\cdot, f^{(\alpha)})\|_\varphi + \|u(\cdot, W_n(f)) - t_n^{(\alpha)}\|_\varphi \\ &\quad + \|W_n^{(\alpha)}(\cdot, f) - u(\cdot, W_n(f))\|_\varphi \\ &:= I_1 + I_2 + I_3. \end{aligned}$$

Since $\|W_n(f)\|_\varphi \leq 4\|f\|_\varphi$, we get

$$\begin{aligned} I_1 &\leq \|f^{(\alpha)}(\cdot) - u(\cdot, f^{(\alpha)})\|_\varphi + \|u(\cdot, f^{(\alpha)}) - W_n(\cdot, f^{(\alpha)})\|_\varphi \\ &= E_n(f^{(\alpha)})_\varphi + \|W_n(\cdot, u(f^{(\alpha)})) - f^{(\alpha)}\|_\varphi \\ &\leq 5E_n(f^{(\alpha)})_\varphi. \end{aligned}$$

From Theorem 1.1, we get

$$I_2 \leq 2(n-1)^\alpha \|u(\cdot, W_n(f)) - t_n\|_\varphi$$

and

$$I_3 \leq 2(2n-2)^\alpha \|W_n(\cdot, f) - u(\cdot, W_n(f))\|_\varphi \leq 2^{\alpha+1} n^\alpha E_n(W_n(f))_\varphi.$$

Now we have

$$\begin{aligned} \|u(\cdot, W_n(f)) - t_n\|_\varphi &\leq \|u(\cdot, W_n(f)) - W_n(\cdot, f)\|_\varphi + \|W_n(\cdot, f) - f(\cdot)\|_\varphi \\ &\quad + \|f(\cdot) - t_n\|_\varphi \\ &\leq E_n(W_n(f))_\varphi + 5E_n(f)_\varphi + cE_n^-(f)_\varphi. \end{aligned}$$

Since

$$E_n(W_n(f))_\varphi \leq \|W_n(f) - u\|_\varphi = \|W_n(f - u)\|_\varphi \leq 4E_n(f)_\varphi,$$

we get

$$\begin{aligned} (1.12) \quad \|f^{(\alpha)} - t_n^{(\alpha)}\|_\varphi &\leq 5E_n(f^{(\alpha)})_\varphi + 2n^\alpha E_n(W_n(f))_\varphi + 10n^\alpha E_n(f)_\varphi \\ &\quad + 2^{\alpha+1} n^\alpha E_n(W_n(f))_\varphi + c2n^\alpha E_n^-(f)_\varphi \\ &\leq 5E_n\left(f^{(\alpha)}\right)_\varphi + (18 + 2^{3+\alpha}) n^\alpha E_n(f)_\varphi + c2n^\alpha E_n^-(f)_\varphi. \end{aligned}$$

Using Theorem 1.2 we get (1.9), and (1.10) can be proved using the same procedure. \square

Direct theorem of trigonometric approximation:

1.4. Theorem. *Let $\varphi \in QC_2^0(0, 1)$ and $r \in \mathbb{R}^+$. If $f \in L_\varphi(\mathbb{T})$, then there is a constant $c > 0$, dependent only on r and φ , such that the inequality*

$$(1.13) \quad E_n(f)_\varphi \leq c\omega_\varphi^r\left(f, \frac{1}{n+1}\right)$$

holds for $n = 0, 1, 2, 3, \dots$

Proof. This is a consequence of [3, Theorem 2] and the property $\omega_\varphi^r(f, \cdot) \leq c\omega_\varphi^s(f, \cdot)$, ($r \geq s \in \mathbb{R}^+$), of the smoothness moduli. \square

1.5. Theorem. *If $r, \delta \in \mathbb{R}^+$ and $f \in B^\alpha$, $\alpha \in \mathbb{R}^+$, then there exists a constant $c > 0$ depending only on r and B such that*

$$(1.14) \quad \omega_B^r(f, \delta) \leq c\delta^r \|f^{(r)}\|_B, \quad \delta \geq 0$$

holds.

Proof. For the function $\chi_r(\cdot, h) \in L^1(\mathbb{T})$ of [6, (20.15), p.376] we define

$$(A_h^r f)(x) := (f * \chi_r(\cdot, h))(x) = \frac{1}{2\pi} \int_{\mathbb{T}} f(x-u) \chi_r(u, h) du, \quad x \in \mathbb{T}, \quad h \in \mathbb{R}^+.$$

Then using Fubini's theorem we get

$$(1.15) \quad \|A_h^r f\|_B \leq \|\chi_r(\cdot, h)\|_{L^1(\mathbb{T})} \|f\|_B \leq c \|f\|_B.$$

Since

$$(\Delta_h^r f)(x) = h^r (A_h^r f)^{(r)}(x) = h^r A_h^r (f^{(r)})(x)$$

we have from (1.15) that

$$\sup_{|h| \leq \delta} \|\Delta_h^r f\|_B = \sup_{|h| \leq \delta} h^r \|A_h^r (f^{(r)})\|_B \leq c\delta^r \|f^{(r)}\|_B,$$

from which we obtain (1.14). \square

The converse theorem of trigonometric approximation:

1.6. Theorem. Let $\varphi \in QC_2^\theta(0, 1)$ and $r \in \mathbb{R}^+$. If $f \in L_\varphi(\mathbb{T})$, then there is a constant $c > 0$, dependent only on r and φ , such that for $n = 0, 1, 2, 3, \dots$

$$\omega_\varphi^r\left(f, \frac{\pi}{n+1}\right) \leq \frac{c}{(n+1)^r} \sum_{\nu=0}^n (\nu+1)^{r-1} E_\nu(f)_\varphi$$

holds.

Proof. The proof goes similarly to that of the proof of [2, Theorem 3]. □

From Theorems 1.4 and 1.6 we have the following corollaries:

1.7. Corollary. Let $\varphi \in QC_2^\theta(0, 1)$ and $r \in \mathbb{R}^+$. If $f \in L_\varphi(\mathbb{T})$ satisfies

$$E_n(f)_\varphi = \mathcal{O}(n^{-\sigma}), \quad \sigma > 0, \quad n = 1, 2, \dots,$$

then

$$\omega_\varphi^r(f, \delta) = \begin{cases} \mathcal{O}(\delta^\sigma) & \text{if } r > \sigma, \\ \mathcal{O}(\delta^\sigma |\log(1/\delta)|) & \text{if } r = \sigma, \\ \mathcal{O}(\delta^r) & \text{if } r < \sigma, \end{cases}$$

holds. □

1.8. Definition. Let $\varphi \in QC_2^\theta(0, 1)$ and $r \in \mathbb{R}^+$. If $f \in L_\varphi(\mathbb{T})$, then for $0 < \sigma < r$ we set $\text{Lip}\sigma(r, \varphi) := \{f \in L_\varphi(\mathbb{T}) : \omega_\varphi^r(f, \delta) = \mathcal{O}(\delta^\sigma), \delta > 0\}$.

The following constructive characterization of the Lipschitz class holds:

1.9. Corollary. Let $0 < \sigma < r$, $M \in QC_2^\theta(0, 1)$ and $f \in L_\varphi(\mathbb{T})$. Then the conditions

- (a) $f \in \text{Lip}\sigma(r, \varphi)$,
- (b) $E_n(f)_\varphi = \mathcal{O}(n^{-\sigma})$, $n = 1, 2, \dots$,

are equivalent. □

1.10. Theorem. Let $\varphi \in QC_2^\theta(0, 1)$ and $f \in L_\varphi(\mathbb{T})$. If $\alpha \in \mathbb{R}^+$ and

$$\sum_{\nu=1}^\infty \nu^{\alpha-1} E_\nu(f)_\varphi < \infty,$$

then there exists a constant $c > 0$ dependent only on α and φ , such that

$$(1.16) \quad E_n(f^{(\alpha)})_\varphi \leq c \left(n^\alpha E_n(f)_\varphi + \sum_{\nu=n+1}^\infty \nu^{\alpha-1} E_\nu(f)_\varphi \right)$$

holds.

Proof. Since

$$\begin{aligned} \|f^{(\alpha)} - S_n(f^{(\alpha)})\|_\varphi &\leq \|S_{2^{m+2}}(f^{(\alpha)}) - S_n(f^{(\alpha)})\|_\varphi \\ &\quad + \sum_{k=m+2}^\infty \|S_{2^{k+1}}(f^{(\alpha)}) - S_{2^k}(f^{(\alpha)})\|_\varphi \end{aligned}$$

we have for $2^m < n < 2^{m+1}$ that

$$\|S_{2^{m+2}}(f^{(\alpha)}) - S_n(f^{(\alpha)})\|_\varphi \leq c 2^{(m+2)\alpha} E_n(f)_\varphi \leq cn^\alpha E_n(f)_\varphi.$$

On the other hand we find

$$\begin{aligned} & \sum_{k=m+2}^{\infty} \|S_{2^{k+1}}(f^{(\alpha)}) - S_{2^k}(f^{(\alpha)})\|_{\varphi} \\ & \leq c \sum_{k=m+2}^{\infty} 2^{(k+1)\alpha} E_{2^k}(f)_{\varphi} \leq c \sum_{k=m+2}^{\infty} \sum_{\mu=2^{k-1}+1}^{2^k} \mu^{\alpha-1} E_{\mu}(f)_{\varphi} \\ & = c \sum_{\nu=2^{m+1}+1}^{\infty} \nu^{\alpha-1} E_{\nu}(f)_{\varphi} \leq c \sum_{\nu=n+1}^{\infty} \nu^{\alpha-1} E_{\nu}(f)_{\varphi}. \end{aligned}$$

Therefore

$$E_n(f^{(\alpha)})_{\varphi} \leq c \left(n^{\alpha} E_n(f)_{\varphi} + \sum_{\nu=n+1}^{\infty} \nu^{\alpha-1} E_{\nu}(f)_{\varphi} \right), \quad \square$$

As a corollary of Theorems 1.4, 1.6 and 1.10,

1.11. Theorem. *Let $f \in W_{\varphi}^{\alpha}(\mathbb{T})$, $r \in (0, \infty)$, and*

$$\sum_{\nu=1}^{\infty} \nu^{\alpha-1} E_{\nu}(f)_{\varphi} < \infty$$

for some $\alpha > 0$. In this case, for $n = 0, 1, 2, \dots$ there exists a constant $c > 0$, dependent only on α, r and φ such that

$$\omega_{\varphi}^r \left(f^{(\alpha)}, \frac{\pi}{n+1} \right) \leq c \left(\frac{1}{(n+1)^r} \sum_{\nu=0}^n (\nu+1)^{\alpha+r-1} E_{\nu}(f)_{\varphi} + \sum_{\nu=n+1}^{\infty} \nu^{\alpha-1} E_{\nu}(f)_{\varphi} \right)$$

holds. □

As a corollary of Theorem 1.4,

1.12. Theorem. *Let $\varphi \in QC_2^{\theta}(0, 1)$, $r \in \mathbb{R}^+$ and $1 \leq \beta < \infty$. If $f \in W_{\varphi}^{\beta}(\mathbb{T})$ is real valued and $0 \leq \alpha \leq \beta - 1$, then there is a constant $c > 0$, dependent only on r and φ , such that the inequality*

$$E_n^{\pm}(f^{(\alpha)})_{\varphi} \leq \frac{c}{n^{\beta-\alpha}} \omega_{\varphi}^r \left(f^{(\beta)}, \frac{\pi}{n} \right)$$

holds for $n = 1, 2, 3, \dots$ □

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