SOME CONVEXITY PROPERTIES FOR TWO NEW P-VALENT INTEGRAL OPERATORS

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Received 11:11:2010 : Accepted 09:05:2011

Abstract
In this paper, we define two new general p-valent integral operators in the unit disc U, and obtain the convexity properties of these integral operators of p-valent functions on some classes of β-uniformly p-valent starlike and β-uniformly p-valent convex functions of complex order. As special cases, the convexity properties of the operators ∫₀^z (f(t)/t) p dt and ∫₀^z (g'(t))^α dt are given.

Keywords: Analytic functions, Integral operators, β-uniformly p-valent starlike and β-uniformly p-valent convex functions, Complex order.


1. Introduction and preliminaries
Let Ap denote the class of functions of the form
(1.1) f(z) = z^p + ∑_k=p+1^∞ a_k z^k, (p ∈ ℕ = {1, 2, …, }),
which are analytic in the open disc U = {z ∈ ℂ : |z| < 1}.

A function f ∈ S^p_γ(γ, α) is p-valently starlike of complex order γ (γ ∈ ℂ − {0}) and type α (0 ≤ α < p), that is, f ∈ S^p_γ(γ, α), if it satisfies the following inequality;
(1.2) ℜ \left\{ p + \frac{1}{γ} \left( \frac{f'}{f} \frac{z}{f(z)} - p \right) \right\} > α, (z ∈ U).

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Furthermore, a function \( f \in \mathcal{C}_p(\gamma, \alpha) \) is \( p \)-valently convex of complex order \( \gamma \) \((\gamma \in \mathbb{C} - \{0\})\) and type \( \alpha \) \((0 \leq \alpha < p)\), that is, \( f \in \mathcal{C}_p(\gamma, \alpha) \) if it satisfies the following inequality:

\[
(1.3) \quad \Re \left\{ p + \frac{1}{\gamma} \left( 1 + \frac{zf''(z)}{f'(z)} - p \right) \right\} > \alpha, \quad (z \in U).
\]

In particular cases, for \( p = 1 \) in the classes \( \mathcal{S}_p^*(\gamma, \alpha) \) and \( \mathcal{C}_p(\gamma, \alpha) \) of starlike functions of complex order \( \gamma \) \((\gamma \in \mathbb{C} - \{0\})\) and type \( \alpha \) \((0 \leq \alpha < p)\), and convex functions of complex order \( \gamma \) \((\gamma \in \mathbb{C} - \{0\})\) and type \( \alpha \) \((0 \leq \alpha < p)\), respectively, which were introduced and studied by Frasin [12].

Also, for \( \alpha = 0 \) in the classes \( \mathcal{S}_p^*(\gamma, \alpha) \) and \( \mathcal{C}_p(\gamma, \alpha) \), we obtain the classes \( \mathcal{S}_p^*(\gamma) \) and \( \mathcal{C}_p(\gamma) \), which are called \( p \)-valently starlike of complex order \( \gamma \) \((\gamma \in \mathbb{C} - \{0\})\), and \( p \)-valently convex of complex order \( \gamma \) \((\gamma \in \mathbb{C} - \{0\})\), respectively.

Setting \( p = 1 \) and \( \alpha = 0 \), we obtain the classes \( \mathcal{S}^*(\gamma) \) and \( \mathcal{C}(\gamma) \). The class \( \mathcal{S}^*(\gamma) \) of starlike functions of complex order \( \gamma \) \((\gamma \in \mathbb{C} - \{0\})\) was defined by Nasr and Aouf (see [18]), while the class \( \mathcal{C}(\gamma) \) of convex functions of complex order \( \gamma \) \((\gamma \in \mathbb{C} - \{0\})\) was considered earlier by Wiatrowski (see [25]). Note that \( \mathcal{S}_p^*(1, \alpha) = \mathcal{S}_p^*(\alpha) \) and \( \mathcal{C}_p(1, \alpha) = \mathcal{C}_p(\alpha) \) are, respectively, the classes of \( p \)-valently starlike and \( p \)-valently convex functions of order \( \alpha \) \((0 \leq \alpha < p)\) in \( U \). Also, we note that \( \mathcal{S}_p^*(\gamma) \) and \( \mathcal{C}_p(\gamma) \) are, respectively, the usual classes of starlike and convex functions of order \( \alpha \) \((0 \leq \alpha < 1)\) in \( U \). In special cases, \( \mathcal{S}_1^*(0) = \mathcal{S}^* \) and \( \mathcal{C}_1 = \mathcal{C} \) are, respectively, the familiar classes of starlike and convex functions in \( U \).

A function \( f \in \beta - \mathcal{US}_p(\alpha) \) is \( \beta \)-uniformly \( p \)-valently starlike of order \( \alpha \) \((-1 \leq \alpha < p)\), that is, \( f \in \beta - \mathcal{US}_p(\alpha) \) if it satisfies the following inequality:

\[
(1.4) \quad \Re \left\{ \frac{zf'(z)}{f(z)} \right\} > \beta \left| \frac{zf'(z)}{f(z)} - p \right| + \alpha, \quad (\beta \geq 0, \ z \in U).
\]

Furthermore, a function \( f \in \beta - \mathcal{UC}_p(\alpha) \) is \( \beta \)-uniformly \( p \)-valently convex of order \( \alpha \) \((-1 \leq \alpha < p)\), that is, \( f \in \beta - \mathcal{UC}_p(\alpha) \) if it satisfies the following inequality:

\[
(1.5) \quad \Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \beta \left| 1 + \frac{zf''(z)}{f'(z)} - p \right| + \alpha, \quad (\beta \geq 0, \ z \in U).
\]

These classes generalize various other classes which are worthy of mention here. For example for \( p = 1 \), the classes \( \beta - \mathcal{US}(\alpha) \) and \( \beta - \mathcal{UC}(\alpha) \) introduced by Bharti, Parvatham and Swaminathan (see [2]). Also, the class \( \beta - \mathcal{UC}(0) = \beta - \mathcal{UC} \) is the known class of \( \beta \)-uniformly convex functions [15]. Using an Alexander type relation, we can obtain the class \( \beta - \mathcal{US}_p(\alpha) \) in the following way:

\[
f \in \beta - \mathcal{UC}_p(\alpha) \iff \frac{zf'(z)}{p} \in \beta - \mathcal{US}_p(\alpha).
\]

The class \( \mathcal{UC}_1(0) = \mathcal{UC} \) of uniformly convex functions was defined by Goodman [14], while the class \( 1 - \mathcal{US}_1(0) = \mathcal{S}^p \) was considered by Rønning [24].

For \( f \in \mathcal{A}_p \), given by (1.1) and \( g(z) \) given by

\[
g(z) = z^n + \sum_{k=p+1}^{\infty} b_k z^k
\]

their convolution (or Hadamard product), denoted by \((f * g)\), is defined as

\[
(1.7) \quad (f * g)(z) = z^n + \sum_{k=p+1}^{\infty} a_k b_k z^k = (g * f)(z), \quad (z \in U).
\]
The \( n \)-th order Ruscheweyh derivative \( R^n : A_p \to A_p \) is defined by

\[
R^n f(z) = \frac{z^n}{(1 - z)^{n+p}} \ast f(z), \quad (n > -p).
\]

In terms of the binomial coefficients, we can rewrite (1.8) as follows:

\[
R^n f(z) = z^n + \sum_{k=p+1}^{\infty} \binom{n+k-1}{k-p} a_k z^k, \quad (n > -p).
\]

In particular, when \( n = \lambda \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \), it is easily observed from (1.8) and (1.9) that

\[
R^n f(z) = \frac{z^n (z^\lambda - p f(z))^\lambda}{\lambda!}, \quad (\lambda \in \mathbb{N}_0, \ p \in \mathbb{N}).
\]

The symbol \( R^n \) is called the Ruscheweyh derivative of \( n \)th order defined by Goel and Sohi [13].

By using the operator \( R^\lambda \ (\lambda \in \mathbb{N}_0) \) defined by (1.10), we introduce the new classes \( \beta-\mathcal{US}_p(\lambda, \gamma, \alpha) \) and \( \beta-\mathcal{UC}_p(\lambda, \gamma, \alpha) \) as follows:

1.1. Definition. Let \(-1 \leq \alpha < p, \beta \geq 0 \) and \( \gamma \in \mathbb{C} - \{0\} \). A function \( f \in A_p \) is in the class \( \beta-\mathcal{US}_p(\lambda, \gamma, \alpha) \) if and only if for all \( z \in \mathbb{U} \),

\[
\Re \left\{ p + \frac{1}{\gamma} \left( \frac{z (R^\lambda f(z))'}{(R^\lambda f(z))} - p \right) \right\} > \Re \left\{ \frac{1}{\gamma} \left( \frac{z (R^\lambda f(z))'}{(R^\lambda f(z))} + 1 - p \right) \right\} + \alpha.
\]

1.2. Definition. Let \(-1 \leq \alpha < p, \beta \geq 0 \) and \( \gamma \in \mathbb{C} - \{0\} \). A function \( f \in A_p \) is in the class \( \beta-\mathcal{UC}_p(\lambda, \gamma, \alpha) \) if and only if for all \( z \in \mathbb{U} \),

\[
\Re \left\{ p + \frac{1}{\gamma} \left( \frac{z (R^\lambda f(z))''}{(R^\lambda f(z))} + 1 - p \right) \right\} > \Re \left\{ \frac{1}{\gamma} \left( \frac{z (R^\lambda f(z))''}{(R^\lambda f(z))} + 1 - p \right) \right\} + \alpha.
\]

We note that by specializing the parameters \( \lambda, p, \gamma, \alpha \) in the classes \( \beta-\mathcal{US}_p(\lambda, \gamma, \alpha) \) and \( \beta-\mathcal{UC}_p(\lambda, \gamma, \alpha) \), these classes reduce to several well-known subclasses of analytic functions. For example, for \( p = 1 \) and \( \lambda = 0 \) the classes \( \beta-\mathcal{US}_p(\lambda, \gamma, \alpha) \) and \( \beta-\mathcal{UC}_p(\lambda, \gamma, \alpha) \) reduces to the classes \( \beta-\mathcal{US}(\gamma, \alpha) \) and \( \beta-\mathcal{UC}(\gamma, \alpha) \), respectively. The reader can find more information about these classes in Deniz, Orhan and Sokol [10], Orhan, Deniz and Raducanu [19] and Oros [20].

1.3. Definition. Let \( l = (l_1, l_2, \ldots, l_m) \in \mathbb{N}_0^m \), \( \mu = (\mu_1, \mu_2, \ldots, \mu_m) \in \mathbb{R}_+^m \) for all \( i = 1, m, m \in \mathbb{N} \). We define the following general integral operators

\[
J_{p, m, l, \mu} (f_1, f_2, \ldots, f_m) : A_p^m \to A_p
\]

and

\[
J_{p, m, l, \mu} (g_1, g_2, \ldots, g_m) : A_p^m \to A_p.
\]

where \( f_i, g_i \in A_p \) for all \( i = 1, m \) and \( R^i \) is defined by (1.10).
1.4. Remark. We note that if \( l_1 = l_2 = \cdots = l_m = 0 \) for all \( i = 1, m \), then the integral operator \( J_{p,m,t,\mu}(z) \) reduces to the operator \( F_p(z) \), which was studied by Frasin (see [11]). Upon setting \( p = 1 \) in the operator (1.13), we can obtain the integral operator \( F_m(z) \) which was studied by Oros and Oros (see [21]). For \( p = 1 \) and \( l_1 = l_2 = \cdots = l_m = 0 \) in (1.13), the integral operator \( J_{p,m,t,\mu}(z) \) reduces to the operator \( F_m(z) \) which was studied by Breaz and Breaz (see [6]). Observe that when \( p = m = 1, l_1 = 0 \) and \( \mu_1 = \mu \), we obtain the integral operator \( I_\mu(f)(z) \) which was studied by Pescar and Owa (see [22]), for \( \mu_1 = \mu \in (0, 1] \) a special case of the operator \( I_\mu(f)(z) \) was studied by Miller, Mocanu and Reade (see [17]). For \( p = m = 1, l_1 = 0 \) and \( \mu_1 = 1 \) in (1.13), we have the Alexander integral operator \( I(f)(z) \) in [1].

1.5. Remark. For \( l_1 = l_2 = \cdots = l_m = 0 \) in (1.14) the integral operator \( J_{p,m,t,\mu}(z) \) reduces to the operator \( G_p(z) \) which was studied by Frasin (see [11]). For \( p = 1 \) and \( l_1 = l_2 = \cdots = l_m = 0 \) in (1.14), the integral operator \( J_{p,m,t,\mu}(z) \) reduces to the operator \( G_{\mu_1,\mu_2,\ldots,\mu_m}(z) \) which was studied by Breaz, Owa and Breaz (see [8]). If \( p = m = 1, l_1 = 0 \) and \( \mu_1 = \mu \), we obtain the integral operator \( G(z) \) which was introduced and studied by Pfaltzgraff (see [23]) and Kim and Merkes (see [16]).

In this paper, we consider the integral operators \( J_{p,m,t,\mu}(z) \) and \( J_{p,m,t,\mu}(z) \) defined by (1.13) and (1.14), respectively, and study their properties on the classes \( \beta-\mu\mathcal{L}_p(\lambda, \gamma, \alpha) \) and \( \beta-\mu\mathcal{E}_p(\lambda, \gamma, \alpha) \). As special cases, the order of convexity of the operators \( \int_0^z \left( \frac{t}{y} \right)^\mu \, dt \) and \( \int_0^z (g^\prime(t))^\mu \, dt \) are given.

2. Sufficient conditions on the integral operator \( J_{p,m,t,\mu}(z) \)

First, in this section we prove a sufficient condition for the integral operator \( J_{p,m,t,\mu}(z) \) to be \( p \)-valently convex.

2.1. Theorem. Let \( l = (l_1, l_2, \ldots, l_m) \in \mathbb{N}_0^m \), \( \mu = (\mu_1, \mu_2, \ldots, \mu_m) \in \mathbb{R}_+^m \), \( -1 \leq \alpha_i < p \), \( \beta_i \geq 0 \), \( \gamma \in \mathbb{C} - \{0\} \) and \( f_i \in \beta_i-\mu\mathcal{L}_p(l_i, \gamma, \alpha_i) \) for all \( i = 1, m \). Moreover, suppose that these numbers satisfy the following inequality

\[
0 \leq p + \sum_{i=1}^m \mu_i (\alpha_i - p) < p.
\]

Then the integral operator \( J_{p,m,t,\mu}(z) \) defined by (1.13) is \( p \)-valently convex of complex order \( \gamma \) (\( \gamma \in \mathbb{C} - \{0\} \)) and type \( p + \sum_{i=1}^m \mu_i (\alpha_i - p) \).

Proof. From the definition (1.13), we observe that \( J_{p,m,t,\mu}(z) \in \mathcal{A}_p \). On the other hand, it is easy to see that

\[
J'_{p,m,t,\mu}(z) = pz^{p-1} \prod_{i=1}^m \left( \frac{R_i f_i(z)}{z^p} \right)^{\mu_i}.
\]

Now we differentiate (2.2) logarithmically and multiply by \( z \) to obtain

\[
\frac{z J''_{p,m,t,\mu}(z)}{J'_{p,m,t,\mu}(z)} + 1 - p = \sum_{i=1}^m \mu_i \left( \frac{z \left( R_i f_i \right)'(z)}{(R_i f_i)(z)} - p \right).
\]

Then multiplying the relation (2.3) with \( \frac{1}{\gamma} \),

\[
\frac{1}{\gamma} \left( \frac{z J''_{p,m,t,\mu}(z)}{J'_{p,m,t,\mu}(z)} + 1 - p \right) = \sum_{i=1}^m \frac{1}{\mu_i} \frac{1}{\gamma} \left( \frac{z \left( R_i f_i \right)'(z)}{(R_i f_i)(z)} - p \right).
\]
The relation (2.4) is equivalent to

$$\text{(2.5)} \quad p + \frac{1}{\gamma} \left( \frac{zF''_{p,m,l,\mu}(z)}{F'_{p,m,l,\mu}(z)} + 1 - p \right) = p + \sum_{i=1}^{m} \mu_i \left( p + \frac{1}{\gamma} \left( \frac{z(R_i^l f_i)'(z)}{(R_i^l f_i)(z)} - p \right) \right) - p \sum_{i=1}^{m} \mu_i.$$

Lastly, we calculate the real part of both sides of (2.5) and obtain

$$\Re \left\{ p + \frac{1}{\gamma} \left( \frac{zF''_{p,m,l,\mu}(z)}{F'_{p,m,l,\mu}(z)} + 1 - p \right) \right\} = \sum_{i=1}^{m} \mu_i \Re \left\{ p + \frac{1}{\gamma} \left( \frac{z(R_i^l f_i)'(z)}{(R_i^l f_i)(z)} - p \right) \right\} - p \sum_{i=1}^{m} \mu_i + p.$$

Since $f_i \in \beta_i - US_p(l_i, \gamma, \alpha_i)$ for all $i = 1, m$, from (1.11) and (2.6), we have

$$\Re \left\{ p + \frac{1}{\gamma} \left( \frac{zF''_{p,m,l,\mu}(z)}{F'_{p,m,l,\mu}(z)} + 1 - p \right) \right\} > \sum_{i=1}^{m} \mu_i \Re \left\{ z \frac{(R_i^l f_i)'(z)}{(R_i^l f_i)(z)} - p \right\} + \sum_{i=1}^{m} \mu_i (\alpha_i - p).$$

Because $\sum_{i=1}^{m} \mu_i \Re \left\{ z \frac{(R_i^l f_i)'(z)}{(R_i^l f_i)(z)} - p \right\} > 0$, for all $i = 1, m$, from (2.7), we obtain

$$\Re \left\{ p + \frac{1}{\gamma} \left( \frac{zF''_{p,m,l,\mu}(z)}{F'_{p,m,l,\mu}(z)} + 1 - p \right) \right\} > \sum_{i=1}^{m} \mu_i (\alpha_i - p).$$

Therefore, the operator $F_{p,m,l,\mu}(z)$ is $p$-valently convex of complex order $\gamma$ ($\gamma \in C - \{0\}$) and type $p + \sum_{i=1}^{m} \mu_i (\alpha_i - p)$. This evidently completes the proof of Theorem 2.1. □

2.2. Remark.

(1) Letting $\gamma = 1$ and $l_i = 0$ for all $i = 1, m$ in Theorem 2.1, we obtain [11, Theorem 2.1].

(2) Letting $p = 1$, $\gamma = 1$ and $l_i = 0$ for all $i = 1, m$ in Theorem 2.1, we obtain [4, Theorem 1].

(3) Letting $p = 1$, $\gamma = 1$ and $\alpha_i = l_i = 0$ for all $i = 1, m$ in Theorem 2.1, we obtain [7, Theorem 2.5].

(4) Letting $p = 1$, $\beta = 0$ and $l_i = 0$ for all $i = 1, m$ in Theorem 2.1, we obtain [3, Theorem 1].

(5) Letting $p = 1$, $\beta = 0$, $\alpha_i = \mu$ and $l_i = 0$ for all $i = 1, m$ in Theorem 2.1, we obtain [9, Theorem 1].

(6) Letting $p = 1$, $\beta = 0$, $\alpha_i = 0$ and $l_i = 0$ for all $i = 1, m$ in Theorem 2.1, we obtain [5, Theorem 1].

Putting $p = m = 1, l_i = 0, \mu_1 = \mu, \alpha_1 = \alpha, \beta_1 = \beta$ and $f_1 = f$ in Theorem 2.1, we have

2.3. Corollary. Let $\mu > 0$, $-1 \leq \alpha < 1$, $\beta \geq 0$, $\gamma \in C - \{0\}$ and $f \in \beta-US(\gamma, \alpha)$. If $0 \leq 1 + \mu (\alpha - 1) < 1$, then $\int_{0}^{\gamma} \left( \frac{f(t)}{\gamma} \right)^{\mu} dt$ is convex of complex order $\gamma$ ($\gamma \in C - \{0\}$) and type $\mu (\alpha - 1) + 1$ in $U$. □

2.4. Theorem. Let $l = (l_1, l_2, \ldots, l_m) \in N_0^m$, $\mu = (\mu_1, \mu_2, \ldots, \mu_m) \in \mathbb{R}_+^m$, $-1 \leq \alpha_i < p$, $\beta_i > 0$, $\gamma \in C - \{0\}$ for all $i = 1, m$ and

$$\text{(2.8)} \quad \left| \frac{z(R_i^l f_i)'(z)}{(R_i^l f_i)(z)} - p \right| > \frac{p + \sum_{i=1}^{m} \mu_i (\alpha_i - p)}{\sum_{i=1}^{m} \mu_i |\gamma|}.$$
for all $i = \overline{1,m}$, then the integral operator $\mathcal{G}_{p,m,l,\mu}(z)$ defined by (1.13) is $p$-valently convex of complex order $\gamma$ ($\gamma \in \mathbb{C} - \{0\}$).

Proof. From (2.7) and (2.8) we easily get $\mathcal{G}_{p,m,l,\mu}(z)$ is $p$-valently convex of complex order $\gamma$. $\square$

From Theorem 2.4, we easily get

2.5. Corollary. Let $l = (l_1, l_2, \ldots, l_m) \in \mathbb{N}_0^m$, $\mu = (\mu_1, \mu_2, \ldots, \mu_m) \in \mathbb{R}_+^m$, $-1 \leq \alpha_i < p$, $\beta_i > 0$, $\gamma \in \mathbb{C} - \{0\}$ for all $i = \overline{1,m}$ and

$$ R \left( \frac{z (R^i f_i)'(z)}{(R^i f_i)(z)} \right) > p - \frac{p + \sum_{i=1}^{m} \mu_i (\alpha_i - p)}{\sum_{i=1}^{m} \mu_i |\beta_i|}, $$

that is $R^i f_i \in S^p(\sigma)$, where $\sigma = (p + \sum_{i=1}^{m} \mu_i (\alpha_i - p)) / \sum_{i=1}^{m} \mu_i |\beta_i| > 0$ for all $i = \overline{1,m}$, then the integral operator $\mathcal{G}_{p,m,l,\mu}(z)$ is $p$-valently convex of complex order $\gamma$ ($\gamma \in \mathbb{C} - \{0\}$).

Putting $p = m = 1$, $l_1 = 0$, $\mu_1 = \mu$, $\alpha_1 = \alpha$, $\beta_1 = \beta$ and $f_1 = f$ in Corollary 2.5, we have

2.6. Corollary. Let $\mu > 0$, $-1 \leq \alpha < 1$, $\beta > 0$, $\gamma \in \mathbb{C} - \{0\}$ and $f \in S^p(\rho)$, where $\rho = |\mu (1 - \alpha) | |\gamma| / |\beta| \leq \rho < p$ and $0 \leq \rho < 1$, then the integral operator $\int_0^1 \left( \frac{f'(z)}{f(z)} \right)^p dz$ is convex of complex order $\gamma$ ($\gamma \in \mathbb{C} - \{0\}$) in $\mathcal{U}$.

$\square$

3. Sufficient conditions on the integral operator $\mathcal{G}_{p,m,l,\mu}(z)$

Next, in this section we give a sufficient condition for the integral operator $\mathcal{G}_{p,m,l,\mu}(z)$ to be $p$-valently convex.

3.1. Theorem. Let $l = (l_1, l_2, \ldots, l_m) \in \mathbb{N}_0^m$, $\mu = (\mu_1, \mu_2, \ldots, \mu_m) \in \mathbb{R}_+^m$, $-1 \leq \alpha_i < p$, $\beta_i > 0$, $\gamma \in \mathbb{C} - \{0\}$ and $f_i \in \mathcal{A}_\gamma(l_i, \gamma, \alpha_i)$ for all $i = \overline{1,m}$. Moreover, suppose that these numbers satisfy the following inequality

$$ 0 \leq p + \sum_{i=1}^{m} \mu_i (\alpha_i - p) < p. $$

Then the integral operator $\mathcal{G}_{p,m,l,\mu}(z)$ defined by (1.14) is $p$-valently convex of complex order $\gamma$ ($\gamma \in \mathbb{C} - \{0\}$) and type $p + \sum_{i=1}^{m} \mu_i (\alpha_i - p)$.

Proof. From the definition (1.14), we observe that $\mathcal{G}_{p,m,l,\mu}(z) \in \mathcal{A}_p$. On the other hand, it is easy to see that

$$ \mathcal{G}_{p,m,l,\mu}(z) = \left( \frac{R^i g_i(z)}{p^{z-1}} \right)^{\mu_i}. $$

Now, we differentiate (3.1) logarithmically to obtain

$$ \frac{\mathcal{G}_{p,m,l,\mu}'(z)}{\mathcal{G}_{p,m,l,\mu}(z)} = \frac{p - 1}{z} + \sum_{i=1}^{m} \mu_i \left( \frac{R^i g_i''(z)}{(R^i g_i)'(z)} \right), $$

Then multiplying this relation (3.2) with $z$, we obtain

$$ \frac{1}{\gamma} \left( \frac{z \mathcal{G}_{p,m,l,\mu}''(z)}{\mathcal{G}_{p,m,l,\mu}(z)} + 1 - p \right) = \sum_{i=1}^{m} \mu_i \frac{1}{\gamma} \left( \frac{z (R^i g_i)'(z)}{(R^i g_i)(z)} + 1 - p \right) $$
or

\[(3.3) \quad p + \frac{1}{\gamma} \left( \frac{zG_{p,m,l,\mu}''(z)}{G_{p,m,l,\mu}'(z)} + 1 - p \right) = p + \sum_{i=1}^{m} \mu_i \frac{1}{\gamma} \left( \frac{z(R^i g_i)''(z)}{(R^i g_i)'(z)} + 1 - p \right).\]

Taking the real part of both sides of (3.3), we have

\[
\Re \left\{ p + \frac{1}{\gamma} \left( \frac{zG_{p,m,l,\mu}''(z)}{G_{p,m,l,\mu}'(z)} + 1 - p \right) \right\}
\]
\[
P + p + \sum_{i=1}^{m} \mu_i \Re \left\{ \frac{1}{\gamma} \left( \frac{z(R^i g_i)''(z)}{(R^i g_i)'(z)} + 1 - p \right) \right\}
\]
\[
= p + p \sum_{i=1}^{m} \mu_i + \sum_{i=1}^{m} \mu_i \left\{ \beta_i \left( \frac{1}{\gamma} \left( \frac{z(R^i g_i)''(z)}{(R^i g_i)'(z)} + 1 - p \right) \right) \right\} + \alpha_i
\]
\[
> p + \sum_{i=1}^{m} \mu_i (\alpha_i - p) + \sum_{i=1}^{m} \frac{\mu_i \beta_i}{\gamma} \left| \frac{z(R^i g_i)''(z)}{(R^i g_i)'(z)} \right| + 1 - p
\]
\[
> p + \sum_{i=1}^{m} \mu_i (\alpha_i - p).
\]

Therefore, the operator \( G_{p,m,l,\mu}(z) \) is \( p \)-valently convex of complex order \( \gamma \) \((\gamma \in \mathbb{C} - \{0\})\) and type \( p + \sum_{i=1}^{m} \mu_i (\alpha_i - p) \). This evidently completes the proof of Theorem 3.1. \(\square\)

3.2. Remark.

1. Letting \( \gamma = 1 \) and \( l_i = 0 \) for all \( i = 1, m \) in Theorem 3.1, we obtain [11, Theorem 3.1].
2. Letting \( p = 1, \beta = 0 \) and \( l_i = 0 \) for all \( i = 1, m \) in Theorem 3.1, we obtain [3, Theorem 3].
3. Letting \( p = 1, \beta = 0, \alpha_i = \mu \) and \( l_i = 0 \) for all \( i = 1, m \) in Theorem 3.1, we obtain [9, Theorem 3].
4. Letting \( p = 1, \beta = 0, \alpha_i = 0 \) and \( l_i = 0 \) for all \( i = 1, m \) in Theorem 3.1, we obtain [5, Theorem 2].

Putting \( p = m = 1, l_i = 0, \mu_1 = \mu, \alpha_1 = \alpha, \beta_1 = \beta \) and \( g_1 = g \) in Theorem 3.1, we have

3.3. Corollary. Let \( \mu > 0, -1 \leq \alpha < 1, \beta \geq 0, \gamma \in \mathbb{C} - \{0\} \) and \( g \in \beta \text{-} \mathfrak{U} \mathfrak{C}(\gamma, \alpha) \). If \( 0 \leq 1 + \mu (\alpha - 1) < 1 \), then \( \int_0^1 (g'(t))^\mu \ dt \) is convex of complex order \( \gamma \) \((\gamma \in \mathbb{C} - \{0\})\) and type \( \mu (\alpha - 1) + 1 \) in \( \mathfrak{U} \).

3.4. Theorem. Let \( l = (l_1, l_2, \ldots, l_m) \in \mathbb{N}_0^m, \mu = (\mu_1, \mu_2, \ldots, \mu_m) \in \mathbb{R}_+^m, -1 \leq \alpha_i < p, \beta_i > 0, \gamma \in \mathbb{C} - \{0\} \) for all \( i = 1, m \) and

\[
\left| \frac{z(R^i g_i)''(z)}{(R^i g_i)'(z)} + 1 - p \right| > \frac{p + \sum_{i=1}^{m} \mu_i (\alpha_i - p)}{\sum_{i=1}^{m} \mu_i \alpha_i}.
\]
for all $i = \overline{1, m}$, then the integral operator $S_{p, m, l, \mu}(z)$ defined by (1.14) is $p$-valently convex of complex order $\gamma$ ($\gamma \in \mathbb{C} - \{0\}$).

Proof. From the proof of Theorem 3.1 and (3.5) we easily get $S_{p, m, l, \mu}(z)$ is $p$-valently convex of complex order $\gamma$. \hfill $\Box$

From Theorem 3.4, we easily get

3.5. Corollary. Let $l = (l_1, l_2, \ldots, l_m) \in \mathbb{N}_0^m$, $\mu = (\mu_1, \mu_2, \ldots, \mu_m) \in \mathbb{R}_+^m$, $-1 \leq \alpha_i < p$, $\beta_i > 0$, $\gamma \in \mathbb{C} - \{0\}$ for all $i = \overline{1, m}$ and $R^i g_i \in \mathbb{C}_\mu(\sigma)$, where $\sigma = p - \left( p + \sum_{i=1}^m \mu_i (\alpha_i - p) \right)/ \sum_{i=1}^m \mu_i \beta_i$, $0 \leq \sigma < p$ for all $i = \overline{1, m}$, then the integral operator $S_{p, m, l, \mu}(z)$ is $p$-valently convex of complex order $\gamma$ ($\gamma \in \mathbb{C} - \{0\}$). \hfill $\Box$

Putting $p = m = 1$, $l_1 = 0$, $\mu_1 = \mu$, $\alpha_1 = \alpha$, $\beta_1 = \beta$ and $g_1 = g$ in Corollary 3.5, we have

3.6. Corollary. Let $\mu > 0$, $-1 \leq \alpha < 1$, $\beta > 0$, $\gamma \in \mathbb{C} - \{0\}$ and $g \in \mathbb{C}(\rho)$, where $\rho = [\mu(\beta + (1 - \alpha) |\gamma|) - |\gamma|]/\mu \beta$; $0 \leq \rho < 1$, then the integral operator $\int_0^1 (g'(t))^\mu \ dt$ is convex of complex order $\gamma$ ($\gamma \in \mathbb{C} - \{0\}$) in $U$. \hfill $\Box$

Acknowledgement

The present investigation was supported by Atatürk University Rectorship under BAP Project (The Scientific and Research Project of Atatürk University) Project No: 2010/28.

References


