INEQUALITIES FOR ONE SIDED APPROXIMATION IN ORLICZ SPACES

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Abstract

In the present article some inequalities of trigonometric approximation are proved in Orlicz spaces generated by a quasiconvex Young function. Also, the main one-sided approximation problems are investigated.

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1. Introduction

A function Φ is called a Young function if Φ is even, continuous, nonnegative in \( \mathbb{R} := (-\infty, +\infty) \), increasing on \( \mathbb{R}^+ := (0, \infty) \) and such that

\[ \Phi(0) = 0, \quad \lim_{x \to \infty} \Phi(x) = \infty. \]

A function \( \varphi : [0, \infty) \to [0, \infty) \) is said to be quasiconvex if there exist a convex Young function \( \Phi \) and a constant \( c_1 \geq 1 \) such that

\[ \Phi(x) \leq \varphi(x) \leq \Phi(c_1 x) \quad \forall x \geq 0. \]

Set \( T := [0, 2\pi] \) and let \( \varphi \) be a quasiconvex Young function. We denote by \( \varphi(L) \) the class of complex valued Lebesgue measurable functions \( f : T \to \mathbb{C} \) satisfying the condition

\[ \int_T \varphi(|f(x)|) \, dx < \infty. \]

The class of functions \( f : T \to \mathbb{C} \) having the property

\[ \int_T \varphi(c_2 |f(x)|) \, dx < \infty \]

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for some \( c_2 \in \mathbb{R}^+ \) is denoted by \( L_\varphi^+ (T) \). The set \( L_\varphi (T) \) becomes a normed space with the Orlicz norm

\[
\|f\|_\varphi := \sup \left\{ \int_T |f(x)g(x)| \, dx : \int_T \tilde{\varphi}(|g|) \, dx \leq 1 \right\},
\]

where \( \tilde{\varphi}(y) := \sup_{x \geq 0} (xy - \varphi(x)) \), \( y \geq 0 \), is the complementary function of \( \varphi \).

For a quasiconvex function \( \varphi \) we define the index \( p(\varphi) \) of \( \varphi \) as

\[
\frac{1}{p(\varphi)} := \inf \{ p : p > 0, \varphi^p \text{ is quasiconvex} \}
\]

and the conjugate index of \( \varphi \) as

\[
p'(\varphi) := \frac{p(\varphi)}{p(\varphi) - 1}.
\]

It can be easily seen that the functions in \( L_\varphi (T) \) are summable on \( T \), \( L_\varphi (T) \subset L^1 (T) \) and \( L_\varphi (T) \) becomes a Banach space with the Orlicz norm. The Banach space \( L_\varphi (T) \) is called the Orlicz space.

A Young function \( \Phi \) is said to satisfy the \( \Delta_2 \) condition if there is a constant \( c_3 > 0 \) such that

\[
\Phi(2x) \leq c_3 \Phi(x)
\]

for all \( x \in \mathbb{R} \).

We will denote by \( QC_{\theta}^g (0,1) \) the class of functions \( g \) satisfying the condition \( \Delta_2 \) such that \( g^\theta \) is quasiconvex for some \( \theta \in (0,1) \).

In the present work we consider the trigonometric polynomial approximation problems for functions and their fractional derivatives in the spaces \( L_\varphi (T) \), where \( \varphi \in QC_{\theta}^g (0,1) \). We prove a Jackson type direct theorem, and a converse theorem of trigonometric approximation with respect to the fractional order moduli of smoothness in Orlicz spaces. As a particular case, we obtain a constructive description of the Lipschitz class in Orlicz spaces. A direct theorem of one sided trigonometric approximation is also obtained.

Let

\[
(f(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx} \quad \text{and} \quad \hat{f}(x) = \sum_{k=-\infty}^{\infty} (-i\text{sgn} k) c_k e^{ikx}
\]

be the Fourier and the conjugate Fourier series of \( f \in L^1 (T) \), respectively. We define

\[
S_n (f) := S_n (x,f) := \sum_{k=-n}^{n} c_k e^{ikx}, \quad n = 0, 1, 2, \ldots
\]

For a given \( f \in L^1 (T) \), assuming \( c_0 = 0 \) in (1.1), we define the \( \alpha \)th fractional \( (\alpha \in \mathbb{R}^+) \) integral of \( f \) as in [7, v.2, p.134] by

\[
I_\alpha (x,f) := \sum_{k \in \mathbb{Z}} c_k (ik)^{-\alpha} e^{ikx},
\]

where \( \mathbb{Z} \) is the set of integers, \( \mathbb{Z}^* := \{ z \in \mathbb{Z} : z \neq 0 \} \), and

\[
(ik)^{-\alpha} := |k|^{-\alpha} e^{(-1/2)i\text{sgn} k}
\]

as principal value.

Let \( \alpha \in \mathbb{R}^+ \) be given. We define the fractional derivative of a function \( f \in L^1 (T) \), satisfying \( c_0 = 0 \) in (1.1), as

\[
f^{(\alpha)} (x) := \frac{d^{[\alpha]+1}}{dx^{[\alpha]+1}} I_{1+\alpha-[\alpha]} (x,f),
\]
provided the righthand side exists, where \([x]\) denotes the integer part of the real number \(x\).

Setting \(h \in T\), \(r \in \mathbb{R}^+\), \(\varphi \in QC^2_2 (0, 1)\) and \(f \in L_\varphi (T)\), we define
\[
\Delta_h^\alpha f (\cdot) := (T_h - I)^\alpha f (\cdot) = \sum_{k=0}^{\infty} (-1)^k \binom{r}{k} f (\cdot + (r - k) h),
\]
where \(\binom{r}{k} := \frac{r(r - 1) \ldots (r - k + 1)}{k!}\) for \(k > 1\), \(\binom{r}{1} := r\) and \(\binom{r}{0} := 1\) are the binomial coefficients, \(T_h f (x) := f (x + h)\) is the translation operator and \(I\) the identity operator.

Since \(\sum_{k=0}^{\infty} \left| \binom{r}{k} \right| < \infty\) we get
\[
(1.2) \quad \|\Delta_h^\alpha f\|_\varphi \leq c\|f\|_\varphi < \infty
\]
under the condition \(f \in L_\varphi (T)\), where \(\varphi \in QC^2_2 (0, 1)\).

Here and in the following we will denote by \(B\) a translation invariant Banach Function Space. Also, the notation \(\| \cdot \|_B\) stands for the norm of \(B\).

For \(r \in \mathbb{R}^+\), we define the fractional modulus of smoothness of order \(r\) for \(f \in B\), as
\[
\omega_B^r (f, \delta) := \sup_{|h| \leq \delta} \|\Delta_h^r f\|_B, \quad \delta > 0.
\]
If \(\varphi \in QC^2_2 (0, 1)\) and \(B = L_\varphi (T)\), we will set \(\omega_B^r (f, \cdot) := \omega_\varphi^r (f, \cdot)\). Hence for \(\varphi \in QC^2_2 (0, 1)\) and \(f \in L_\varphi (T)\), we have by (1.2) that
\[
\omega_\varphi^r (f, \delta) \leq c\|f\|_\varphi,
\]
where the constant \(c > 0\) dependent only on \(r\) and \(\varphi\).

Let \(T_n\) be the class of trigonometric polynomials of degree not greater than \(n\). We begin with the fractional Nikolski-Civin inequality:

**1.1. Theorem.** Suppose that \(\alpha \in \mathbb{R}^+\), \(T_n \in \mathcal{T}_n\) and \(0 < h < 2\pi/n\). Then
\[
\|T_n^{(\alpha)}\|_B \leq \left(\frac{n}{2 \sin (nh/2)}\right)^\alpha \|\Delta_h^\alpha T_n\|_B.
\]
In particular, if \(h = \pi/n\), then
\[
(1.3) \quad \|T_n^{(\alpha)}\|_B \leq 2^{-\alpha} n^\alpha \|\Delta_h^\alpha T_n\|_B.
\]

**Proof.** Let \(T_n (x) = \frac{\pi}{2} + \sum_{\nu \in \mathbb{Z}_n^*} c_\nu e^{i\nu x}\), where \(\mathbb{Z}_n^* := \{z \in \mathbb{Z} : z < n, z > -n, z \neq 0\}\). Then
\[
T_n^{(\alpha)} (x) = \sum_{\nu \in \mathbb{Z}_n^*} (i\nu)^\alpha c_\nu e^{i\nu x}, \quad \Delta_h^\alpha T_n \left(x + \frac{\alpha}{2} h\right) = \sum_{\nu \in \mathbb{Z}_n^*} \left(2 i \sin \frac{h}{2} \nu\right)^\alpha c_\nu e^{i\nu x}.
\]
We set
\[
\varphi (t) := \left(2 i \sin \frac{h}{2} t\right)^\alpha, \quad g (t) := \left(\frac{t}{2 \sin \frac{h}{2} t}\right)^\alpha \text{ for } -n \leq t \leq n \text{ and } g (0) := h^{-\alpha}.
\]
Then for \(x \in \mathbb{R}\), \(h \in (0, 2\pi/n)\), we obtain
\[
\Delta_h^\alpha T_n \left(x + \frac{\alpha}{2} h\right) = \sum_{\nu \in \mathbb{Z}_n^*} \varphi (\nu) c_\nu e^{i\nu x}
\]
Hence we conclude

\[ T_n^{(\alpha)}(x) = \sum_{\nu \in \mathbb{Z}_n^*} \varphi(\nu)g(\nu) c_\nu e^{i\nu x}. \]

The convergence

\[ g(t) = \sum_{k=-\infty}^{\infty} d_k e^{ik\pi t/n} \]

is uniform for \( t \in [-n, n] \). Since \((-1)^k d_k \geq 0\), we find

\[ T_n^{(\alpha)}(x) = \sum_{\nu \in \mathbb{Z}_n^*} \varphi(\nu) \sum_{k=-\infty}^{\infty} d_k e^{ik\pi \nu} c_\nu e^{i\nu x} \]

is continuous (AC). Similarly, the best trigonometric approximation error of \( f \in L_\varphi(T) \) is defined as \( E_n(\varphi) := \inf_{T \in I_n}(f) \|T - f\|_\varphi \). We denote by \( B_\alpha^\varphi \), \( \alpha > 0 \), the linear space of 2\pi-periodic complex valued functions \( f \in B \) such that \( f^{(\alpha-1)} \) is absolutely continuous (AC), and \( f^{(\alpha)} \in B \). If \( \varphi \in QC_2(0, 1) \) and \( B = L_\varphi(T) \) we will let \( B_\alpha^\varphi := W_\alpha^\varphi(T) \).

We set \( \mathcal{T}_n^\varphi := \{ f \in L_\infty : f \text{ is real valued and bounded on } T \} \). If \( f \in \mathcal{T}_n^\varphi \) we define

\[ \mathcal{T}_n^\varphi(f) := \{ t \in \mathcal{T}_n : t \text{ is real valued } 2\pi \text{ periodic and } t(x) \leq f(x) \text{ for every } x \in \mathbb{R} \}, \]

\[ \mathcal{T}_n^\varphi(f) := \{ T \in \mathcal{T}_n : T \text{ is real valued } 2\pi \text{ periodic and } f(x) \leq T(x) \text{ for every } x \in \mathbb{R} \}, \]

\[ E_n^-(f) := \inf_{T \in \mathcal{T}_n^\varphi(f)} \|f - T\|_\varphi, \quad E_n^+(f) := \inf_{T \in \mathcal{T}_n^\varphi(f)} \|T - f\|_\varphi. \]

The quantities \( E_n^-(f) \) and \( E_n^+(f) \) are, respectively, called the best lower (upper) one sided approximation errors for \( f \in \mathcal{T}_n^\varphi \). Similarly, the best trigonometric approximation error of \( f \in \mathcal{T}_n^\varphi(T) \) is defined as \( E_n(\varphi) := \inf_{S \in \mathcal{T}_n} \|f - S\|_\varphi \). We note that \( E_n^-(f) \leq E_n^+(f) \).

If \( f \in QC_2(0, 1), f \in \mathcal{T}_n^\varphi(T), g \in L^1(T) \), we introduce the convolution

\[ (f * g)(x) = \frac{1}{2\pi} \int_T f(x - u) g(u) \, du. \]

This convolution exists for every \( x \in \mathbb{R} \) and is a measurable function. Furthermore

\[ \|f * g\|_\varphi \leq \|f\|_\varphi \|g\|_{L^1(T)}. \]

If \( f \) is continuous (AC) then \( f * g \) is continuous (AC).
1.2. Theorem. Let \( \varphi \in QC_1^\infty (0, 1) \), \( 1 \leq \beta < \infty \) and \( f \in W_\beta^1 (T) \). If \( 0 \leq \alpha \leq \beta \) and \( n = 1, 2, 3, \ldots \), then there exists a constant \( c > 0 \) depending only on \( \alpha \) and \( \beta \) such that

\[
E_n(f^{(\alpha)}) \leq \frac{c}{n^{\beta-\alpha}} E_n(f^{(\beta)}) \tag{1.4}
\]

holds. If \( f \) is real valued, \( 0 \leq \alpha \leq \beta - 1 \) and \( n = 1, 2, 3, \ldots \), then

\[
E_n^+(f^{(\alpha)}) \leq \frac{c}{n^{\beta-\alpha}} E_n(f^{(\beta)}) \tag{1.5}
\]

holds.

Proof. 1° First we prove that \( f^{(\alpha)} \) is AC for \( 0 \leq \alpha \leq \beta - 1 \) and \( f^{(\alpha)} \in L_\varphi (T) \) for \( \beta - 1 \leq \alpha \leq \beta \). It is well known that the function \( \Psi_\alpha (u) := \lim_{n \to \infty} \sum_{\nu \in \mathbb{Z}} c_{\nu u} \frac{1}{(1 + \nu)^\alpha} \) is defined for every \( u \in \mathbb{R} \) if \( 1 \leq \alpha < \infty \) (for \( u \neq 2k\pi \), \( k \in \mathbb{Z} \) if \( 0 < \alpha < 1 \)) and \( \Psi_\alpha \) is of class \( L^1 (T) \). In this case

\[
f(x) = (f^{(\beta)} * \Psi_\alpha)(x) \quad \text{for every } x \in \mathbb{R}.
\]

Furthermore,

\[
f^{(\alpha)}(x) = (f^{(\beta)} * \Psi_{\beta-\alpha})(x) \tag{1.6}
\]

is satisfied for every \( x \in \mathbb{R} \) if \( 0 \leq \alpha < \beta - 1 \) (for almost every \( x \in \mathbb{R} \) if \( \beta - 1 < \alpha < \beta \)). Now (1.6) implies that if \( \beta \geq 1 \), then \( f \) is absolutely continuous, and (1.7) implies that \( f^{(\alpha)} \) is AC for \( 0 \leq \alpha \leq \beta - 1 \) and \( f^{(\alpha)} \in L_\varphi (T) \) for \( \beta - 1 \leq \alpha \leq \beta \).

2° If \( \alpha = \beta \), then (1.4) is obvious. If \( \alpha = 0 \), then (1.4) was proved in [3]. Let \( 0 \leq \alpha < \beta \). We choose a \( S_{\alpha,n} \in T_n \) with \( \|S_{\alpha,n} - \Psi_{\beta-\alpha}\|_{L^1(T)} = E_n(\Psi_{\beta-\alpha})_{L^1(T)} \). Let \( U_{\alpha,n} [f] = f^{(\beta)} * S_{\alpha,n} \), \( n = 1, 2, 3, \ldots \). Then

\[
f^{(\alpha)}(x) - U_{\alpha,n} [f] (x) = \frac{1}{2\pi} \int_{T} f^{(\beta)} (u) \{ \Psi_{\beta-\alpha} (x - u) - S_{\alpha,n} (x - u) \} \, du
\]

holds a.e. Therefore,

\[
\|f^{(\alpha)} - U_{\alpha,n} [f]\|_\varphi \leq \|\Psi_{\beta-\alpha} - S_{\alpha,n}\|_{L^1(T)} \|f^{(\beta)}\|_\varphi.
\]

Since by [4]

\[
\|\Psi_{\beta-\alpha} - S_{\alpha,n}\|_{L^1(T)} \leq cn^{\alpha-\beta}
\]

we get (since \( U_{\alpha,n} [f] \in T_n \)) that

\[
E_n(f^{(\alpha)}) \varphi \leq cn^{\alpha-\beta} \|f^{(\beta)}\|_\varphi.
\]

Let \( Q_n \in T_n \) be such that

\[
\|f^{(\beta)} - Q_n\|_\varphi = E_n(f^{(\beta)}) \varphi, \quad n = 1, 2, 3, \ldots
\]

We suppose

\[
\phi(x) = f(x) - I_\beta [Q_n] (x), \quad x \in \mathbb{R}.
\]

Then

\[
\phi^{(\beta)}(x) = f^{(\beta)}(x) - Q_n(x),
\]

and hence

\[
\|\phi^{(\beta)}\|_\varphi = \|f^{(\beta)} - Q_n\|_\varphi = E_n(f^{(\beta)}) \varphi.
\]

Therefore we find

\[
E_n(\phi^{(\alpha)}) \varphi \leq cn^{\alpha-\beta} \|\phi^{(\beta)}\|_\varphi \leq cn^{\alpha-\beta} E_n(f^{(\beta)}) \varphi.
\]
Since
\[ E_n(\phi^{(\alpha)})_\nu = E_n(f^{(\alpha)})_\nu, \]
we conclude that (1.4) holds.

3° Let
\[ f^{(\beta)}_+(u) = \frac{1}{2} \left\{ |f^{(\beta)}(u)| + f^{(\beta)}(u) \right\} \text{ and } f^{(\beta)}_-(u) = \frac{1}{2} \left\{ |f^{(\beta)}(u)| - f^{(\beta)}(u) \right\} \]
for \( u \in \mathbb{R} \). Then
\[ f(x) = (f^{(\beta)}_+ \ast \Psi_\beta)(x) - (f^{(\beta)}_- \ast \Psi_\beta)(x), \]
\[ f^{(\alpha)}(x) = (f^{(\beta)}_+ \ast \Psi_{\beta - \alpha})(x) - (f^{(\beta)}_- \ast \Psi_{\beta - \alpha})(x) \]
for every \( 0 < \alpha \leq \beta - 1 \). Let \( t_{\alpha,n} \in \mathcal{T}_n^- (\Psi_{\beta - \alpha}), T_{\alpha,n} \in \mathcal{T}_n^+ (\Psi_{\beta - \alpha}) \) be such that
\[ \|f - t_{\alpha,n}\|_\varphi = E_n^- (\Psi_{\beta - \alpha})_{L^1(T)} \text{ and } \|T_{\alpha,n} - f\|_\varphi = E_n^+ (\Psi_{\beta - \alpha})_{L^1(T)} \]
for \( n = 1, 2, 3, \ldots \). Let also
\[ U^+_0 \left[ f \right] = (f^{(\beta)}_+ \ast T_{\alpha,n}) - (f^{(\beta)}_- \ast t_{\alpha,n}), \quad U^-_0 \left[ f \right] = (f^{(\beta)}_+ \ast t_{\alpha,n}) - (f^{(\beta)}_- \ast T_{\alpha,n}), \]
and for \( 0 < \alpha < \beta - 1 \) we set
\[ U^+_{\alpha,n} \left[ f \right] = (f^{(\beta)}_+ \ast T_{\alpha,n}) - (f^{(\beta)}_- \ast t_{\alpha,n}), \quad U^-_{\alpha,n} \left[ f \right] = (f^{(\beta)}_+ \ast t_{\alpha,n}) - (f^{(\beta)}_- \ast T_{\alpha,n}). \]
Hence,
\[ U^+_{\alpha,n} \left[ f \right] (x) - f^{(\alpha)}(x) = \frac{1}{2\pi} \int_T f^{(\beta)}_+(u) \left\{ T_{\alpha,n} (x - u) - \Psi_{\beta - \alpha} (x - u) \right\} du + \frac{1}{2\pi} \int_T f^{(\beta)}_-(u) \left\{ \Psi_{\beta - \alpha} (x - u) - t_{\alpha,n} (x - u) \right\} du \]
for every \( x \in \mathbb{R} \). Then
\[ U^+_{\alpha,n} \left[ f \right] (x) \geq f^{(\alpha)}(x) \]
for every \( x \in \mathbb{R} \). This implies that
\[ U^+_{\alpha,n} \left[ f \right] \in \mathcal{T}_n^+ (f^{(\alpha)}), \quad 0 \leq \alpha \leq \beta - 1. \]
Similarly,
\[ U^-_{\alpha,n} \left[ f \right] \in \mathcal{T}_n^- (f^{(\beta)}), \quad 0 \leq \alpha \leq \beta - 1. \]
We obtain
\[ \|U^\pm_{\alpha,n} \left[ f \right] - f^{(\alpha)}\|_\varphi \leq cn^{\alpha - \beta} \|f^{(\beta)}\|_\varphi, \]
and hence
\[ \|U^\pm_{\alpha,n} \left[ f \right] - f^{(\alpha)}\|_\varphi \leq cn^{\alpha - \beta} E_n \{f^{(\beta)}\}_\nu \]
for \( 0 \leq \alpha \leq \beta - 1 \). Since
\[ U_{\alpha,n} \left[ \phi \right] (x) \leq \phi^{(\alpha)}(x) \leq U^+_{\alpha,n} \left[ \phi \right] (x) \]
and
\[ \phi^{(\alpha)}(x) = f^{(\alpha)}(x) - Q_n^{(\alpha - \beta)}(x) \]
we have
\[ U_{\alpha,n} \left[ \phi \right] (x) + Q_n^{(\alpha - \beta)}(x) \leq f^{(\alpha)}(x) \leq U^+_{\alpha,n} \left[ \phi \right] (x) + Q_n^{(\alpha - \beta)}(x) \]
for every \( x \in \mathbb{R} \). Therefore
\[
E_n^\pm (f^{(\alpha)})_\varphi \leq \| U^{\pm}_{n, \alpha} [\varphi] - Q_n^{(\alpha-\beta)} - f^{(\alpha)} \|_\varphi
= \| U^{\pm}_{n, \alpha} [\varphi] - \varphi^{(\alpha)} \|_\varphi
\leq c n^{\alpha-\beta} E_n (f^{(\beta)})_\varphi,
\]
and the required result holds. \( \square \)

**1.3. Theorem.** Let \( \varphi \in QC_2^0 (0, 1) \). If \( 1 \leq \beta < \infty \), \( f \in W_\varphi^\beta (T) \) and \( \beta \geq \alpha \geq 0 \), then for \( n = 1, 2, 3, \ldots \) there is a constant \( c > 0 \) dependent only on \( \alpha \), \( \beta \) and \( \varphi \) such that
\[
\| f^{(\alpha)} (\cdot) - S_n^{(\alpha)} (\cdot, f) \|_\varphi \leq \frac{c}{n^{\beta-\alpha}} E_n (f^{(\beta)})_\varphi
\]
holds. If \( f \) is real valued and there exist polynomials \( t_n \in \mathcal{T}_n (f) \), \( T_n \in \mathcal{T}_n^+ (f) \) such that \( \| f - t_n \|_\varphi \leq c E_n (f)_\varphi \), \( \| T_n - f \|_\varphi \leq c E_n (f)_\varphi \), then for \( 0 \leq \alpha \leq \beta \) and \( n = 1, 2, 3, \ldots \),
\[
\| f^{(\alpha)} (\cdot) - t_n^{(\alpha)} \|_\varphi \leq \frac{c}{n^{\beta-\alpha}} E_n (f^{(\beta)})_\varphi, \quad \text{and}
\]
\[
\| T_n^{(\alpha)} - f^{(\alpha)} \|_\varphi \leq \frac{c}{n^{\beta-\alpha}} E_n (f^{(\beta)})_\varphi
\]
hold.

**Proof.** If \( \alpha = 0 \), then the results follows from Theorem 1.2. If \( \alpha = \beta \), then it was proved in [2] that
\[
\| f^{(\alpha)} (\cdot) - S_n^{(\alpha)} (\cdot, f) \|_\varphi \leq c E_n (f^{(\alpha)})_\varphi.
\]
From Theorem 1.2 and last inequality, (1.8) follows. Therefore
\[
W_n (f) := W_n (x, f) := \frac{1}{n+1} \sum_{\nu=n}^{2n} S_\nu (x, f), \quad n = 0, 1, 2, \ldots.
\]
Suppose that \( u := u (\cdot, f) \in \mathcal{T}_n (f) \) satisfies \( \| f - u \|_\varphi = E_n (f)_\varphi \). Since
\[
W_n (\cdot, f^{(\alpha)}) = W_n^{(\alpha)} (\cdot, f)
\]
we have
\[
\| f^{(\alpha)} (\cdot) - t_n^{(\alpha)} \|_\varphi \leq \| f^{(\alpha)} (\cdot) - W_n (\cdot, f^{(\alpha)}) \|_\varphi + \| u (\cdot, W_n (f)) - t_n^{(\alpha)} \|_\varphi
+ \| W_n^{(\alpha)} (\cdot, f) - u (\cdot, W_n (f)) \|_\varphi
:= I_1 + I_2 + I_3.
\]
Since \( \| W_n (f) \|_\varphi \leq 4 \| f \|_\varphi \), we get
\[
I_1 \leq \| f^{(\alpha)} (\cdot) - u (\cdot, f^{(\alpha)}) \|_\varphi + \| u (\cdot, f^{(\alpha)}) - W_n (\cdot, f^{(\alpha)}) \|_\varphi
= E_n (f^{(\alpha)})_\varphi + \| W_n (\cdot, u (f^{(\alpha)}) - f^{(\alpha)}) \|_\varphi
\leq 5 E_n (f^{(\alpha)})_\varphi.
\]
From Theorem 1.1, we get
\[
I_2 \leq 2 (n-1)^{\alpha} \| u (\cdot, W_n (f)) - t_n \|_\varphi
\]
and
\[
I_3 \leq 2 (2n - 2)^{\alpha} \| W_n (\cdot, f) - u (\cdot, W_n (f)) \|_\varphi \leq 2^{\alpha+1} n^{\alpha} E_n (W_n (f))_\varphi.
\]
Now we have
\[ \| u(\cdot, W_n(f)) - t_n \|_\varphi \leq \| u(\cdot, W_n(f)) - W_n(\cdot, f) \|_\varphi + \| W_n(\cdot, f) - f(\cdot) \|_\varphi + \| f(\cdot) - t_n \|_\varphi \]
\[ \leq E_n (W_n(f))_\varphi + 5E_n (f)_\varphi + cE_n^\infty (f)_\varphi. \]
Since
\[ E_n (W_n(f))_\varphi \leq \| W_n(f) - u \|_\varphi = \| W_n(f - u) \|_\varphi \leq 4E_n (f)_\varphi, \]
we get
\[ \| f^{(\alpha)} - t_n^{(\alpha)} \|_\varphi \leq 5E_n (f^{(\alpha)})_\varphi + 2n^\alpha E_n (W_n(f))_\varphi + 10n^\alpha E_n (f)_\varphi + 2n^{\alpha+1} \]
\[ \leq 5E_n (f^{(\alpha)})_\varphi + (18 + 2^{3+\alpha}) n^\alpha E_n (f)_\varphi + c2n^\alpha E_n^\infty (f)_\varphi. \]
Using Theorem 1.2 we get (1.9), and (1.10) can be proved using the same procedure. □

Direct theorem of trigonometric approximation:

1.4. Theorem. Let \( \varphi \in QC^q_2 (0, 1) \) and \( r \in \mathbb{R}^+ \). If \( f \in L_\varphi (T) \), then there is a constant \( c > 0 \), dependent only on \( r \) and \( \varphi \), such that the inequality
\[ E_n (f)_\varphi \leq c \omega_r^\varphi \left( f, \frac{1}{n+1} \right) \]
holds for \( n = 0, 1, 2, 3, \ldots \).

Proof. This is a consequence of [3, Theorem 2] and the property \( \omega_r^\varphi (f, \cdot) \leq c \omega_s^\varphi (f, \cdot) \), \( (r \geq s \in \mathbb{R}^+) \), of the smoothness moduli. □

1.5. Theorem. If \( r, \delta \in \mathbb{R}^+ \) and \( f \in B^\alpha \), \( \alpha \in \mathbb{R}^+ \), then there exists a constant \( c > 0 \) depending only on \( r \) and \( \varphi \) such that
\[ \omega_B^r (f, \delta) \leq c \delta^r \| f^{(r)} \|_B, \ \delta \geq 0 \]
holds.

Proof. For the function \( \chi_r (\cdot, h) \in L_1 (T) \) of [6, (20.15), p.376] we define
\[ (A_h^r f) (x) := (f + \chi_r (\cdot, h)) (x) = \frac{1}{2\pi} \int_T f (x - u) \chi_r (u, h) \, du, \ x \in T, \ h \in \mathbb{R}^+. \]
Then using Fubini’s theorem we get
\[ \| A_h^r f \|_B \leq \| \chi_r (\cdot, h) \|_{L_1 (T)} \| f \|_B \leq c \| f \|_B. \]
Since
\[ (\Delta_h^r f) (x) = h^r (A_h^r f)^{(r)} (x) = h^r A_h^r \left( f^{(r)} \right) (x) \]
we have from (1.15) that
\[ \sup_{|h| \leq \delta} \| \Delta_h^r f \|_B = \sup_{|h| \leq \delta} h^r \| A_h^r \left( f^{(r)} \right) \|_B \leq c \delta^r \| f^{(r)} \|_B, \]
from which we obtain (1.14). □

The converse theorem of trigonometric approximation:
1.6. Theorem. Let $\varphi \in QC^d_2(0,1)$ and $r \in \mathbb{R}^+$. If $f \in L_r(T)$, then there is a constant $c > 0$, dependent only on $r$ and $\varphi$, such that for $n = 0, 1, 2, 3, \ldots$

$$\omega^r_{\varphi}\left(f, \frac{n}{n+1}\right) \leq \frac{c}{(n+1)^r} \sum_{\nu=0}^{n-1} (n+1)^{r-1} E_{\nu}(f)_{\varphi}$$

holds.

Proof. The proof goes similarly to that of the proof of [2, Theorem 3]. \hfill \Box

From Theorems 1.4 and 1.6 we have the following corollaries:

1.7. Corollary. Let $\varphi \in QC^d_2(0,1)$ and $r \in \mathbb{R}^+$. If $f \in L_r(T)$ satisfies

$$E_n(f)_{\varphi} = 0 \left( n^{-\sigma} \right), \quad \sigma > 0, \quad n = 1, 2, \ldots,$$

then

$$\omega^r_{\varphi}(f, \delta) = \begin{cases} 0(\delta^n) & \text{if } r > \sigma, \\ 0(\delta^{|\log(1/\delta)|}) & \text{if } r = \sigma, \\ 0(\delta^r) & \text{if } r < \sigma, \end{cases}$$

holds. \hfill \Box

1.8. Definition. Let $\varphi \in QC^d_2(0,1)$ and $r \in \mathbb{R}^+$. If $f \in L_r(T)$, then for $0 < \sigma < r$ we set $\text{Lip}(r, \varphi) := \{ f \in L_r(T) : \omega^r_{\varphi}(f, \delta) = 0(\delta^r), \delta > 0 \}$.

The following constructive characterization of the Lipschitz class holds:

1.9. Corollary. Let $0 < \sigma < r$, $M \in QC^d_2(0,1)$ and $f \in L_r(T)$. Then the conditions

(a) $f \in \text{Lip}(r, \varphi)$,
(b) $E_n(f)_{\varphi} = 0 \left( n^{-\sigma} \right), \quad n = 1, 2, \ldots$,

are equivalent. \hfill \Box

1.10. Theorem. Let $\varphi \in QC^d_2(0,1)$ and $f \in L_r(T)$. If $\alpha \in \mathbb{R}^+$ and

$$\sum_{\nu=1}^{\infty} \nu^{\alpha-1} E_{\nu}(f)_{\varphi} < \infty,$$

then there exists a constant $c > 0$ dependent only on $\alpha$ and $\varphi$, such that

(1.16) \quad $E_n(f^{(\alpha)})_{\varphi} \leq c \left( n^{\alpha} E_n(f)_{\varphi} + \sum_{\nu=n+1}^{\infty} \nu^{\alpha-1} E_{\nu}(f)_{\varphi} \right)$

holds.

Proof. Since

$$\|f^{(\alpha)} - S_n(f^{(\alpha)})\|_{\varphi} \leq \|S_{2^{m+2}}(f^{(\alpha)}) - S_n(f^{(\alpha)})\|_{\varphi} + \sum_{k=m+2}^{\infty} \|S_{2^{k+1}}(f^{(\alpha)}) - S_{2^k}(f^{(\alpha)})\|_{\varphi}$$

we have for $2^m < n < 2^{m+1}$ that

$$\|S_{2^{m+2}}(f^{(\alpha)}) - S_n(f^{(\alpha)})\|_{\varphi} \leq c2^{(m+2)\alpha} E_n(f)_{\varphi} \leq c n^\alpha E_n(f)_{\varphi}.$$
On the other hand we find
\[
\sum_{k=m+2}^{\infty} \|S_{2k+1}(f^{(\alpha)}) - S_{2k}(f^{(\alpha)})\|_{\varphi} \\
\leq c \sum_{k=m+2}^{\infty} 2^{(k+1)\alpha} E_{2k}(f)_{\varphi} \leq c \sum_{k=m+2}^{\infty} \sum_{\mu=2k-1+1}^{2k} \mu^{\alpha-1} E_{\mu}(f)_{\varphi} \\
= c \sum_{\nu=2m+1+1}^{\infty} \nu^{\alpha-1} E_{\nu}(f)_{\varphi} \leq c \sum_{\nu=m+1}^{\infty} \nu^{\alpha-1} E_{\nu}(f)_{\varphi}.
\]

Therefore
\[
E_{n}(f^{(\alpha)})_{\varphi} \leq c \left( n^{\alpha} E_{n}(f)_{\varphi} + \sum_{\nu=n+1}^{\infty} \nu^{\alpha-1} E_{\nu}(f)_{\varphi} \right),
\]
\square

As a corollary of Theorems 1.4, 1.6 and 1.10,

1.11. Theorem. Let \( f \in W_{\varphi}^{\alpha}(T) \), \( r \in (0, \infty) \), and
\[
\sum_{\nu=1}^{\infty} \nu^{\alpha-1} E_{\nu}(f)_{\varphi} < \infty
\]
for some \( \alpha > 0 \). In this case, for \( n = 0, 1, 2, \ldots \) there exists a constant \( c > 0 \), dependent only on \( \alpha, r \) and \( \varphi \) such that
\[
\omega_{\varphi}^{r} \left( f^{(\alpha)}_{n+1} \right) \leq c \left( \frac{1}{(n+1)^{\alpha}} \sum_{\nu=0}^{n} (\nu+1)^{\alpha+r-1} E_{\nu}(f)_{\varphi} + \sum_{\nu=n+1}^{\infty} \nu^{\alpha-1} E_{\nu}(f)_{\varphi} \right)
\]
holds.
\square

As a corollary of Theorem 1.4.

1.12. Theorem. Let \( \varphi \in QC_{2}^{\beta}(0, 1) \), \( r \in \mathbb{R}^{+} \) and \( 1 \leq \beta < \infty \). If \( f \in W_{\varphi}^{\beta}(T) \) is real valued and \( 0 \leq \alpha \leq \beta - 1 \), then there is a constant \( c > 0 \), dependent only on \( r \) and \( \varphi \), such that the inequality
\[
E_{n}^{\pm}(f^{(\alpha)})_{\varphi} \leq \frac{c}{n^{\beta-\alpha}} \omega_{\varphi}^{r} \left( f^{(\beta)}_{n+1} \right)
\]
holds for \( n = 1, 2, 3, \ldots \).
\square

References

[4] Doronin V. G. and Ligun A. A. Best one-sided approximation of the classes \( W_{r} V \) \((r > -1)\) by trigonometric polynomials in the \( L_{1} \) metric, Mat. Zametki 22 (3) (in Russian), 357–370, 1977.