A COMMON FIXED POINT THEOREM FOR WEAKLY COMPATIBLE MAPPINGS IN SYMMETRIC SPACES SATISFYING AN INTEGRAL TYPE CONTRACTIVE CONDITION

Rakesh Tiwari*, S. K. Shrivastava† and V. K. Pathak§

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Abstract


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*Department Of Mathematics, Govt. V. Y. T. PG. Autonomous College, Durg (C.G.), 491001 India. E-mail: rakeshtiwari66@gmail.com
†Corresponding Author.
‡Department of Mathematics, D. D. U. University, Gorakhpur (U.P.), 273009 India. E-mail: sudhirpr66@gmail.com
§Department Of Mathematics, Govt. PG. College, Dhamtari (C.G.), 493773 India. E-mail: vkpath21162@yahoo.co.in
1. Introduction

A symmetric on a set $X$ is a nonnegative real valued function $d$ on $X \times X$ such that

(i) $d(x, y) = 0$ if and only if $x = y$,

(ii) $d(x, y) = d(y, x)$.

Hicks and Rhoades [4] established some common fixed point theorems in symmetric spaces using the fact that some of the properties of metrics are not required in the proofs of certain metric theorems.

Let $d$ be a symmetric on a set $X$ and for $r > 0$ and any $x \in X$, let $B(x, r) = \{y \in X : d(x, y) < r\}$. A topology $t(d)$ on $X$ is given by $U \in t(d)$ if and only if for each $x \in U$, $B(x, r) \subset U$ for some $r > 0$. A symmetric $d$ is a semi-metric if for each $x \in X$ and each $r > 0$, $B(x, r)$ is a neighborhood of $x$ in the topology $t(d)$. Note that $\lim_{n \to \infty} d(x_n, x) = 0$ if and only if $x_n \to x$ in the topology $t(d)$.

The following two axioms appeared in Wilson [9] for a symmetric space $(X, d)$.

(W.3) Given $\{x_n\}$, $x$ and $y$ in $X$,

$$\lim_{n \to \infty} d(x_n, x) = 0 \text{ and } \lim_{n \to \infty} d(x_n, y) = 0 \text{ imply } x = y.$$

(W.4) Given $\{x_n\}$, $\{y_n\}$ and $x$ in $X$,

$$\lim_{n \to \infty} d(x_n, x) = 0 \text{ and } \lim_{n \to \infty} d(x_n, y_n) = 0 \text{ imply that } \lim_{n \to \infty} d(y_n, x) = 0.$$

Commuting, weakly commuting, compatible and weakly compatible mappings have been frequently used to prove existence theorems in common fixed point theory. Recall that, Jungck and Rhoades [6] defined $S$ and $T$ to be weakly compatible if they commute at their coincidence points, i.e. if $Su = Tu$ for some $u \in X$, then $STu = TSu$.

Many examples in the literature illustrate that commuting implies weakly commuting implies compatible implies weakly compatible maps, but the converse need not be true (see [5] and [8]).

Amri and Moutawakil [2] have established some common fixed point theorems under strict contractive conditions on a metric space for mappings satisfying the property (E.A). Again recall that the pair $(S, T)$ satisfies the property (E.A) if there exists a sequence $\{x_n\}$ in $X$ such that

$$\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t \text{ for some } t \in X.$$

Most recently, W. Liu et al. [7] defined a common (E.A) property for pair of mappings as follows:

1.1. Definition. Let $A, B, S, T : X \to X$. The pairs $(A, S)$ and $(B, T)$ satisfy the common (E.A) property if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in $X$ such that

$$\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} By_n = \lim_{n \to \infty} Ty_n = t \in X.$$

If $B = A$ and $S = T$ in the above, we obtain the definition of the property (E.A).

Now we present an example of the above definition as follows:
1.2. Example. Let \( A, B, S \) and \( T \) be self maps on \( X = [0, 1] \), with the usual metric \( d(x, y) = |x - y| \), defined by:

\[
\begin{align*}
Ax &= \begin{cases} 
1 - \frac{x}{2} & \text{when } x \in \left[0, \frac{1}{2}\right), \\
1 & \text{when } x \in \left[\frac{1}{2}, 1\right],
\end{cases} \\
Bx &= 1 - x, \text{ for all } x \in X,
\end{align*}
\]

\[
\begin{align*}
Sx &= \begin{cases} 
1 - 2x & \text{when } x \in \left[0, \frac{1}{2}\right), \\
1 & \text{when } x \in \left[\frac{1}{2}, 1\right],
\end{cases} \\
T x &= 1 - \frac{x}{3}, \text{ for all } x \in X.
\end{align*}
\]

Let \( \{x_n\} \) and \( \{y_n\} \) be the sequences defined by \( x_n = \frac{1}{n+1} \) and \( y_n = \frac{1}{n^2+1} \). Then

\[
\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} By_n = \lim_{n \to \infty} Ty_n = 1 \in X.
\]

Thus the common (E.A) property is satisfied.

2. Preliminaries

In the sequel, we need a function \( \phi : \mathbb{R}_+ \to \mathbb{R}_+ \) satisfying the condition \( 0 < \phi(t) < t \) for each \( t > 0 \).

2.1. Definition. [3, Definition 4] Let \((X, d)\) be a symmetric space. We say that \((X, d)\) satisfies property (H.E) if given \(\{x_n\}, \{y_n\}\) and \(x\) in \(X\),

\[
\lim_{n \to \infty} d(x_n, x) = 0 \text{ and } \lim_{n \to \infty} d(y_n, x) = 0 \text{ imply } \lim_{n \to \infty} d(x_n, y_n) = 0.
\]

Recently Abdelkrim Aliouche [3] proved a common fixed point theorem for self mappings in a symmetric space under a contractive condition of integrals and satisfying a new property introduced recently in [2].

The main object of this paper is to prove a common fixed point theorem for weakly compatible mappings in the setting of symmetric spaces satisfying an integral type contractive condition and the common (E.A) property. In this process, we are dropping two conditions, namely the properties (H.E) and (W.4). Thus, this Theorem generalizes and improves the results of Aamri et al. [1, 2] and Abdelkrim Aliouche [3].

3. Main results

Now we present our main result.

3.1. Theorem. Let \( d \) be a symmetric for \( X \) which satisfies (W.3). Let \( A, B, S \) and \( T \) be self mappings of \( X \) such that

\[
\begin{align*}
&A(X) \subset T(X) \quad \text{and} \quad B(X) \subset S(X), \\
&\int_0^A d(Ax, By) \varphi(t) \, dt \leq \phi\left( \int_0^B aL(x, y) + (1-a)M(x, y) \varphi(t) \, dt \right),
\end{align*}
\]

for all \( x, y \in X \), where \( \varphi : \mathbb{R}_+ \to \mathbb{R}_+ \) is a Lebesgue-integrable mapping which is summable, non-negative and such that

\[
\int_0^\epsilon \varphi(t) \, dt > 0, \text{ for all } \epsilon > 0,
\]
where,

\[ L(x, y) = \max \{ d(Sx, Ty), d(Sx, By), d(By, Ty) \}, \]
\[ M(x, y) = \left[ \max \{ d^2(Sx, Ty), d(Sx, By) \} \right]^{1/2}, \]

and \( 0 \leq a \leq 1 \). Suppose that \((A, S)\) and \((B, T)\) satisfy the common (E.A) property and are weakly compatible. If one of the subspaces \(AX\), \(BX\), \(SX\) and \(TX\) of \(X\) is complete, then \(A, B, S\) and \(T\) have a unique common fixed point in \(X\).

**Proof.** Since \((A, S)\) and \((B, T)\) satisfies the common (E.A) property, there exist two sequences \(\{x_n\}\) and \(\{y_n\}\) in \(X\) such that

\[ \lim_{n \to \infty} d(Bx_n, z) = \lim_{n \to \infty} d(Tx_n, z) = \lim_{n \to \infty} d(Ay_n, z) = \lim_{n \to \infty} d(Sy_n, z) = 0 \]

for some \(z \in X\).

Now suppose that \(SX\) is a complete subspace of \(X\). Then \(z = Su\) for some \(u \in X\). Consequently, we have

\[ \lim_{n \to \infty} d(Ay_n, Su) = \lim_{n \to \infty} d(Bx_n, Su) = \lim_{n \to \infty} d(Tx_n, Su) = \lim_{n \to \infty} d(Sy_n, Su) = 0. \]

If \(Au \neq z\), using (3.2) we get

\[ \int_0^{d(Au, Bx_n)} \phi(t) dt \leq \phi \left( \int_0^{aL(u, x_n) + (1-a)M(u, x_n)} \phi(t) dt \right) \]

\[ < \int_0^{aL(u, x_n) + (1-a)M(u, x_n)} \phi(t) dt, \]

where

\[ L(u, x_n) = \max \{ d(Su, Tx_n), d(Su, Bx_n), d(Bx_n, Tx_n) \}, \]
\[ M(u, x_n) = \left[ \max \{ d^2(Su, Tx_n), d(Su, Bx_n) \} \right]^{1/2}. \]

Taking \(n \to \infty\), we get \(L(u, x_n) = 0\), and \(M(u, x_n) = 0\) respectively. Now (3.4) becomes

\[ \lim_{n \to \infty} \int_0^{d(Au, Bx_n)} \phi(t) dt = 0, \]

and (3.3) implies that \( \lim_{n \to \infty} d(Au, Bx_n) = 0 \). By (W.3), we have \(z = Au = Su\).

Since \(AX \subset TX\), there exists \(v \in X\) such that \(z = Au = Tv\).

If \(Bv \neq z\), using (3.2) we have

\[ \int_0^{d(z, Bv)} \phi(t) dt = \int_0^{d(Au, Bv)} \phi(t) dt \]
\[ \leq \phi \left( \int_0^{aL(u, v) + (1-a)M(u, v)} \phi(t) dt \right), \]
from which we get \( L(u, v) = d(z, Bv) \), and \( M(u, v) = \left[ d^2(z, Bv) \right]^{1/2} \), respectively. Now (3.5) becomes

\[
\int_0^{d(Au, Bv)} \varphi(t) dt \leq \phi \left( \int_0^{ad(z, Bv) + (1-a)d(z, Bv)} \varphi(t) dt \right),
\]

\[
= \phi \left( \int_0^{d(z, Bv)} \varphi(t) dt \right),
\]

\[
< \int_0^{d(z, Bv)} \varphi(t) dt,
\]

which is a contradiction. Hence

\[
\int_0^{d(z, Bv)} \varphi(t) dt = 0,
\]

and (3.3) implies that \( z = Bv = Tv \).

The pair \((A, S)\) is weakly compatible, so \( ASu = SAu \) whenever \( Au = Su \), which implies \( Az = Sz \).

Let us show that \( z \) is a common fixed point of \( A, B, S \) and \( T \).

If \( z \neq Az \), using (3.2) we get

\[
\int_0^{d(z, Az)} \varphi(t) dt = \int_0^{d(Az, Bv)} \varphi(t) dt
\]

\[
\leq \phi \left( \int_0^{aL(z, v) + (1-a)M(z, v)} \varphi(t) dt \right),
\]

so \( L(z, v) = d(z, Az) \), and \( M(z, v) = \left[ d^2(z, Az) \right]^{1/2} \) respectively. Now (3.6) becomes

\[
\int_0^{d(Az, z)} \varphi(t) dt \leq \phi \left( \int_0^{ad(Az, z) + (1-a)d(Az, z)} \varphi(t) dt \right),
\]

\[
< \int_0^{d(z, Az)} \varphi(t) dt,
\]

which is a contradiction. Therefore

\[
\int_0^{d(z, Az)} \varphi(t) dt = 0,
\]

and (3.3) implies that \( z = Az = Sz \).

Similarly, the weak compatibility of \( B \) and \( T \) implies \( BTv = TBv \), i.e. \( Bz = Tz \). If \( z \neq Bz \), by using (3.2) and (3.3) a similar calculation to the above yields \( z = Bz = Tz \). Thus \( z \) is a common fixed point of \( A, B, S \) and \( T \).

When \( TX \) is assumed to be complete subspace of \( X \), then the proof is similar. On the other hand the cases in which \( AX \) or \( BX \) is a complete subspace of \( X \) are similar to the cases in which \( TX \) or \( SX \) is complete, respectively, by (3.1).
For the uniqueness of the common fixed point \( z \), let \( w \neq z \) be another common fixed point of \( A, B, S \) and \( T \). Then using (3.2), we obtain
\[
\int_0^{d(z,w)} \varphi(t) dt = \int_0^{d(Ax,Bw)} \varphi(t) dt \\
\leq \phi \left( \int_0^{aL(z,w) + (1-a)M(z,w)} \varphi(t) dt \right) \\
= \phi \left( \int_0^{aL(z,w) + (1-a)d(z,w)} \varphi(t) dt \right) \\
\leq \phi \left( \int_0^{d(z,w)} \varphi(t) dt \right) \\
< \int_0^{d(z,w)} \varphi(t) dt,
\]
which is a contradiction. Therefore \( \int_0^{d(z,w)} \varphi(t) dt = 0 \), and (3.3) implies that \( z = w \). □

3.2. Remark. If we take \( a = 1 \) in Theorem 3.1 we obtain an improved and more general version of Aliouche [3, Theorem 1].

For \( \varphi(t) = 1 \) in Theorem 3.1, we obtain the following corollary.

3.3. Corollary. Let \( d \) be a symmetric for \( X \) which satisfies (W3). Let \( A, B, S \) and \( T \) be self mappings of \( X \) such that
\[
(3.7) \quad A(X) \subset T(X) \quad \text{and} \quad B(X) \subset S(X),
\]
\[
(3.8) \quad d(Ax, By) \leq \phi(aL(x, y) + (1 - a)M(x, y))
\]
for all \( x, y \in X \), where
\[
L(x, y) = \max\{d(Sx, Ty), d(Sx, By), d(By, Ty)\},
\]
\[
M(x, y) = \left[ \max \{d^2(Sx, Ty), d(Sx, By)d(By, Ty), d(Sx, Ty)d(Sx, By), \right.
\]
\[
\left. d(Sx, Ty)d(By, Ty), d^2(By, Ty)\} \right]^{1/2}
\]
and \( 0 < a < 1 \). Suppose that \( (A, S) \) and \( (B, T) \) satisfies the common (E.A) property and are weakly compatible. If one of the subspaces \( AX, BX, SX \) or \( TX \) of \( X \) is complete, then \( A, B, S \) and \( T \) have a unique common fixed point of \( X \). □

3.4. Remark. If we take \( a = 1 \) in Corollary 3.3, we obtain an improved version of [1, Theorem 2.2].

Again, for \( B = A \) and \( T = S \) in Theorem 3.1, we get the following corollary.

3.5. Corollary. Let \( d \) be a symmetric on \( X \) which satisfies (W3). Let \( A \) and \( S \) be self mappings of \( X \) such that
\[
(3.9) \quad A(X) \subset S(X),
\]
\[
(3.10) \quad \int_0^{d(Ax, Ay)} \varphi(t) dt \leq \phi \left( \int_0^{aL(x, y) + (1-a)M(x, y)} \varphi(t) dt \right),
\]
for all \( x, y \in X \), where \( \varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) is a Lebesgue-integrable mapping which is summable, non-negative and satisfies (3.3),
\[
L(x, y) = \max\{d(Sx, Sy), d(Sx, Ay), d(Ay, Sy)\},
\]
\[
M(x, y) = \left[ \max \{d^2(Sx, Sy), d(Sx, Ay)d(Ay, Sy), d(Sx, Sy)d(Sx, Ay), \right. \]
\[
\left. d(Sx, Sy)d(Ay, Sy), d^2(Ay, Sy)\} \right]^{1/2},
\]
and $0 \leq a \leq 1$. Suppose that $(A,S)$ satisfies the property (E.A) and is weakly compatible. If one of the subspaces $AX$ or $SX$ of $X$ is complete, then $A$ and $S$ have a unique common fixed point in $X$. □

Again for $\varphi(t) = 1$ in Corollary 3.5, we obtain the following corollary.

3.6. Corollary. Let $A$ and $B$ be self mappings of $X$ such that
\begin{align}
A(X) & \subset S(X), \tag{3.11} \\
d(Ax, Ay) & \leq \varphi(aL(x, y) + (1 - a)M(x, y)) \tag{3.12}
\end{align}
for all $x, y \in X$, where
$$
L(x, y) = \max\{d(Sx, Sy), d(Sx, Ay), d(Ay, Sy)\}, \\
M(x, y) = \left[ \max\{d^2(Sx, Sy), d(Sx, Ay)d(Ay, Sy), d(Sx, Sy)d(Sx, Ay), d(Sx, Ay)d(Ay, Sy), d^2(Ay, Sy)\} \right]^{1/2}
$$
and $0 \leq a \leq 1$. Suppose that $(A,S)$ satisfies the property (E.A) and is weakly compatible. If one of the subspaces $AX$ or $SX$ of $X$ is complete, then $A$ and $S$ have a unique common fixed point in $X$. □

3.7. Remark. If we take $a = 1$ in Corollary 3.5, we obtain a generalized and improved form of [1, Theorem 2.1].

Since the non-compatibility of two mappings in a symmetric space implies that they satisfy the common (E.A) property, we get the following corollary.

3.8. Corollary. Let $d$ be a symmetric for $X$ which satisfies (W.3). Let $A, B, S$ and $T$ be self mappings of $X$ satisfying (3.1) and (3.2) for all $x, y \in X$, where $\varphi: \mathbb{R}_+ \to \mathbb{R}_+$ is a Lebesgue-integrable mapping which is summable, non-negative and satisfies (3.3). Suppose that $(A,S)$ and $(B,T)$ are weakly compatible but noncompatible. If one of the subspaces $AX$, $BX$, $SX$ or $TX$ of $X$ is complete, then $A$, $B$, $S$ and $T$ have a unique common fixed point in $X$. □

3.9. Remark. If we take $\varphi(t) = 1$ in Corollary 3.8, we obtain a generalized and improved form of [2, Theorem 2].

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