AN OVERVIEW OF INTUITIONISTIC FUZZY SUPRATOPOLOGICAL SPACES

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Abstract

The purpose of this paper is to present some fundamental results on the so called “intuitionistic fuzzy supratopological spaces”, and then give some introductory results about several connectedness concepts and a notion of compactness in these spaces. The product of two intuitionistic fuzzy supratopological spaces is also considered and some results on the productivity of connectedness and compactness are presented.

Keywords: Intuitionistic fuzzy set, Intuitionistic fuzzy topology, Intuitionistic fuzzy supratopology, Intuitionistic fuzzy supratopological space, Fuzzy supracontinuity, Fuzzy super supraconnectedness, Fuzzy C5-supraconnectedness, Fuzzy strong supraconnectedness, Fuzzy Cs-supraconnectedness, Intuitionistic fuzzy supracompactness.

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1. Introduction

First we give the concept of “intuitionistic fuzzy set” defined by Atanassov as a generalization of the concept of fuzzy set given by Zadeh [12].

1.1. Definition. [1, 2] Let X be a nonempty set. An intuitionistic fuzzy set (IFS for short) on X is an object of the form

\[ A = \{(x, \mu_A(x), \gamma_A(x)) : x \in X\} \]

where the fuzzy sets \( \mu_A : X \rightarrow [0, 1] \) and \( \gamma_A : X \rightarrow [0, 1] \), denote, respectively, the membership function and the nonmembership function of A, and satisfy \( 0 \leq \mu_A(x) + \gamma_A(x) \leq 1 \) for each \( x \in X \). An IFS \( A = \{(x, \mu_A(x), \gamma_A(x)) : x \in X\} \) can be written in the form \( A = \langle x, \mu_A, \gamma_A \rangle \), or simply \( A = \langle \mu_A, \gamma_A \rangle \).

1.2. Definition. [2] Let \( A = \{(x, \mu_A(x), \gamma_A(x)) : x \in X\} \) and \( B = \{(x, \mu_B(x), \gamma_B(x)) : x \in X\} \) be IFS’s on X. Then

(a) \( \overline{A} = \{(x, \gamma_A(x), \mu_A(x)) : x \in X\} \) (the complement of A);

(b) \( A \cap B = \{(x, \mu_A(x) \land \mu_B(x), \gamma_A(x) \lor \gamma_B(x)) : x \in X\} \) (the meet of A and B);

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(c) \( A \cup B = \{ x, \mu_A(x) \lor \mu_B(x), \gamma_A(x) \land \gamma_B(x) : x \in X \} \) (the join of \( A \) and \( B \));
(d) \( A \subseteq B \Leftrightarrow \mu_A(x) \leq \mu_B(x) \) and \( \gamma_A(x) \geq \gamma_B(x) \) for each \( x \in X \);
(e) \( A = B \Leftrightarrow A \subseteq B \) and \( B \subseteq A \);
(f) \( \{A = \{\{x, \mu_A(x), 1-\mu_A(x)\} : x \in X\} \) and \( \emptyset A = \{\{x, 1-\gamma_A(x), \gamma_A(x)\} : x \in X\};
(g) \( \underline{1} = \{(x, 1, 0) : x \in X\}, \underline{0} = \{(x, 0, 1) : x \in X\}

Now we shall give some fundamental definitions concerning intuitionistic fuzzy supra-topological spaces:

1.3. Definition. [10] A family \( \tau \) of IFS’s on \( X \) is called an intuitionistic fuzzy supratopology (IFST for short) on \( X \) if \( \emptyset, \underline{1} \in \tau \) and \( \tau \) is closed under arbitrary suprema. Then we call the pair \( (X, \tau) \) an intuitionistic fuzzy supratopological space (IFSTS for short).

Each member of \( \tau \) is called an intuitionistic fuzzy supraopen set and the complement of an intuitionistic fuzzy supraopen set is called an intuitionistic fuzzy supraclosed set. The intuitionistic fuzzy supraopen set of an IFS \( A \) is denoted by s-cl \( (A) \). Here s-cl \( (A) \) is the intersection of all intuitionistic fuzzy supraclosed sets containing \( A \). The intuitionistic fuzzy supratopology of \( A \) will be denoted by s-int \( (A) \). Here, s-int \( (A) \) is the union of all intuitionistic fuzzy supraopen sets contained in \( A \).

1.4. Example. Let \( X = \{a, b\} \) and
\[
A = \langle x, (0.3, 0.4), (0.4, 0.5) \rangle, \quad B = \langle x, (0.4, 0.2), (0.5, 0.3) \rangle.
\]
Then \( \tau = \{\emptyset, \underline{1}, A, B, A \cup B\} \) is an IFST on \( X \).

Now let \( IF(X) \) denote the family of all intuitionistic fuzzy sets in \( X \).

1.5. Proposition. If \( \beta \) is an intuitionistic fuzzy supratopology on \( X \), then
\[
\tau_\beta = \{T \in IF(X) : (\forall A)(A \in \beta \implies T \cap A \in \beta)\}
\]
is an intuitionistic fuzzy topology on \( X \) and \( \tau_\beta \subseteq \beta \).

Proof. (1). \( A \in \beta \implies \emptyset \cap A = \emptyset \in \beta \) and \( \underline{1} \cap A = A \in \beta \). Hence \( \emptyset, \underline{1} \in \tau_\beta \).

(2). Let \( \{A_i\} \subseteq \tau_\beta \) and \( A \in \beta \). Since
\[
\Big( \bigcup A_i \Big) \cap A = \Big( \bigcup (A_i \cap A) \Big) \in \beta,
\]
\[
\Big( \bigvee \mu_{A_i}, \bigwedge \gamma_{A_i} \Big) \cap \langle \mu_A, \gamma_A \rangle = \langle \bigvee (\mu_{A_i} \land \mu_A), \bigwedge (\gamma_{A_i} \lor \gamma_A) \rangle
\]
\[
= \bigvee (\mu_{A_i} \land \mu_A, \gamma_{A_i} \lor \gamma_A)
\]
\[
= \bigcup (A_i \cap A) \in \beta,
\]
we have \( \bigcup A_i \in \tau_\beta \).

(3). Let \( A_1, A_2, A \in \tau_\beta \). Then
\[
(A_1 \cap A_2) \cap A = (\mu_{A_1} \land \mu_{A_2}, \gamma_{A_1} \lor \gamma_{A_2}) \cap (\mu_A, \gamma_A)
\]
\[
= (\mu_{A_1} \land \mu_{A_2}) \land \mu_A, (\gamma_{A_1} \lor \gamma_{A_2}) \lor \gamma_A
\]
\[
= (\mu_{A_1} \land (\mu_{A_2} \land \mu_A), (\gamma_{A_1} \lor (\gamma_{A_2} \lor \gamma_A))
\]
\[
= A_1 \cap (A_2 \cap A)
\]
Since \( A_2 \cap A \in \beta \) and \( A_1 \in \tau_\beta \), \( A_1 \cap (A_2 \cap A) \in \beta \). Hence, \( A_1 \cap A_2 \in \tau_\beta \). Thus \( \tau_\beta \) is an intuitionistic fuzzy topology on \( X \). If \( T \in \tau_\beta \) then \( T \in \beta \), since \( \underline{1} \in \beta \) and \( T \cap \underline{1} = T \in \beta \). \( \square \)
Let \((X, \tau)\) be an intuitionistic fuzzy supratopological space. Then \(\tau_1 = \{ \mu_A : A \in \tau \}\) is a fuzzy supratopological space on \(X\) in the sense of El-Monsef and Ramadan [8], and \(\tau_2 = \{ \gamma_A : A \in \tau \}\) is the family of all fuzzy supraclosed sets of the fuzzy supratopological space \(\tau_2 = \{ 1 - \gamma_A : A \in \tau \}\) on \(X\). Furthermore, in \((X, \tau_1)\) and \((X, \tau_2)\), the fuzzy sets \(\mu_A\) and \(\gamma_A\) are fuzzy supraopen and fuzzy supraclosed sets, respectively. On the other hand, since \(\tau_1\) is a fuzzy supratopology on \(X\),

\[
T_{\tau_1} = \{ P \in \mathcal{P}^X : \mu_A \in \tau_1 \implies P \wedge \mu_A \in \tau_1 \}
\]

is a fuzzy topology on \(X\), and \(T_{\tau_1} \subseteq \tau_1\) (see Turanlı and Çoker [9]).

1.6. Definition. [6] Let \((X, \tau)\) be an IFSTS. The intuitionistic fuzzy set \(A\) is called

(a) **intuitionistic fuzzy semiopen** in \(X\) iff there exists \(B \in \tau\) such that \(B \subseteq A \subseteq s-cl(B)\).

(b) **intuitionistic fuzzy semiclosed** in \(X\) if there exists an intuitionistic fuzzy supraclosed set \(B\) in \(X\) such that \(s-int(B) \subseteq A \subseteq B\).

Arbitrary unions of intuitionistic fuzzy semiopen sets are again intuitionistic fuzzy semiopen sets. But the intersection of two intuitionistic fuzzy semiopen sets is not necessarily an intuitionistic fuzzy semiopen set (cf. [6]). Therefore the family of all intuitionistic fuzzy semiopen sets in \((X, \tau)\) is again an intuitionistic fuzzy supratopology on \(X\).

1.7. Definition. Let \(A\) be an IFS in an intuitionistic fuzzy supratopological space.

(a) If \(s-int(s-cl(A)) = A\), then we say that \(A\) is an **intuitionistic fuzzy regular supraopen set** in \(X\).

(b) If \(s-cl(s-int(A)) = A\), then we say that \(A\) is an **intuitionistic fuzzy regular supraclosed set** in \(X\).

The union of two intuitionistic fuzzy regular supraopen sets need not be an intuitionistic fuzzy regular supraopen set; hence the family of all intuitionistic fuzzy regular supraclosed sets in an IFSTS is not an IFST on \(X\), in general.

1.8. Definition. Let \((X, \tau)\) be an intuitionistic fuzzy supratopological space. An IFS \(A \in IF(X)\) is called

(a) **intuitionistic fuzzy semi-supraopen** iff \(A \subseteq s-cl(s-int(A))\).

(b) **intuitionistic fuzzy \(\alpha\)-supraopen** iff \(A \subseteq s-int(s-cl(s-int(A)))\).

(c) **intuitionistic fuzzy \(\beta\)-supraopen** iff \(A \subseteq s-cl(s-int(s-cl(A)))\).

(d) **intuitionistic fuzzy \(\psi\)-supraopen** iff \(A \subseteq s-int(s-cl(A))\).

1.9. Example. Let

\[
X = \{ a, b \} \quad E = \{ x, (0.3, 0.3), (0.3, 0.3) \} , \quad F = \{ x, (0.4, 0.4), (0.4, 0.4) \} .
\]

The family \(\tau = \{ \emptyset, X, E, F, E \cup F \}\) is an IFST on \(X\). Here \(E\) and \(F\) are intuitionistic fuzzy regular supraopen sets.

1.10. Definition. [4] Let \(A = \langle x, \mu_A, \gamma_A \rangle\) and \(B = \langle x, \mu_B, \gamma_B \rangle\) be two intuitionistic fuzzy sets in \(X\). \(A\) is said to be **quasi-coincident with** \(B\) (written \(A \equiv B\)) iff there exists an element \(x\) in \(X\) such that \(\mu_A(x) > \gamma_B(x)\) or \(\gamma_A(x) < \mu_B(x)\).

1.11. Example. Let \(X = \{ a, b \}\) and take the IFS’s

\[
A = \langle x, (0.4, 0.4), (0.4, 0.3) \rangle \quad \text{and} \quad B = \langle x, (0.3, 0.2), (0.5, 0.5) \rangle .
\]

Since, for all \(x\) in \(X\), \(\mu_A(x) > \gamma_B(x)\) or \(\gamma_A(x) < \mu_B(x)\), \(A \equiv B\) follows.

1.12. Proposition. [4] If \(A\) and \(B\) are intuitionistic fuzzy sets in \(X\), then

(a) \(A \equiv B \iff A \subseteq B\),

(b) \(A \equiv B \iff A \not\subseteq B\).
Here $\bar{q}$ denotes the negation of the relation $q$.

1.13. Corollary. $A \nsubseteq B \implies A \cap B \neq \emptyset$.

1.14. Definition. Let $(X, \tau)$ be an intuitionistic fuzzy supratopological space. The intuitionistic fuzzy sets $A$ and $B$ are said to be weakly supraseparated, if $s-cl(A) \subseteq \overline{B}$ and $s-cl(B) \subseteq \overline{A}$.

1.15. Definition. [5] Let $X$ and $Y$ be two nonempty sets and $f : X \to Y$ a function.
   
   (a) If $B = \{(y, \mu_B(y), \gamma_B(y)) : y \in Y\}$ is an IFS in $Y$, then the preimage of $B$ under $f$, denoted $f^{-1}(B)$, is the IFS in $X$ defined by
      \[ f^{-1}(B) = \{ (x, f^{-1}(\mu_B)(x), f^{-1}(\gamma_B)(x)) : x \in X \}. \]
   
   (b) If $A = \{ (x, \lambda_A(x), \eta_A(x)) : x \in X \}$ is an IFS in $X$, then the image of $A$ under $f$, denoted by $f(A)$, is the IFS in $Y$ defined by
      \[ f(A) = \{ (y, f(\lambda_A)(y), (1 - f(1 - \eta_A))(y)) : y \in Y \}. \]

   For the sake simplicity, the symbol $f_{-} (\eta_A)$ is often used to denote $1 - f(1 - \eta_A)$ [5].

1.16. Definition. Let $(X, \tau)$ and $(Y, \delta)$ be IFSTS’s, and $f : (X, \tau) \to (Y, \delta)$ a function. Then $f$ is said to be fuzzy supracontinuous if the preimage of each IFS in $\delta$ is in $\tau$.

2. Types of Fuzzy Supraconnectedness in IFSTS’s

Here we shall give several types of fuzzy connectedness in IFSTS’s as a generalization of Turanli-Çoker [9, 11]:

2.1. Definition. An intuitionistic fuzzy supratopological space $(X, \tau)$ is said to be fuzzy strongly supraconnected if $X$ has no non-zero intuitionistic fuzzy supraclosed sets $A$ and $B$ in $X$ such that $\mu_A + \mu_B \leq 1$ and $\gamma_A + \gamma_B \geq 1$.

2.2. Note. Since $\mu_A + \gamma_A \leq 1$ and $\mu_B + \gamma_B \leq 1$ it follows that $\gamma_B + \gamma_B \geq 1 \implies \mu_A + \mu_B \leq 1$. Hence in Definition 2.1 the condition $\mu_A + \mu_B \leq 1$ is redundant.

2.3. Theorem. Let $(X, \tau)$ be an IFSTS. Then the following are equivalent:
   
   (1) $(X, \tau)$ is fuzzy strongly supraconnected.
   
   (2) There exist no intuitionistic fuzzy supraopen sets $A$, $B$ in $X$ such that $A \neq \emptyset \neq B$ and $\mu_A + \mu_B \geq 1$.
   
   (3) There exist no $C \in \tau$ satisfying $C \neq \emptyset$, $\frac{1}{2} \leq \mu_C$, and there exist no $A, B \in \tau$ with $A \neq \emptyset \neq B$ and $A \cup B = X$.

Proof. (1) $\implies$ (2). Trivial in view of Note 2.2.

(2) $\implies$ (3). The second condition is a special case of (2) and the first condition is obtained by considering $A = B = C$.

(3) $\implies$ (1). If $(X, \tau)$ is not fuzzy strongly supraconnected then we have $A, B \in \tau$ with $A \neq \emptyset \neq B$ and $\mu_A + \mu_B \geq 1$. Then $C = A \cup B \in \tau$, $C \neq \emptyset$ and $2\mu_C \geq \mu_A + \mu_B \geq 1$, which contradicts (3).

2.4. Example. Let
   \[ X = (a, b), \quad A = (x, (0.3, 0.4), (0.4, 0.5)), \quad B = (x, (0.4, 0.5), (0.5, 0.3)). \]

Then $\tau = \emptyset, \emptyset, A, B, A \cup B$ is an intuitionistic fuzzy supratopology on $X$. There exist no intuitionistic fuzzy supraopen sets $A$, $B$ in $X$ such that $\mu_A + \mu_B \geq 1$, $\gamma_A + \gamma_B \leq 1$. Therefore $(X, \tau)$ is fuzzy strongly supraconnected.
2.5. Theorem. Let \((X, \tau), (Y, \delta)\) be intuitionistic fuzzy supratopological spaces and \(f : (X, \tau) \rightarrow (Y, \delta)\) a surjective fuzzy supracontinuous function. If \((X, \tau)\) is fuzzy strongly supraconnected, then \((Y, \delta)\) is also fuzzy strongly supraconnected.

Proof. Suppose that \(X\) is fuzzy strongly supraconnected but that there exists \(A, B \in \delta\) satisfying \(A \neq 1_X = B\) and \(\mu_A + \mu_B \geq 1\). Then for all \(x \in X\) we have
\[
\mu_{f^{-1}(A)}(x) + \mu_{f^{-1}(B)}(x) = f^{-1}(\mu_A)(x) + f^{-1}(\mu_B)(x) = \mu_A(f(x)) + \mu_B(f(x)) \geq 1,
\]
which contradicts Theorem 2.3 since \(f^{-1}(A), f^{-1}(B) \in \tau\) and clearly \(f^{-1}(A) \neq 1_X \neq f^{-1}(B)\).

\[\square\]

2.6. Definition. Let \((X, \tau)\) be an IFSTS. \(X\) is said to be fuzzy super supraconnected, if there exist no intuitionistic fuzzy regular supraopen set \(A\) in \(X\) such that \(0 \neq A \neq 1\).

2.7. Theorem. (cf. [11]) The following properties are equivalent:

(a) \((X, \tau)\) is fuzzy super supraconnected.

(b) For each intuitionistic fuzzy supraopen set \(A \neq 0\) in \(X\), \(s\text{-cl}(A) = \frac{1}{\tau}\).

(c) For each intuitionistic fuzzy supraclosed set \(A \neq 1\) in \(X\), \(s\text{-int}(A) = 0\).

(d) There exist no intuitionistic fuzzy supraopen sets \(A\) and \(B\) in \(X\) such that \(A \neq 0 \neq B\) and \(A \subseteq \overline{B}\).

(e) There exist no intuitionistic fuzzy supraopen sets \(A\) and \(B\) in \(X\) such that \(A \neq 0 \neq B\) and \(B = s\text{-cl}(A), A = s\text{-cl}(B)\).

(f) There exist no intuitionistic fuzzy supraopen sets \(A\) and \(B\) in \(X\) such that \(A \neq 1 \neq B\) and \(B = \overline{s\text{-int}(A), A = \overline{s\text{-int}(B)}}\).

2.8. Definition. Let \((X, \tau)\) be an IFSTS.

(a) \(X\) is called fuzzy \(C_\tau\)-supradisconnected if there exist two nonzero IFS’s \(A\) and \(B\) in \((X, \tau)\) such that \(A\) and \(B\) are weakly separated and \(\frac{1}{\tau} = A \cup B\).

(b) \(X\) is called fuzzy \(C_\tau\)-supraconnected if \((X, \tau)\) is not fuzzy \(C_\tau\)-supradisconnected.

2.9. Definition. Let \((X, \tau)\) be an IFSTS.

(a) \(X\) is said to be fuzzy \(C_\tau\)-supradisconnected if there exists an intuitionistic fuzzy supraopen and fuzzy supraconnected set \(A\) in \(X\) such that \(0 \neq A \neq 1\).

(b) \(X\) is called fuzzy \(C_\tau\)-supraconnected if \(X\) is not fuzzy \(C_\tau\)-supradisconnected.

(c) \(X\) is said to be fuzzy supraconnected if there exist intuitionistic fuzzy supraopen sets \(A \neq 0\) and \(B \neq 0\) such that \(\frac{1}{\tau} = A \cup B\) and \(A \cap B = \frac{0}{\tau}\).

(d) \(X\) is called fuzzy supraconnected if \(X\) is not fuzzy supradisconnected.

2.10. Theorem. Let \((X, \tau)\) be an IFSTS.

(a) If \((X, \tau)\) is fuzzy supradisconnected, then \((X, \tau)\) is fuzzy \(C_\tau\)-supradisconnected.

(b) If \((X, \tau)\) is fuzzy \(C_\tau\)-supraconnected, then \((X, \tau)\) is fuzzy supraconnected.

Proof. Straightforward.

\[\square\]

2.11. Theorem. Let \(f : (X, \tau) \rightarrow (Y, \Phi)\) be a surjective fuzzy supracontinuous function. If \(X\) is fuzzy supraconnected, then \(Y\) is also fuzzy supraconnected.

Proof. Immediate.

\[\square\]

2.12. Theorem. The following are equivalent:
(1) $(X, \tau)$ is fuzzy $C\beta$-supraconnected.
(2) There exist no intuitionistic fuzzy supraopen sets $A, B$ in $X$ such that $A \neq 0 \neq B$.
(3) There exist no intuitionistic fuzzy sets $A$ and $B$ in $X$ such that $A \neq 0 \neq B$.

$B = \mathcal{A}$, $B = \overline{\text{cl}}(A)$ and $A = \overline{\text{cl}}(B)$.

Proof. Straightforward. □

2.13. Remark. There is no relation between fuzzy strongly supraconnected and fuzzy $C\beta$-supraconnectedness in IFSTS’s. In other words, each of these properties does not imply the other.

2.14. Example. Let

$$X = \{a, b\}, \quad A = \{x, (0.3, 0.5), (0.6, 0.5)\}, \quad B = \{x, (0.6, 0.3), (0.3, 0.6)\}.$$ 

Then $\tau = \{\emptyset, 1, A, B, A \cup B\}$ is an IFSTS on $X$, $(X, \tau)$ is not fuzzy $C\beta$-supraconnected, but is fuzzy strongly supraconnected.

2.15. Example. Let

$$X = \{a, b\}, \quad A = \{x, (0.5, 0.6), (0.4, 0.3)\}, \quad B = \{x, (0.5, 0.4), (0.2, 0.4)\}.$$ 

Then $\tau = \{\emptyset, 1, A, B, A \cup B\}$ is an IFSTS on $X$, and for the intuitionistic fuzzy supraopen sets $A, B$ in $X$ we have $\mu_A + \mu_B \geq 1$, $\gamma_A + \gamma_B \leq 1$, whence $(X, \tau)$ is not fuzzy strongly supraconnected. However, it is fuzzy $C\beta$-supraconnected.

2.16. Note. Let $(X, \beta)$ be an IFSTS, $\tau_\beta$ the intuitionistic fuzzy topology on $X$ defined in Proposition 1.5. Since $\tau_\beta \subseteq \beta$, if $(X, \beta)$ has one of the connectedness properties defined above so does $(X, \tau_\beta)$. The converse is false in general. For example if $\beta$ is the fuzzy $C\beta$-supradisconnected IFST of Example 2.15 then $\tau_\beta = \{\emptyset, 1, A \cup B\}$, which is fuzzy $C\beta$-supraconnected.

3. Products of Intuitionistic Fuzzy Supratopological Spaces

Let $(X, \tau)$ and $(Y, \delta)$ be two intuitionistic fuzzy supratopological spaces, $A \in IF(X)$ and $B \in IF(Y)$. The product of $A$ and $B$ is defined by:

$$A \times B = \{(x, y), \mu_A(x) \wedge \mu_B(y), \gamma_A(x) \lor \gamma_B(y) : (x, y) \in X \times Y\}.$$ 

The product intuitionistic fuzzy supratopology on $X \times Y$ is defined as the coarsest intuitionistic fuzzy supratopology on $X \times Y$ making the projections $\pi_1 : X \times Y \to X$, $\pi_1(x, y) = x$ and $\pi_2 : X \times Y \to Y$, $\pi_2(x, y) = y$ fuzzy supracontinuous. Since for $G \in \tau$,

$$\pi_1^{-1}(\mu_G)(x, y) = \mu_G(\pi_1(x, y)) = \mu_G(x),$$

and likewise $\pi_2^{-1}(\gamma_G)(x, y) = \gamma_G(x)$ we see that

$$\pi_1^{-1}(G) = \{(x, y), \mu_G(x), \gamma_G(x)\} = G \times 1_Y.$$ 

In just the same way, for $H \in \delta$,

$$\pi_2^{-1}(H) = \{(x, y), \mu_H(y), \gamma_H(y)\} = 1_X \times H.$$ 

Hence $\{G \times 1_Y \mid G \in \tau\} \cup \{1_X \times H \mid H \in \delta\}$ is a base for the product intuitionistic fuzzy supratopology $\beta$ on $X \times Y$. However, since the families $\{G \times 1_Y \mid G \in \tau\}$ and
\{1 \times H \mid H \in \delta\} are both closed under arbitrary unions because \(\tau\) and \(\delta\) are intuitionistic fuzzy supratopologies, we obtain
\[ \beta = \{(G \times 1) \cup (1 \times H) \mid G \in \tau, H \in \delta\}. \]

3.1. Example. Let \(X = \{a, b\}\) and
\[ A = \langle x, (0.3, 0.4), (0.4, 0.5) \rangle, \quad B = \langle x, (0.4, 0.2), (0.5, 0.3) \rangle. \]
Now \(\tau = \{0X, 1X, A, B, A \cup B\}\) is an intuitionistic fuzzy supratopology on \(X\). Secondly, let \(Y = \{a, b\}\) and
\[ C = \langle y, (0.3, 0.2), (0.6, 0.5) \rangle, \quad D = \langle y, (0.4, 0.1), (0.5, 0.3) \rangle. \]
Then \(\delta = \{1Y, 1X, C, D, C \cup D\}\) is an intuitionistic fuzzy supratopology on \(Y\). The product of these spaces is as follows:
\[ \beta = \{\langle (x, y), (0.3, 0.4), (0.5, 0.6) \rangle, \langle (x, y), (0.4, 0.4), (0.5, 0.3) \rangle, \langle (x, y), (0.4, 0.1), (0.5, 0.2) \rangle, \langle (x, y), (0.4, 0.4), (0.5, 0.3) \rangle, \langle (x, y), (1, 1), (0, 0) \rangle, \langle (x, y), (0, 0), (1, 1) \rangle\} \]
In general a product of two IFSOS’s is not necessarily supraopen. For example in the above example \(A\) and \(D\) are supraopen, but
\[ A \times D = \langle (x, y), (0.3, 0.4), (0.5, 0.4) \rangle \notin \beta \]
On the other hand the supraclosed sets with respect to \((X \times Y, \beta)\) are precisely the products \(F \times K\), where \(F\) is supraclosed in \((X, \tau)\) and \(K\) supraclosed in \((Y, \delta)\). To see this we need only note that if \(G = F \in \tau\) and \(H = K \in \delta\) then
\[ (G \times 1) \cup (1 \times H) = (1 \times F) \cap (X \times H) = \mathcal{G} \times H = F \times K. \]

3.2. Theorem. Let \((X, \tau), (Y, \delta)\) be IFSST’s. Then the product IFSST \((X \times Y, \beta)\) is fuzzy strongly supraconnected if and only if \((X, \tau)\) and \((Y, \delta)\) are fuzzy strongly supraconnected.

Proof. \(\Rightarrow\). This follows by Theorem 2.5 since the projections \(\pi_1\) and \(\pi_2\) are surjective fuzzy supracontinuous functions.
\[ \iff \]
Let \((X, \tau)\) and \((Y, \delta)\) be fuzzy strongly supraconnected. Suppose that there exist \(A, B \in \beta\) with \(A \neq \frac{1}{2} \neq B\) and \(A \cup B = \frac{1}{2}\). By the above comments we have \(A = (G_1 \times 1) \cup (1 \times H_1)\) and \(B = (G_2 \times 1) \cup (1 \times H_2)\) where \(G_1, G_2 \in \tau\) and \(H_1, H_2 \in \delta\). By hypothesis we have \(G_1 \neq 1 \neq G_2, H_1 \neq 1 \neq H_2\) and
\[ 1 \times 1 = (G_1 \times 1) \cup (1 \times H_1) \cup (G_2 \times 1) \cup (1 \times H_2) = ((G_1 \cup G_2) \times 1) \cup (1 \times (H_1 \cup H_2)). \]
It follows easily that \( G_1 \cup G_2 = 1_Y \) or \( H_1 \cup H_2 = 1_Y \) (compare the proof of Theorem 4.5 below), which is a contradiction.

Now suppose we have \( C \in \beta \) with \( C \neq 1_X \) and \( \frac{1}{2} \leq \mu_C \). Again, \( C = (G \times 1_Y) \cup (1_X \times H) \) with \( G \in \tau, H \in \delta \), so \( G \neq 1_X \), \( H \neq 1_Y \) and \( \frac{1}{2} \leq \mu_G(x) \vee \mu_H(y) \) for all \( x \in X \) and \( y \in Y \). Since \((X, \tau)\) is fuzzy strongly supraconnected \( 1_X \supseteq \) by Theorem 2.3, so there exists \( x_0 \in X \) with \( \mu_G(x_0) < \frac{1}{2} \). In the same way we have \( y_0 \in Y \) with \( \mu_H(y_0) < \frac{1}{2} \), which gives the contradiction \( \mu_G(x_0) \vee \mu_H(y_0) < \frac{1}{2} \). Hence \((X \times Y, \beta)\) is fuzzy strongly supraconnected by Theorem 2.3.

If the product IFSTS \((X \times Y, \beta)\) is \( C_5 \)-supraconnected then clearly \((X, \tau)\) and \((Y, \delta)\) are also \( C_5 \)-supraconnected. However the converse does not hold in general, as shown by the following example.

3.3. Example. Let \( X = \{a, b\} \), \( Y = \{c\} \) and define \( G = (x, (0.1, 0.2), (0.2, 0.2)) \), \( G_1 = (x, (0.1, 0.2), (0.2, 0.4)) \) and \( H = (y, 0.2, 0.4) \), \( H_1 = (y, 0.2, 0.5) \). Then \( \tau = \{0_X, 1_X, G, G_1, G \cup G_1\} \) is an IFST on \( X \) and \( \delta = \{0_Y, 1_Y, H, H_1, H \cup H_1\} \) an IFST on \( Y \). Clearly \((X, \tau)\) and \((Y, \delta)\) are \( C_5 \)-supraconnected. If we denote the product of \( \tau \) and \( \delta \) by \( \beta \) then
\[
(G \times 1_Y) \cup (1_X \times H) = ((x, y), (0.2, 0.4), (0.2, 0.2)) \in \beta.
\]

On the other hand
\[
((x, y), (0.2, 0.4), (0.2, 0.2)) = \overline{G_1} \times \overline{H_1}
\]
is a supraclosed set in \((X \times Y, \beta)\) by the remarks above. This shows that \((X \times Y, \beta)\) is \( C_5 \)-supradisconnected.

4. Intuitionistic fuzzy supracompactness

First we present the basic concepts.

4.1. Definition. (cf. [5]) Let \((X, \tau)\) be an IFSTS.

(a) If a family \( \{ (x, \mu_G, \gamma_G) \mid i \in J \} \) of IFSOS’s in \( X \) satisfies the condition
\[
\bigcup \{ (x, \mu_G, \gamma_G) \mid i \in J \} = 1_X,
\]
then it is called a fuzzy supraopen cover of \( X \). A finite subfamily of a fuzzy supraopen cover \( (x, \mu_G, \gamma_G), i \in J \) of \( X \), which is also a fuzzy supraopen cover of \( X \), is called a finite subcover of \( (x, \mu_G, \gamma_G), i \in J \).

(b) A family \( \{ (x, \mu_{K_i}, \gamma_{K_i}) \mid i \in J \} \) of IFSCS’s in \( X \) satisfies the finite intersection property (FIP for short) if every finite subfamily \( \{ (x, \mu_{K_{i_k}}, \gamma_{K_{i_k}}) \mid k = 1, 2, 3, \ldots, n \} \) satisfies the condition
\[
\bigcap_{k=1}^n (x, \mu_{K_{i_k}}, \gamma_{K_{i_k}}) \neq 0_Y.
\]

4.2. Lemma. The following are equivalent:

1. \( \{ (x, \mu_G, \gamma_G) \mid i \in J \} \) is a cover of \( X \).
2. \( \bigvee \{ \mu_G(x) \mid i \in J \} = 1 \forall x \in X \).
3. \( \bigwedge \{ \gamma_G(x) \mid i \in J \} = 0 \forall x \in X \).
4.5. Theorem. Let \( (X, \tau, \delta) \) be IFSTS’s and \( f : X \to Y \) a fuzzy supracontinuous surjection. If \( (X, \tau) \) is fuzzy supracompact, then so is \( (Y, \delta) \).

\[ \text{Proof.} \]

For \( (X, \tau) \), there is \( Y \) and \( \phi \) such that \( X \to Y \) is a fuzzy supracontinuous function. It follows that \( X \) is fuzzy supracompact if and only if \( Y \) is fuzzy supracompact. Hence \( (Y, \delta) \) is fuzzy supracompact, as required. \( \square \)

4.4. Theorem. Let \( (X, \tau) \) be IFSTS’s and \( f : X \to Y \) a fuzzy supracontinuous surjection. If \( (X, \tau) \) is fuzzy supracompact, then so is \( (Y, \phi) \).

\[ \text{Proof.} \]

For \( (X, \tau) \), there is \( Y \) and \( \phi \) such that \( X \to Y \) is a fuzzy supracontinuous function. It follows that \( X \) is fuzzy supracompact if and only if \( Y \) is fuzzy supracompact. Hence \( (Y, \phi) \) is fuzzy supracompact, as required. \( \square \)

4.3. Definition. An IFSTS \((X, \tau)\) is called fuzzy supracompact if every fuzzy supraopen cover of \( X \) has a finite subcover.

Equivalently, \((X, \tau)\) is fuzzy supracompact if every family of IFSC’s with the FIP has a non-empty meet. Clearly, Lemma 4.2 will be useful in establishing fuzzy supracompactness.

It is known that fuzzy supracompactness is preserved under a fuzzy supracontinuous surjection:

4.5. Theorem. Let \((X, \tau)\), \((Y, \delta)\) be IFSTS’s and \((X \times Y, \beta)\) their product. Then the product \((X \times Y, \beta)\) is fuzzy supracompact if and only if \((X, \tau)\) and \((Y, \delta)\) are fuzzy supracompact.

\[ \text{Proof.} \]

\( \implies \). If \((X \times Y, \beta)\) is fuzzy supracompact then by Theorem 4.4 so are \((X, \tau)\) and \((Y, \delta)\), since the projections \(\pi_1\) and \(\pi_2\) are fuzzy supracompact. Hence \((X, \tau)\) and \((Y, \delta)\) are fuzzy supracompact.

\( \iff \). Let \((X, \tau)\), \((Y, \delta)\) be fuzzy supracompact. Since \(\beta = \{(G \times 1_Y) \cup (1_X \times H) \mid G \in \tau, H \in \delta\}\), a general fuzzy supraopen cover of \(X \times Y\) has the form

\[ C = \{(G_i \times 1_Y) \cup (1_X \times H_i) \mid i \in J, \}\]

where \(G_i = \langle x, \mu_{G_i}, \gamma_{G_i} \rangle \in \tau\) and \(H_i = \langle y, \mu_{H_i}, \gamma_{H_i} \rangle \in \delta\). We claim that \(G_i \mid i \in J\) is a cover of \(X\) or \(H_i \mid i \in J\) a cover of \(Y\). By Lemma 4.2 it is sufficient to prove \(\bigvee \{\mu_{G_i}(x) \mid i \in J\} = 1\) or \(\bigvee \{\mu_{H_i}(x) \mid i \in J\} = 1\). However, \(\bigvee \{\mu_{G_i}(x) \mid i \in J\} = (x, \mu_{G_i}(x) \vee \mu_{H_i}(y), \gamma_{G_i}(x) \wedge \gamma_{H_i}(y))\), so \(\bigvee \{\mu_{G_i}(x) \mid i \in J\} = \bigvee \{\mu_{H_i}(x) \mid i \in J\} = 1\), from which the stated result follows. Hence we have a finite subset \(J'\) of \(J\) for which \(\bigvee \{\mu_{G_i}(x) \mid i \in J'\} = 1\) or \(\bigvee \{\mu_{H_i}(x) \mid i \in J'\} = 1\), whence by Lemma 4.2 \((G_i \times 1_Y) \cup (1_X \times H_i) \mid i \in J'\) is a finite subcover of \(C\). Hence \((X \times Y, \beta)\) is fuzzy supracompact. \( \square \)

4.6. Definition. (cf. [5]) Let \((X, \tau)\) be an IFSTS and \(A\) an IF in \(X\).

(a) If \(\{\langle x, \mu_{G_i}, \gamma_{G_i} \rangle \mid i \in J\}\) of IFSC’s in \(X\) satisfies the condition \(A \subseteq \bigcup \{\langle x, \mu_{G_i}, \gamma_{G_i} \rangle \mid i \in J\}\), then it is called a fuzzy supraopen cover of \(A\). A finite subfamily of the fuzzy supraopen cover \(\{\langle x, \mu_{G_i}, \gamma_{G_i} \rangle \mid i \in J\}\) of \(A\), which is also a fuzzy supraopen cover of \(A\), is called a finite subcover of \(\{\langle x, \mu_{G_i}, \gamma_{G_i} \rangle \mid i \in J\}\).
(b) An IFS $A = (x, \mu_A, \gamma_A)$ in an IFSTS $(X, \tau)$ is called fuzzy supracompact if every fuzzy supraopen cover of $A$ has a finite subcover.

4.7. Corollary. (cf.[5]) An IFS $A = (x, \mu_A, \gamma_A)$ in an IFSTS $(X, \tau)$ is fuzzy supracompact iff for each family $\delta = \{G_i : i \in J\}$, $G_i = (x, \mu_{G_i}, \gamma_{G_i})$, $i \in J$, of IFSOS's in $X$ with the properties $\mu_A \subseteq \bigvee_{i \in J} \mu_{G_i}$ and $1 - \gamma_A \subseteq \bigvee_{i \in J} (1 - \gamma_{G_i})$, there exists a finite subfamily $\{G_{i_k} : k = 1, 2, \ldots, n\}$ of $\delta$ such that $\mu_A \subseteq \bigvee_{k=1}^n \mu_{G_{i_k}}$ and $1 - \gamma_A \subseteq \bigvee_{k=1}^n (1 - \gamma_{G_{i_k}})$.

4.8. Corollary. Let $(X, \tau)$, $(Y, \phi)$ be IFSTS's and $f : X \rightarrow Y$ a fuzzy supracontinuous surjection. If $A$ is fuzzy supracompact in $(X, \tau)$ then $f(A)$ is fuzzy supracompact in $(Y, \phi)$.

Proof. Let $B = \{G_i : i \in J\}$, where $G_i = (y, \mu_{G_i}, \gamma_{G_i})$, $i \in J$, be a fuzzy supraopen cover of $f(A)$. Then, by the definition of supracontinuity $A = \{f^{-1}(G_i) : i \in J\}$ is a fuzzy supraopen cover of $A$. Since $A$ is fuzzy supracompact, there exists a finite subcover of $A$, that is $G_{i_k}$, $k = 1, 2, \ldots, n$, such that $A \subseteq \bigcup_{k=1}^n f^{-1}(G_{i_k})$. Hence $f(A) \subseteq \bigcup_{k=1}^n f^{-1}(G_{i_k}) = \bigcup_{k=1}^n f(f^{-1}(G_{i_k})) \subseteq \bigcup_{k=1}^n G_{i_k}$. Therefore, $f(A)$ is fuzzy supracompact. 

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