GENERALIZED SKEW DERIVATIONS ON MULTILINEAR POLYNOMIALS IN RIGHT IDEALS OF PRIME RINGS

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Abstract

Let $R$ be a prime ring, $f(x_1,\ldots,x_n)$ a multilinear polynomial over $C$ in $n$ noncommuting indeterminates, $I$ a nonzero right ideal of $R$, and $F : R \to R$ be a nonzero generalized skew derivation of $R$.

Suppose that $F(f(r_1,\ldots,r_n))f(r_1,\ldots,r_n) \in C$, for all $r_1,\ldots,r_n \in I$. If $f(x_1,\ldots,x_n)$ is not central valued on $R$, then either $\text{char}(R) = 2$ and $R$ satisfies $s_4$ or one of the following holds:

(i) $f(x_1,\ldots,x_n)x_{n+1}$ is an identity for $I$;

(ii) $F(I)I = (0)$;

(iii) $[f(x_1,\ldots,x_n),x_{n+1}]x_{n+2}$ is an identity for $I$, there exist $b,c,q \in Q$ with $q$ an invertible element such that $F(x) = bx - qxq^{-1}c$ for all $x \in R$, and $q^{-1}cI \subseteq I$. Moreover, in this case either $(b-c)I = (0)$ or $b-c \in C$ and $f(x_1,\ldots,x_n)^2$ is central valued on $R$.

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1. Introduction.

Throughout this paper, unless specially stated, $K$ denotes a commutative ring with unit, $R$ is always a prime $K$-algebra with center $Z(R)$, right Martindale quotient ring $Q$ and extended centroid $C$. The definition, axiomatic formulations and properties of this quotient ring can be found in [2] (Chapter 2).

Many results in literature indicate how the global structure of a ring $R$ is often tightly connected to the behaviour of additive mappings defined on $R$. A well known result of Posner [32] states that if $d$ is a derivation of $R$ such that $[d(x), x] \in Z(R)$, for any $x \in R$, then either $d = 0$ or $R$ is commutative. Later in [3], Bresar proved that if $d$ and $\delta$ are derivations of $R$ such that $d(x)x - x\delta(x) \in Z(R)$, for all $x \in R$, then either $d = \delta = 0$ or $R$ is commutative. In [29], Lee and Wong extended Bresar’s result to the Lie case. They proved that if $d(x)x - x\delta(x) \in Z(R)$, for all $x$ in some non-central Lie ideal $L$ of $R$ then either $d = \delta = 0$ or $R$ satisfies $s_4$, the standard identity of degree 4.

Recently in [25], Lee and Zhou considered the case when the derivations $d$ and $\delta$ are replaced respectively by the generalized derivations $H$ and $G$, and proved that if $R \neq M_2(GF(2))$, $H,G$ are two generalized derivations of $R$, and $m,n$ are two fixed positive integers, then $H(x^m)x^n = x^mG(x^n)$ for all $x \in R$ if and only if the following two conditions hold: (1) There exists $w \in Q$ such that $H(x) = xw$ and $G(x) = wx$ for all $x \in R$; (2) either $w \in C$, or $x^m$ and $x^n$ are $C$-dependent for all $x \in R$.

Moreover in this last case a complete description of $H$ and $G$ is given.

Finally, as a partial extension of the above results to the case of derivations and generalized derivations acting on multilinear polynomials, we have the following:

1.1. Fact. (Theorem 2 in [22]) Let $R$ be a prime ring, $f(x_1, \ldots, x_n)$ a multilinear polynomial over $C$ in $n$ noncommuting indeterminates, and $d : R \rightarrow R$ a nonzero derivation of $R$. If $d(f(r_1, \ldots, r_n))f(r_1, \ldots, r_n) \in C$, for all $r_1, \ldots, r_n \in R$ and $f(x_1, \ldots, x_n)$ is not central valued on $RC$, then $\text{char}(R) = 2$ and $R$ satisfies $s_4$.

1.2. Fact. (Lemma 3 in [1]) Let $R$ be a prime ring, $f(x_1, \ldots, x_n)$ a noncentral multilinear polynomial over $C$ in $n$ noncommuting indeterminates, and $G : R \rightarrow R$ a nonzero generalized derivation of $R$. If $G(f(r_1, \ldots, r_n))f(r_1, \ldots, r_n) \in C$, for all $r_1, \ldots, r_n \in R$, then either $\text{char}(R) = 2$ and $R$ satisfies $s_4$ or there exists $b \in C$ such that $G(x) = bx$ for all $x \in R$ and $f(x_1, \ldots, x_n)^2$ is central valued on $R$.

These facts in a prime $K$-algebra are natural tests which evidence that, if $d$ is a derivation of $R$ and $G$ is a generalized derivation of $R$, then the sets $\{d(x)x \mid x \in S\}$ and $\{G(x)x \mid x \in S\}$ are rather large in $R$, where $S$ is either a non-central Lie ideal of $R$, or the set of all the evaluations of a non-central multilinear polynomial over $K$.

In this paper we will continue the study of the set

$$ \{F(f(x_1, \ldots, x_n))f(x_1, \ldots, x_n) \mid x_1, \ldots, x_n \in R\} $$

for a generalized skew derivation $F$ of $R$ instead of a generalized derivation, and for a multilinear polynomial $f(x_1, \ldots, x_n)$ in $n$ noncommuting variables over $C$. For the sake of clearness and completeness we now recall the definition of a generalized skew derivation of $R$. Let $R$ be an associative ring and $\alpha$ be an automorphism of $R$. An additive mapping $d : R \rightarrow R$ is called a skew derivation of $R$ if

$$ d(xy) = d(x)y + \alpha(x)d(y) $$

for all $x, y \in R$. The automorphism $\alpha$ is called an associated automorphism of $d$. An additive mapping $F : R \rightarrow R$ is said to be a generalized skew derivation of $R$ if there
exists a skew derivation $d$ of $R$ with associated automorphism $\alpha$ such that
$$F(xy) = F(x)y + \alpha(x)d(y)$$
for all $x, y \in R$, and $d$ is said to be an associated skew derivation of $F$ and $\alpha$ is called an associated automorphism of $F$. For fixed elements $a$ and $b$ of $R$, the mapping $F : R \to R$ defined as $F(x) = ax - \sigma(x)b$ for all $x \in R$ is a generalized skew derivation of $R$. A generalized skew derivation of this form is called an inner generalized skew derivation. The definition of generalized skew derivations is a unified notion of skew derivation and generalized derivation, which have been investigated by many researchers from various viewpoints (see \cite{S, G, L, K}).

The main result of this paper is the following:

**Theorem.** Let $R$ be a prime ring, $f(x_1, \ldots, x_n)$ a multilinear polynomial over $C$ in $n$ noncommuting indeterminates, $I$ a nonzero right ideal of $R$, and $F : R \to R$ a nonzero generalized skew derivation of $R$.

Suppose that $F(f(r_1, \ldots, r_n))f(r_1, \ldots, r_n) \in C$, for all $r_1, \ldots, r_n \in I$. If the polynomial $f(x_1, \ldots, x_n)$ is not central valued on $R$, then either $\text{char}(R) = 2$ and $F$ satisfies $s_4$ or one of the following holds:

(i) $f(x_1, \ldots, x_n)x_{n+1}$ is an identity for $I$;

(ii) $F(I)I = (0)$;

(iii) $[f(x_1, \ldots, x_n), x_{n+1}][x_{n+2}]$ is an identity for $I$, there exist $b, c, q \in Q$ with $q$ an invertible element such that $F(x) = bx - qxq^{-1}c$ for all $x \in R$, and $q^{-1}cI \subseteq I$.

Moreover, in this case either $(b - c)I = (0)$ or $b - c \in C$ and $f(x_1, \ldots, x_n)^2$ is central valued on $R$.

It is well known that automorphisms, derivations and skew derivations of $R$ can be extended to $Q$. Chang in \cite{S} extended the definition of a generalized skew derivation to the right Martindale quotient ring $Q$ of $R$ as follows: by a (right) generalized skew derivation we mean an additive mapping $F : Q \to Q$ such that $F(xy) = F(x)y + \alpha(x)d(y)$ for all $x, y \in Q$, where $d$ is a skew derivation of $R$ and $\alpha$ is an automorphism of $R$. Moreover, there exists $F(1) = a \in Q$ such that $F(x) = ax + d(x)$ for all $x \in R$ (Lemma 2 in \cite{S}).

**2. X-inner Generalized Skew Derivations on Prime Rings.**

In this section we consider the case when $F$ is an X-inner generalized skew derivation induced by the elements $b, c \in R$, that is, $F(x) = bx - \alpha(x)c$ for all $x \in R$, where $\alpha \in \text{Aut}(R)$ is the associated automorphism of $F$. Here $\text{Aut}(R)$ denotes the group of automorphisms of $R$.

At the outset, we will study the case when $R = M_m(K)$ is the algebra of $m \times m$ matrices over a field $K$. Notice that the set $f(R) = \{f(r_1, \ldots, r_n) : r_1, \ldots, r_n \in R\}$ is invariant under the action of all inner automorphisms of $R$. Hence if we denote $r = (r_1, \ldots, r_n) \in R \times \ldots \times R = R^n$, then for any inner automorphism $\varphi$ of $M_m(K)$, we have that $\varphi = (\varphi(r_1), \ldots, \varphi(r_n)) \in R^n$ and $\varphi(f(r)) = f(\varphi(r))$.

Let us recall some results from \cite{P} and \cite{T}. Let $T$ be a ring with 1 and let $e_{ij} \in M_m(T)$ be the matrix unit having 1 in the $(i, j)$-entry and zero elsewhere. For a sequence $u = (A_1, \ldots, A_n)$ in $M_m(T)$ the value of $u$ is defined to be the product $|u| = A_1A_2 \cdots A_n$ and $u$ is nonvanishing if $|u| \neq 0$. For a permutation $\sigma$ of $\{1, 2, \cdots, n\}$ we write $u^\sigma = (A_{\sigma(1)}, \ldots, A_{\sigma(n)})$. We call $u$ simple if it is of the form $u = (a_1e_{i_1j_1}, \ldots, a_ne_{i_nj_n})$, where $a_i \in T$. A simple sequence $u$ is called even if for some $\sigma$, $|u^\sigma| = be_{i_1} \neq 0$, and odd if for some $\sigma$, $|u^\sigma| = be_{ij} \neq 0$, where $i \neq j$ and $b \in T$. We have:
2.1. Fact. (Lemma in [23]) Let \( T \) be a \( K \)-algebra with 1 and let \( R = M_m(T) \), \( m \geq 2 \). Suppose that \( f(x_1, \ldots , x_n) \) is a multilinear polynomial over \( K \) such that \( h(u) = 0 \) for all odd simple sequences \( u \). Then \( h(x_1, \ldots , x_n) \) is central valued on \( R \).

2.2. Fact. (Lemma 2 in [30]) Let \( T \) be a \( K \)-algebra with 1 and let \( R = M_m(T) \), \( m \geq 2 \). Suppose that \( h(x_1, \ldots , x_n) \) is a multilinear polynomial over \( K \). Let \( u = (A_1, \ldots , A_n) \) be a simple sequence from \( R \).

1. If \( u \) is even, then \( h(u) \) is a diagonal matrix.
2. If \( u \) is odd, then \( h(u) = ae_{pq} \) for some \( a \in T \) and \( p \neq q \).

2.3. Fact. Suppose that \( f(x_1, \ldots , x_n) \) is a multilinear polynomial over a field \( K \) not central valued on \( R = M_m(K) \). Then by Fact 2.1 there exists an odd simple sequence \( r = (r_1, \ldots , r_n) \) from \( R \) such that \( f(r) = f(r_1, \ldots , r_n) \neq 0 \). By Fact 2.2 \( f(r) = \beta e_{pq} \), where \( 0 \neq \beta \in K \) and \( p \neq q \). Since \( f(x_1, \ldots , x_n) \) is a multilinear polynomial and \( K \) is a field, we may assume that \( \beta = 1 \). Now, for distinct \( i \) and \( j \), let \( \sigma \in S_n \) be such that \( \sigma(p) = i \) and \( \sigma(q) = j \), and let \( \psi \) be the automorphism of \( R \) defined by \( \psi(\sum_{s,t} \xi_{st} e_{st}) = \sum_{s,t} \xi_{st} e_{\sigma(s)\sigma(t)} \). Then \( f(\psi(r)) = f(\psi(r_1), \ldots , \psi(r_n)) = \psi(f(r)) = \beta e_{ij} = e_{ij} \).

In all that follows we always assume that \( f(x_1, \ldots , x_n) \) is not central valued on \( R \).

2.4. Lemma. Let \( R = M_m(K) \) be the algebra of \( m \times m \) matrices over the field \( K \) and \( m \geq 2 \), \( f(x_1, \ldots , x_n) \) a multilinear polynomial over \( K \), which is not central valued on \( R \).

If there exist \( b, c, q \in R \) with \( q \) an invertible matrix such that

\[
\left( bf(r_1, \ldots , r_n) - qf(r_1, \ldots , r_n)q^{-1}c \right) f(r_1, \ldots , r_n) \in Z(R)
\]

for all \( r_1, \ldots , r_n \in R \), then either \( \text{char}(R) = 2 \) and \( m = 2 \), or \( q^{-1}c, b - c \in Z(R) \) and \( f(x_1, \ldots , x_n) \) is central valued on \( R \), provided that \( b \neq c \).

Proof. If \( q^{-1}c \in Z(R) \) then the conclusion follows from Fact 1.2. Thus we may assume that \( q^{-1}c \) is not a scalar matrix and proceed to get a contradiction. Say \( q = \sum_{hl} q_{hl} e_{hl} \) and \( q^{-1}c = \sum_{hl} p_{hl} e_{hl} \), for \( q_{hl}, p_{hl} \in K \). By Fact 2.3 \( e_{ij} \in f(R) \) for all \( i \neq j \), then for any \( i \neq j \)

\[
X = (be_{ij} - qe_{ij}q^{-1}c)e_{ij} \in Z(R).
\]

By \( X \), we have \( qe_{ij}q^{-1}ce_{ij} = q_{ij}e_{ij} \in Z(R) \). Then for any \( 1 \leq k \leq m \) \( [q_{ij}e_{ij}, e_{ik}] = 0 \), that is \( q_{ij}p_{ki} \neq 0 \) for some \( k \), we get \( p_{ji} = 0 \) for all \( i \neq j \). Hence \( q^{-1}c \) is a diagonal matrix in \( R \). Let \( i \neq j \) and \( \phi(x) = (1 + e_{ij})(1 - e_{ij}) \) be an automorphism of \( R \). It is well known that \( \phi(f(r_i)) \in f(R) \), then

\[
\left( \phi(b)u - \phi(q)u\phi(q^{-1}c) \right) u \in Z(R)
\]

for all \( u \in f(R) \). By the above argument, \( \phi(q^{-1}c) \) is a diagonal matrix, that is the \((j, i)\)-entry of \( \phi(q^{-1}c) \) is zero. By calculations it follows \( p_{ji} = p_{ij} \), and we get the contradiction that \( q^{-1}c \) is central in \( R \).

2.5. Lemma. Let \( R \) be a prime ring, \( f(x_1, \ldots , x_n) \) be a non-central multilinear polynomial over \( C \). If there exist \( b, c, q \in R \) with \( q \) an invertible element such that

\[
(bf(r_1, \ldots , r_n) - qf(r_1, \ldots , r_n)q^{-1}c)f(r_1, \ldots , r_n) \in C
\]

for all \( r_1, \ldots , r_n \in R \), then either \( \text{char}(R) = 2 \) and \( R \) satisfies \( s_4 \), or \( q^{-1}c, b - c \in Z(R) \) and \( f(x_1, \ldots , x_n)^2 \) is central valued on \( R \), provided that \( b \neq c \).
Proof. Consider the generalized polynomial
\[ \Phi(x_1, \ldots, x_{n+1}) = \left( b f(x_1, \ldots, x_n) - q f(x_1, \ldots, x_n) q^{-1} c \right) f(x_1, \ldots, x_n, x_{n+1}) \]
which is a generalized polynomial identity for \( R \). If \( \{1, q^{-1}c\} \) is linearly \( C \)-dependent, then \( q^{-1}c \in C \). In this case \( R \) satisfies
\[ \Phi(x_1, \ldots, x_{n+1}) = \left[ (b - c) f(x_1, \ldots, x_n) \right] f(x_1, \ldots, x_n, x_{n+1}) \]
and we are done by Fact 2.2.

Hence we here assume that \( \{1, q^{-1}c\} \) is linearly \( C \)-independent. In this case \( \Phi(x_1, \ldots, x_{n+1}) \) is a non-trivial generalized polynomial identity for \( R \) and by [12] \( \Phi(x_1, \ldots, x_{n+1}) \) is a non-trivial generalized polynomial identity for \( Q \). By Martindale’s theorem in [31], \( Q \) is a primitive ring having nonzero socle with the field \( C \) as its associated division ring. By [20] (p. 75) \( Q \) is isomorphic to a dense subring of the ring of linear transformations of a vector space \( V \) over \( C \), containing nonzero linear transformations of finite rank. Assume first that \( \dim V = k \) a finite integer. Then \( Q \cong M_k(C) \) and the conclusion follows from Lemma 2.4. Therefore we may assume that \( \dim V = \infty \). As in Lemma 2 in [33], the set \( f(R) = \{ f(r_1, \ldots, r_n) : r_i \in R \} \) is dense in \( R \) and so from \( \Phi(r_1, \ldots, r_{n+1}) = 0 \) for all \( r_1, \ldots, r_{n+1} \in R \), we have that \( Q \) satisfies the generalized identity
\[ (bx_1 - qx_1q^{-1}c)x_1x_2 \]
In particular for \( x_1 = 1, [b - c, x_2] \) is an identity for \( Q \), that is \( b - c \in C \), say \( b = c + \lambda \) for some \( \lambda \in C \). Thus \( Q \) satisfies
\[ ((c + \lambda)x_1 - qx_1q^{-1}c)x_1x_2 \]
and by replacing \( x_1 \) with \( y_1 + t_1 \) we have that
\[ \left[ \left( c + \lambda \right)y_1 - qx_1q^{-1}c \right] t_1, x_2 + \left[ \left( c + \lambda \right)t_1 - qt_1q^{-1}c \right] y_1, x_2 \]
is an identity for \( Q \). Once again for \( y_1 = 1 \) it follows that \( Q \) satisfies
\[ \lambda t_1 + (c + \lambda)t_1 - qt_1q^{-1}c, x_2 \]
and for \( x_2 = t_1 \)
\[ ct_1 - qt_1q^{-1}c, t_1 \]
By Lemma 3.2 in [17] (or [13] Theorem 1) and since \( R \) cannot satisfy any polynomial identity \( (\dim V = \infty) \), it follows the contradiction \( q^{-1}c \in C \).

2.6. Proposition. Let \( R \) be a prime ring, \( f(x_1, \ldots, x_n) \) a non-central multilinear polynomial over \( C \) in \( n \) non-commuting variables, \( b, c \in R \) and \( \alpha \in \text{Aut}(R) \) such that \( F(x) = bx - \alpha(x)c \) for all \( x \in R \). If \( F(f(r_1, \ldots, r_n))f(r_1, \ldots, r_n) \in C \), for all \( r_1, \ldots, r_n \in R \), and \( F \) is nonzero on \( R \), then either \( \text{char}(R) = 2 \) and \( R \) satisfies \( s_4 \), or \( f(x_1, \ldots, x_n)^2 \) is central valued on \( R \) and there exists \( \gamma \in C \) such that \( F(x) = \gamma x \), for all \( x \in R \). When this last case occurs, we have:

- (i) if \( \alpha \) is X-outer then \( \gamma = b \) and \( c = 0 \);
- (ii) if \( \alpha(x) = qxq^{-1} \) for all \( x \in R \) and for some invertible element \( q \in Q \), then \( \gamma = b - c \) and \( q^{-1}c \in C \).

Proof. In case \( \alpha \) is an X-inner automorphism of \( R \), there exists an invertible element \( q \in Q \) such that \( \alpha(x) = qxq^{-1} \) for all \( x \in R \) and the conclusion follows from Lemma 2.5. So we may assume here that \( \alpha \) is X-outer. Since by [14] \( R \) and \( Q \) satisfy the same generalized identities with automorphisms, then
\[ \Phi(x_1, \ldots, x_{n+1}) = \left[ (bf(x_1, \ldots, x_n) - \alpha(f(x_1, \ldots, x_n))c) f(x_1, \ldots, x_n, x_{n+1}) \right] \]
is satisfied by $Q$, moreover $Q$ is a centrally closed prime $C$-algebra. Note that if $c = 0$ we are done by Fact 1.2. Thus we may assume $c \neq 0$. In this case, by [13] (main Theorem), $\Phi(x_1, \ldots, x_{n+1})$ is a non-trivial generalized identity for $R$ and for $Q$. By Theorem 1 in [21], $RC$ has non-zero socle and $Q$ is primitive. Moreover, since $\alpha$ is an outer automorphism and any $(x_i)^\gamma$-word degree in $\Phi(x_1, \ldots, x_n)$ is equal to 1, then by Theorem 3 in [13], $Q$ satisfies the identity
\[
\left[(bf(x_1, \ldots, x_n) - f^\alpha(y_1, \ldots, y_n)c)f(x_1, \ldots, x_n), x_{n+1}\right],
\]
where $f^\alpha(X_1, \ldots, X_n)$ is the polynomial obtained from $f$ by replacing each coefficient $\gamma$ of $f$ with $\alpha(\gamma)$. By Fact 1.2 we conclude that either $\text{char}(R) = 2$ and $R$ satisfies $s_4$ or $b, c \in C$ and $f(x_1, \ldots, x_n)^{s_4}$ is central valued on $R$. Moreover, in this last case we also have that $Q$ satisfies
\[
c[f(y_1, \ldots, y_n)f(x_1, \ldots, x_n), x_{n+1}] = 0.
\]
Since $c \neq 0$ we have $f(y_1, \ldots, y_n)f(x_1, \ldots, x_n, x_{n+1})$ is a polynomial identity for $Q$. Thus there exists a suitable field $K$ such that $Q$ and the $l \times l$ matrix ring $M_l(K)$ satisfy the same polynomial identities by Lemma 1 in [22]. In particular, $M_l(K)$ satisfies $[f(y_1, \ldots, y_n)f(x_1, \ldots, x_n), x_{n+1}]$. Hence, since $f(x_1, \ldots, x_n)$ is not central valued on $M_l(K)$ (and hence $l \geq 2$), by Fact 2.3 we have that for all $i \neq j$ there exist $r_1, \ldots, r_n, s_1, \ldots, s_n \in M_l(K)$ such that $f(r_1, \ldots, r_n) = e_{ij}$ and $f(s_1, \ldots, s_n) = e_{ij}$. As a consequence we get $0 = [e_{ij}, x_{n+1}] = [e_{ii}, x_{n+1}]$, which is a contradiction for a suitable choice of $x_{n+1} \in M_l(K)$ (for example $x_{n+1} = e_{ij}$).

2.7. Fact. (Theorem 1 in [15]) Let $R$ be a prime ring, $D$ be an $X$-outer skew derivation of $R$ and $\alpha$ be an $X$-outer automorphism of $R$. If $\Phi(x_1, D(x_2), \alpha(x_3))$ is a generalized polynomial identity for $R$, then $R$ also satisfies the generalized polynomial identity $\Phi(x_1, y_2, z_3)$, where $x_1$, $y_2$, and $z_3$ are distinct indeterminates.

We close this section by collecting the results we obtained so far in the following

2.8. Proposition. Let $R$ be a prime ring, $f(x_1, \ldots, x_n)$ a non-central multilinear polynomial over $C$ in $n$ non-commuting variables, $F : R \to R$ a nonzero $X$-inner generalized skew derivation of $R$.

If $F(f(r_1, \ldots, r_n))f(r_1, \ldots, r_n) \in C$, for all $r_1, \ldots, r_n \in R$, then either $\text{char}(R) = 2$ and $R$ satisfies $s_4$, or $f(x_1, \ldots, x_n)^{s_4}$ is central valued on $R$ and there exists $\gamma \in C$ such that $F(x) = \gamma x$, for all $x \in R$.

Proof. We can write $F(x) = bx + d(x)$ for all $x \in R$ where $b \in Q$ and $d$ is a skew derivation of $R$ (see [8]). We denote $f(x_1, \ldots, x_n) = \sum_{\sigma \in S_n} \gamma(\sigma)x_{\sigma(1)} \cdots x_{\sigma(n)}$ with $\gamma(\sigma) \in C$. By Theorem 2 in [13] $Q$ and $R$ satisfy the same generalized polynomial identities with a single skew derivation, then $Q$ satisfies
\[
(2.1) \quad \left[(bf(x_1, \ldots, x_n) + d(f(x_1, \ldots, x_n)))f(x_1, \ldots, x_n), x_{n+1}\right].
\]
Since $F$ is $X$-inner then $d$ is $X$-inner, that is there exist $c \in Q$ and $\alpha \in \text{Aut}(Q)$ such that $d(x) = cx - \alpha(x)c$, for all $x \in R$. Hence $F(x) = (b + c)x - \alpha(x)c$ and we conclude by Proposition 2.6.

2.9. Corollary. Let $R$ be a prime ring, $f(x_1, \ldots, x_n)$ a non-vanishing multilinear polynomial over $C$ in $n$ non-commuting variables, $F : R \to R$ a non-zero $X$-inner generalized skew derivation of $R$. If $F(f(r_1, \ldots, r_n))f(r_1, \ldots, r_n) = 0$, for all $r_1, \ldots, r_n \in R$, then $\text{char}(R) = 2$ and $R$ satisfies $s_4$. 

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We premit the following:

3.1. Fact. (Main Theorem in [1]) Let \( R \) be a prime ring, \( I \) a nonzero right ideal of \( R \), \( f(x_1, \ldots, x_n) \) a multilinear polynomial over \( C \) in \( n \) non-commuting indeterminates, which is not an identity for \( R \), and \( g : R \to R \) a nonzero generalized derivation of \( R \) with the associated derivation \( d : R \to R \), that is \( g(x) = ax + d(x) \), for all \( x \in R \) and a fixed \( a \in Q \).

Suppose that \( g(f(r_1, \ldots, r_n))f(r_1, \ldots, r_n) \in C \), for all \( r_1, \ldots, r_n \in I \). Then either \( \text{char}(R) = 2 \) and \( R \) satisfies \( s_4 \) or \( f(x_1, \ldots, x_n)x_{n+1} \) is an identity for \( I \), or there exist \( b, c \in Q \) such that \( g(x) = bx + xc \) for all \( x \in R \) and one of the following holds:

(i) \( b, c \in C \) and \( f(x_1, \ldots, x_n)^2 \) is central valued on \( R \);
(ii) there exists \( \lambda \in C \) such that \( b = \lambda - c \) and \( f(x_1, \ldots, x_n) \) is central valued on \( R \);
(iii) \( (b + c)I = (0) \) and \( I \) satisfies the identity \( [f(x_1, \ldots, x_n), x_{n+1}]x_{n+2} \);
(iv) \( (b + c)I = (0) \) and there exists \( \gamma \in C \) such that \( (c - \gamma)I = (0) \).

3.2. Fact. (Theorem 1 in [1]) Under the same situation as in above Fact, we notice that in case \( g(f(r_1, \ldots, r_n))f(r_1, \ldots, r_n) = 0 \), for all \( r_1, \ldots, r_n \in I \), the conclusions (i) and (ii) cannot occur. Hence we have that either \( \text{char}(R) = 2 \) and \( R \) satisfies \( s_4 \) or \( f(x_1, \ldots, x_n)x_{n+1} \) is an identity for \( I \), or there exist \( b, c \in Q \) such that \( g(x) = bx + xc \) for all \( x \in R \) and one of the following holds:

(i) \( (b + c)I = (0) \) and \( I \) satisfies the identity \( [f(x_1, \ldots, x_n), x_{n+1}]x_{n+2} \);
(ii) \( (b + c)I = (0) \) and there exists \( \gamma \in C \) such that \( (c - \gamma)I = (0) \).

3.3. Proposition. Let \( R \) be a prime ring, \( f(x_1, \ldots, x_n) \) a non-central multilinear polynomial over \( C \) in \( n \) non-commuting indeterminates, \( I \) a nonzero right ideal of \( R \), \( F : R \to R \) an \( X \)-outer generalized skew derivation of \( R \). If

\[
F(f(r_1, \ldots, r_n))f(r_1, \ldots, r_n) \in C,
\]

for all \( r_1, \ldots, r_n \in I \), then either \( \text{char}(R) = 2 \) and \( R \) satisfies \( s_4 \) or \( f(x_1, \ldots, x_n)x_{n+1} \) is an identity for \( I \).

Proof. As above we write \( F(x) = bx + d(x) \) for all \( x \in R \), \( b \in Q \) and \( d \) is an \( X \)-outer skew derivation of \( R \). Let \( \alpha \in \text{Aut}(Q) \) be the automorphism which is associated with \( d \). Notice that in case \( \alpha \) is the identity map on \( R \), then \( d \) is a usual derivation of \( R \) and so \( F \) is a generalized derivation of \( R \). Therefore by Fact 3.1 we obtain the required conclusions. Hence in what follows we always assume that \( \alpha \neq 1 \in \text{Aut}(R) \).

We denote by \( f^d(x_1, \ldots, x_n) \) the polynomial obtained from \( f(x_1, \ldots, x_n) \) by replacing each coefficient \( \gamma_\sigma \) with \( d(\gamma_\sigma) \). Notice that

\[
d(\gamma_\sigma x_{\sigma(1)} \cdots x_{\sigma(n)}) = d(\gamma_\sigma)x_{\sigma(1)} \cdots x_{\sigma(n)}
+ \alpha(\gamma_\sigma) \sum_{j=0}^{n-1} \alpha(x_{\sigma(1)} \cdots x_{\sigma(j)})d(x_{\sigma(j+1)}x_{\sigma(j+2)} \cdots x_{\sigma(n)})
\]

so that

\[
d(f(x_1, \ldots, x_n)) = f^d(x_1, \ldots, x_n)
+ \sum_{\sigma \in S_n} \alpha(\gamma_\sigma) \sum_{j=0}^{n-1} \alpha(x_{\sigma(1)} \cdots x_{\sigma(j)})d(x_{\sigma(j+1)}x_{\sigma(j+2)} \cdots x_{\sigma(n)}).\]
Since $IQ$ satisfies (3.1), then for all $0 \neq u \in I$, $Q$ satisfies
\[
\left[ bf(uX_1, \ldots, uX_n) + f^d(uX_1, \ldots, uX_n) \right] f(uX_1, \ldots, uX_n, x_{n+1})
+ \left[ \sum_{\sigma \in S_n} \alpha(\gamma_{\sigma}) \sum_{j=0}^{n-1} \alpha(uX_{\sigma(1)} \ldots uX_{\sigma(j)}) d(uX_{\sigma(j+1)} uX_{\sigma(j+2)} \ldots uX_{\sigma(n)}) \right] f(uX_1, \ldots, uX_n, x_{n+1}).
\]
By Theorem 1 in [15], $Q$ satisfies
\[
\left[ bf(uX_1, \ldots, uX_n) + f^d(uX_1, \ldots, uX_n) \right] f(uX_1, \ldots, uX_n, x_{n+1})
+ \left[ \sum_{\sigma \in S_n} \alpha(\gamma_{\sigma}) \sum_{j=0}^{n-1} \alpha(uX_{\sigma(1)} \ldots uX_{\sigma(j)}) d(u) uX_{\sigma(j+1)} uX_{\sigma(j+2)} \ldots uX_{\sigma(n)}) \right] f(uX_1, \ldots, uX_n, x_{n+1})
+ \left[ \sum_{\sigma \in S_n} \alpha(\gamma_{\sigma}) \sum_{j=0}^{n-1} \alpha(uX_{\sigma(1)} \ldots uX_{\sigma(j)}) \alpha(u) y_{\sigma(j+1)} uX_{\sigma(j+2)} \ldots uX_{\sigma(n)}) \right] f(uX_1, \ldots, uX_n, x_{n+1}).
\]
In particular $Q$ satisfies
\[
(3.2)
\left[ \sum_{\sigma \in S_n} \alpha(\gamma_{\sigma}) \sum_{j=0}^{n-1} \alpha(uX_{\sigma(1)} \ldots uX_{\sigma(j)}) \alpha(u) y_{\sigma(j+1)} uX_{\sigma(j+2)} \ldots uX_{\sigma(n)}) \right] f(uX_1, \ldots, uX_n, x_{n+1}).
\]
Here we suppose that either $char(R) \neq 2$ or $R$ does not satisfy $s_4$, moreover $f(x_1, \ldots, x_n)x_{n+1}$ is not an identity for $I$, if not we are done. Hence suppose there exist $a_1, \ldots, a_{n+1} \in I$ such that $f(a_1, \ldots, a_n)a_{n+1} \neq 0$. We proceed to get a number of contradictions.

Since $0 \neq \alpha(u)$ is a fixed element of $Q$, we notice that (3.2) is a non-trivial generalized polynomial identity for $Q$, then $Q$ has nonzero socle $H$ which satisfies the same generalized polynomial identities of $Q$ (see [12]). In order to prove our result, we may replace $Q$ by $H$, and by Lemma 1 in [19], we may assume that $Q$ is a regular ring. Thus there exists $0 \neq e = e^2 \in IQ$ such that $\sum_{i=1}^{n+1} a_i Q = eQ$, and $a_i = ea_i$ for each $i = 1, \ldots, n+1$. Notice that $eQ$ satisfies the same generalized identities with skew derivations and automorphisms of $I$. So that we may assume $e \neq 1$, if not $eQ = Q$ and the conclusion follows from Proposition 2.6.

Assume that $\alpha$ is $X$-outer. Thus, by Fact 2.7 and (3.2), $Q$ satisfies
\[
(3.3)
\left[ \sum_{\sigma \in S_n} \alpha(\gamma_{\sigma}) \sum_{j=0}^{n-1} \alpha(e) t_{\sigma(1)} \cdots \alpha(e) t_{\sigma(j)} \alpha(e) y_{\sigma(j+1)} eX_{\sigma(j+2)} \cdots eX_{\sigma(n)}) \right] f(ex_1, \ldots, ex_n, x_{n+1})
\]
and in particular
\[
(3.4)
\left[ \sum_{\sigma \in S_n} \alpha(\gamma_{\sigma}) \alpha(e) y_{\sigma(1)} \cdots \alpha(e) y_{\sigma(n)}) \right] f(ex_1, \ldots, ex_n, x_{n+1}).
\]
We also denote by $f^\alpha(x_1, \ldots, x_n)$ the polynomial obtained from $f(x_1, \ldots, x_n)$ by replacing each coefficient $\gamma_{\sigma}$ with $\alpha(\gamma_{\sigma})$. Therefore we may rewrite (3.4) as follows:
\[
(3.5)
\left[ f^\alpha(\alpha(e)r_1, \ldots, \alpha(e)r_n) f(es_1, \ldots, es_n), X \right] = 0
\]
for all $r_1, \ldots, r_n, s_1, \ldots, s_n, X \in Q$. Choose in (3.5) $X = Y(1 - \alpha(e))$, then we get
\[
f^\alpha(\alpha(e)r_1, \ldots, \alpha(e)r_n) f(es_1, \ldots, es_n) Y(1 - \alpha(e)) = 0
\]
and by the primeness of $Q$ and since $e \neq 1$, it follows that $Q$ satisfies
\[ f^\alpha(\alpha(e)y_1, \ldots, \alpha(e)y_n) f(ex_1, \ldots, ex_n) \]
that is
\[ f^\alpha(\alpha(e)Q) f(eQ) = (0), \]
where $\alpha(e)Q$ and $eQ$ are both right ideals of $Q$ and $f^\alpha$ and $f$ are distinct polynomials over $C$ (since $\alpha \neq 1$). In this situation, applying the result in [16] (see the proof of Lemma 3, pp. 181), it follows that either
\[ f^\alpha(\alpha(e)r_1, \ldots, \alpha(e)r_n)\alpha(e) = 0 \]
and by the primeness of $\alpha$ and since
\[ \alpha \]
By Fact 3.1 it follows that one of the following holds:
\[ (3.8) \]
that is
\[ 0 = \alpha^{-1} f^\alpha(\alpha(e)r_1, \ldots, \alpha(e)r_n)\alpha(e) = f(e\alpha^{-1}(r_1), \ldots, e\alpha^{-1}(r_n))e \]
and since $\alpha^{-1}$ is an automorphism of $Q$, it follows that $f(es_1, \ldots, es_n)e = 0$, for all $s_1, \ldots, s_n \in Q$, which is again a contradiction.

Finally consider the case when there exists an invertible element $q \in Q$ such that
\[ \alpha(x) = qxq^{-1}, \]
for all $x \in Q$. Thus from (3.2) we have that $Q$ satisfies
\[ (3.6) \]
\[ \left[ \sum_{\sigma \in S_n} \alpha(\gamma_{\sigma}) \sum_{j=0}^{n-1} q(e \sigma_{(1)} \cdots e \sigma_{(j)}) e^{-1} y_{\sigma_{(j+1)} \sigma_{(j+2)} \cdots e \sigma_{(n)}} f(ex_1, \ldots, ex_n), x_{n+1} \right] \]
Since $\alpha(\gamma_{\sigma}) = \gamma_{\sigma}$ and by replacing $y_{\sigma(i)}$ with $q e \sigma_{(i)}$, for all $\sigma \in S_n$ and for all $i = 1, \ldots, n$, it follows that $Q$ satisfies
\[ (3.7) \]
\[ \left[ \sum_{\sigma \in S_n} \gamma_{\sigma} q e \sigma_{(1)} \cdots e \sigma_{(j)} e \sigma_{(j+1)} \cdots e \sigma_{(n)} f(ex_1, \ldots, ex_n), x_{n+1} \right] \]
that is
\[ (3.8) \]
\[ \left[ q f(ex_1, \ldots, ex_n) f(ex_1, \ldots, ex_n), x_{n+1} \right]. \]
By Fact [3.4] it follows that one of the following holds:
1. $\text{char}(Q) = 2$ and $Q$ satisfies $s_4$;
2. $f(x_1, \ldots, x_n)x_{n+1}$ is an identity for $eQ$;
3. $q \in C$;
4. $qeQ = (0)$.
Since in any case we get a contradiction, we are done. \[ \square \]

3.4. Lemma. Let $R$ be a prime ring, $f(x_1, \ldots, x_n)$ a non-central multilinear polynomial over $C$ in $n$ non-commuting indeterminates, $I$ a nonzero right ideal of $R$, $b, c \in Q$ and
\[ \alpha \in \text{Aut}(R) \]
be an automorphism of $R$ such that $F(x) = bx - \alpha(x)c$, for all $x \in R$. Assume that $F(f(r_1, \ldots, r_n))f(r_1, \ldots, r_n) \in C$, for all $r_1, \ldots, r_n \in I$. If $R$ does not satisfy any non-trivial generalized polynomial identity then $F(I)I = (0)$.

Proof. Let $u$ be any nonzero element of $I$. By the hypothesis $R$ satisfies the following:
\[ \left[ b(f(ux_1, \ldots, ux_n)) - \alpha(f(ux_1, \ldots, ux_n))c \right] \]
Also here we denote by $f^\alpha(x_1, \ldots, x_n)$ the polynomial obtained from $f(x_1, \ldots, x_n)$ by replacing each coefficient $\gamma_{\sigma}$ of $f(x_1, \ldots, x_n)$ with $\alpha(\gamma_{\sigma})$. Thus $R$ satisfies
\[ (3.9) \]
\[ \left[ b(f(ux_1, \ldots, ux_n)) - f^\alpha(\alpha(u)\alpha(x_1), \ldots, \alpha(u)\alpha(x_n))c \right] \]
In case $\alpha$ is $X$-outer, by Theorem 3 in [14] and (3.9) we have that $R$ satisfies
\[
\left( b f(u x_1, \ldots, u x_n) - f^\alpha(\alpha(u) y_1, \ldots, \alpha(u) y_n) c \right) f(u x_1, \ldots, u x_n), x_{n+1} \right]
\]and in particular $R$ satisfies both
\[(3.10) \quad \left[ b f(u x_1, \ldots, u x_n)^2, x_{n+1} \right] \]
and
\[(3.11) \quad \left[ f^\alpha(\alpha(u) y_1, \ldots, \alpha(u) y_n) c f(u x_1, \ldots, u x_n), x_{n+1} \right]. \]
Since (3.10) and (3.11) must be trivial generalized polynomial identities for $R$ and in particular $R$ satisfies both
\[(3.13) \quad \left[ q f(u x_1, \ldots, u x_n)(\lambda - q^{-1} c) f(u x_1, \ldots, u x_n), x_{n+1} \right]. \]
Once again (3.13) is a trivial identity for $R$, moreover $q u \neq 0$. This implies that $(\lambda - q^{-1} c) u = 0$ and hence $(\lambda_u - q^{-1} c) u = 0$ for all $u \in I$ and for some $\lambda_u \in C$. Then $u$ and $q^{-1} c u$ are $C$-dependent for all $u \in I$. By a standard argument we conclude that $(\lambda - q^{-1} c) I = (0)$ for some $\lambda \in C$, and thus $F(I) I = (0)$. \hfill $\square$

3.5. Lemma. Let $R$ be a prime ring, $f(x_1, \ldots, x_n)$ a non-central multilinear polynomial over $C$ in $n$ non-commuting indeterminates, $I$ a nonzero right ideal of $R$, $b, c \in Q$ and $\alpha \in \text{Aut}(R)$ be an $X$-outer automorphism of $R$ such that $F(x) = bx - \alpha(x)c$, for all $x \in R$. If $F(f(r_1, \ldots, r_n)) f(r_1, \ldots, r_n) \in C$, for all $r_1, \ldots, r_n \in I$, then either $\text{char}(R) = 2$ and $R$ satisfies $S_4$ or one of the following holds:

(i) $f(x_1, \ldots, x_n)x_{n+1}$ is an identity for $I$;
(ii) $F(I) I = (0)$;
(iii) $c I = (0)$, $b \in C$ and $f(x_1, \ldots, x_n)^2$ is central valued on $R$.

Proof. Firstly we notice that in case $c I = (0)$, then $b f(r_1, \ldots, r_n)^2 \in C$, for all $r_1, \ldots, r_n \in I$. Thus by Fact 3.11 it follows that either $c I = (0)$, $b \in C$ and $f(x_1, \ldots, x_n)^2$ is central valued on $R$, or $c I = b I = (0)$ that is $F(I) I = (0)$. Hence in the following we assume $c I \neq (0)$. By previous Lemma we may assume that $R$ satisfies some non-trivial generalized polynomial identity. As above let $u$ be any nonzero element of $I$. By the hypothesis $R$ satisfies the following:

\[
(3.14) \quad \left[ b f(u x_1, \ldots, u x_n) - f^\alpha(\alpha(u) \alpha(x_1), \ldots, \alpha(u) \alpha(x_n)) c \right] f(u x_1, \ldots, u x_n), x_{n+1} \right].
\]
Since $\alpha$ is $X$-outer, by Theorem 3 in [14], $R$ satisfies
\[(3.15) \quad \left[ b f(u x_1, \ldots, u x_n) - f^\alpha(\alpha(u) y_1, \ldots, \alpha(u) y_n) c \right] f(u x_1, \ldots, u x_n), x_{n+1} \right]
\]and in particular $R$ as well as $Q$ satisfy the component
\[(3.16) \quad f^\alpha(\alpha(u) y_1, \ldots, \alpha(u) y_n) c f(u x_1, \ldots, u x_n), x_{n+1} \right].
By \( Q \) is a primitive ring having nonzero socle \( H \) with the field \( C \) as its associated division ring. Moreover \( H \) and \( Q \) satisfy the same generalized polynomial identities with automorphisms (Theorem 1 in [14]). Therefore \( H \) satisfies \((3.14)\) and so we may replace \( Q \) by \( H \). Suppose there exist \( a_1, \ldots, a_{n+2} \in I \) such that \( f(a_1, \ldots, a_n)_{a_{n+1}} \neq 0 \) and \( ca_{n+2} \neq 0 \). Since \( Q \) is a regular GPI-ring, there exists an idempotent element \( e \in IQ \) such that \( eQ = \sum_{i=1}^{n+2} a_i Q \) and \( a_i = ea_i \), for any \( i = 1, \ldots, n+2 \). Therefore, by \((3.14)\), \( Q \) satisfies
\[
(3.17) \quad \left[ \left(\begin{align*}
bf & \left(\begin{array}{c}
\alpha(v) & \ldots \alpha(v) & \alpha(u) & \ldots \alpha(u) & \alpha(w) & \ldots \alpha(w)
\end{array}\right) - & \left(\begin{array}{c}
f(x_1, \ldots, x_n) - f^a(\alpha(v)\alpha(x_1), \ldots, \alpha(v)\alpha(x_n))c
\end{array}\right) f(x_1, \ldots, x_n, x_{n+1})
\end{align*}\right]
\]
Moreover assume \( e \neq 1 \), if not \( eQ = Q \) and by Proposition \[2.6\] we get \( b \in C \), \( c = 0 \) and \( f(x_1, \ldots, x_n)^2 \) is central valued on \( R \). Since \( \alpha \) is \( X \)-outer, as above by \((3.17)\), \( Q \) satisfies
\[
\left[ \left(\begin{align*}
\alpha(v) & \ldots \alpha(v) & \alpha(u) & \ldots \alpha(u) & \alpha(w) & \ldots \alpha(w)
\end{align*}\right) - \left(\begin{array}{c}
f(x_1, \ldots, x_n) - f^a(\alpha(v)\alpha(x_1), \ldots, \alpha(v)\alpha(x_n))c
\end{array}\right) f(x_1, \ldots, x_n, x_{n+1})
\end{align*}\right]
\]
In particular \( Q \) satisfies
\[
\left(\begin{align*}
f^a(\alpha(v)\alpha(x_1), \ldots, \alpha(v)\alpha(x_n))cf(x_1, \ldots, x_n, 1 - \alpha(e))
\end{align*}\right)
\]that is \( Q \) satisfies
\[
f^a(\alpha(v)\alpha(x_1), \ldots, \alpha(v)\alpha(x_n))g(x_1, \ldots, x_n) x_{n+1}^2(1 - \alpha(e))
\]and since \( Q \) is prime and \( e \neq 0, 1 \), it follows \( f^a(\alpha(v)\alpha(x_1), \ldots, \alpha(v)\alpha(x_n))c\) is an identity for \( I \), for all \( r_1, \ldots, r_n, s_1, \ldots, s_n \in Q \). Since \( f(a_1, \ldots, a_n)_{a_{n+1}} \neq 0 \) and \( cea_{n+2} \neq 0 \) and by using the result in [16], it follows that \( f^a(\alpha(v)\alpha(x_1), \ldots, \alpha(v)\alpha(x_n))c\) is an identity for \( Q \). This implies that \( f(\alpha^{-1}(y_1), \ldots, \alpha^{-1}(y_n)) \) is also an identity for \( Q \). Moreover it is clear that \( \alpha^{-1} \) is \( X \)-outer, therefore \( f(x_1, \ldots, x_n) \) is an identity for \( Q \), a contradiction.

\[3.6. \text{Lemma.} \] Let \( R \) be a prime ring, \( f(x_1, \ldots, x_n) \) a non-central multilinear polynomial over \( C \) in \( n \) non-commuting indeterminates, \( I \) a nonzero right ideal of \( R \), \( b, c, q \in Q \) such that \( F(x) = bx - qxq^{-1}c \), for all \( x \in R \). If
\[
F(f(r_1, \ldots, r_n))f(r_1, \ldots, r_n) = 0,
\]
for all \( r_1, \ldots, r_n \in I \), then either \( char R = 2 \) and \( R \) satisfies \( s_4 \) or one of the following holds:
\[
(i) \quad f(x_1, \ldots, x_n)_{x_{n+1}} \text{ is an identity for } I; \quad (ii) \quad f(x_1, \ldots, x_n, x_{n+1})_{x_{n+2}} \text{ is an identity for } I, (b-c)I = (0) \quad \text{and } q^{-1}cI \subseteq I; \quad (iii) \quad F(I)I = (0).
\]

\[\text{Proof.} \] Here \( I \) satisfies
\[
(3.18) \quad \left[ \left(\begin{align*}
bf & \left(\begin{array}{c}
\alpha(v) & \ldots \alpha(v) & \alpha(u) & \ldots \alpha(u) & \alpha(w) & \ldots \alpha(w)
\end{array}\right) - & \left(\begin{array}{c}
f(x_1, \ldots, x_n) - qf(x_1, \ldots, x_n)q^{-1}c
\end{array}\right) f(x_1, \ldots, x_n)
\end{align*}\right]
\]
and left multiplying by \( q^{-1} \), \( I \) satisfies
\[
(3.19) \quad \left[ \left(\begin{align*}
bf & \left(\begin{array}{c}
\alpha(v) & \ldots \alpha(v) & \alpha(u) & \ldots \alpha(u) & \alpha(w) & \ldots \alpha(w)
\end{array}\right) - & \left(\begin{array}{c}
f(x_1, \ldots, x_n) - (f(x_1, \ldots, x_n)q^{-1}c)
\end{array}\right) f(x_1, \ldots, x_n).
\end{align*}\right]
\]
Since we assume \( f(x_1, \ldots, x_n) \) is not central valued on \( R \), by Fact \[3.2\] we have that either \( char R = 2 \) and \( R \) satisfies the standard identity \( s_4 \), or \( f(x_1, \ldots, x_n)_{x_{n+1}} \) is an identity for \( I \), or one of the following holds:

1. there exists \( \gamma \in C \) such that \( q^{-1}bx = \gamma x = q^{-1}cx \), for all \( x \in I \) (this is the case \( F(I)I = (0) \)).
2. \( q^{-1}(b-c)I = (0) \), that is \( (b-c)I = (0) \), moreover \( f(x_1, \ldots, x_n, x_{n+1})_{x_{n+2}} \) is an identity for \( I \).
In this last case, by \(3.19\) it follows that \(I\) satisfies
\[
(3.20) \quad \left( bf(wx_1, \ldots, wx_n) - qf(wx_1, \ldots, wx_n)q^{-1}b \right) f(wx_1, \ldots, wx_n)
\]
and moreover, since \(I\) satisfies the polynomial identity \([f(x_1, \ldots, x_n), x_{n+1}]x_{n+2}\), in view of Proposition in \(23\), \(I = eQ\) for some idempotent \(e\) in the socle of \(Q\). Here we write \(f(x_1, \ldots, x_n) = \sum t_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)x_i\), where any \(t_i\) is a multilinear polynomial in \(n-1\) variables and \(x_i\) never appears in \(t_i\). Of course, if \(t_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)e\) is an identity for \(Q\), then \(f(x_1, \ldots, x_n)x_{n+1}\) is an identity for \(I\) and we are done. Thus assume there exists \(i \in \{1, \ldots, n\}\) such that \(t_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)e \neq 0\) for some \(r_1, \ldots, r_n \in I\). In particular,
\[
f(ex_1, \ldots, ex_{i-1}, ex_i(1-e), ex_{i+1}, \ldots, ex_n) = t_i(ex_1, \ldots, ex_n)ex_i(1-e)
\]
and by \(3.20\) \(Q\) satisfies
\[
bt_i(ex_1, \ldots, ex_n)ex_i(1-e)t_i(ex_1, \ldots, ex_n)ex_i(1-e)
\]
that is \(Q\) satisfies
\[
(3.21) \quad \left( -qt_i(ex_1, \ldots, ex_n)ex_i(1-e)q^{-1}b \right) t_i(ex_1, \ldots, ex_n)ex_i(1-e)
\]
and left multiplying by \((1-e)q^{-1}bq^{-1}\), we easily have that \(Q\) satisfies
\[
(3.22) \quad (1-e)q^{-1}bt_i(ex_1, \ldots, ex_n)eX(1-e)q^{-1}bt_i(ex_1, \ldots, ex_n)eX(1-e).
\]
By Lemma 2 in \( \text{[22]} \) and since \(e \neq 1\), it follows that
\[
(1-e)q^{-1}bt_i(ex_1, \ldots, ex_{i-1}, ex_{i+1}, \ldots, ex_n)e
\]
is an identity for \(Q\), that is \((1-e)q^{-1}bt_i(x_1e, \ldots, x_{i-1}e, x_{i+1}e, \ldots, x_ne)\) is an identity for \(Q\). In this case, since \(t_i(x_1e, \ldots, x_{i-1}e, x_{i+1}e, \ldots, x_ne)\) is not an identity for \(Q\), we get in view of the result in \( \text{[16]} \), \((1-e)q^{-1}be = 0\), that is \(q^{-1}bI \subseteq I\) and also \(q^{-1}eI \subseteq I\). \( \square \)

3.7. Theorem. Let \(R\) be a prime ring, \(f(x_1, \ldots, x_n)\) a multilinear polynomial over \(C\) in \(n\) non-commuting variables, \(I\) a non-zero right ideal of \(R\), \(F : R \to R\) be a non-zero generalized skew derivation of \(R\). Suppose that
\[
F(f(r_1, \ldots, r_n))f(r_1, \ldots, r_n) \in C,
\]
for all \(r_1, \ldots, r_n \in I\). If \(f(x_1, \ldots, x_n)\) is not central valued on \(R\), then either \(\text{char}(R) = 2\) and \(R\) satisfies \(s_4\) or one of the following holds:

(i) \(f(x_1, \ldots, x_n)x_{n+1}\) is an identity for \(I\);
(ii) \(F(I)I = (0)\);
(iii) \([f(x_1, \ldots, x_n), x_{n+1}]x_{n+2}\) is an identity for \(I\), there exist \(b, c, q \in Q\) with \(q\) invertible such that \(F(x) = bx - qxq^{-1}c\) for all \(x \in R\), and \(q^{-1}eI \subseteq I\); moreover in this case either \((b - c)I = (0)\) or \(b - c \in C\) and \(f(x_1, \ldots, x_n)^2\) is central valued on \(R\) provided that \(b \neq c\).

Proof. In view of all previous Lemmas and Propositions, we may assume \(I \neq R\) and \(F(x) = bx - qxq^{-1}c\), for all \(x \in R\). Moreover we may assume that there exist \(s_1, \ldots, s_n \in I\) such that \(F(f(s_1, \ldots, s_n))f(s_1, \ldots, s_n) \neq 0\). Therefore
\[
(bf(x_1, \ldots, x_n) - qf(x_1, \ldots, x_n)q^{-1}c)f(x_1, \ldots, x_n)
\]
is a central generalized polynomial identity for \(I\). Thus \(R\) is a PI-ring and so \(RC\) is a finite dimensional central simple \(C\)-algebra (the proof of this fact is the same of Theorem
1 in [7]). By Wedderburn-Artin theorem, $RC \cong M_k(D)$ for some $k \geq 1$ and $D$ a finite-dimensional central division $C$-algebra. By Theorem 2 in [24]

$$(bf(x_1, \ldots, x_n) - qf(x_1, \ldots, x_n)q^{-1}c)f(x_1, \ldots, x_n) \in C$$

for all $x_1, \ldots, x_n \in IC$. Without loss of generality we may replace $R$ with $RC$ and assume that $R = M_k(D)$. Let $E$ be a maximal subfield of $D$, so that $M_k(D) \otimes C E \cong M_t(E)$ where $t = k \cdot [E : C]$. Hence $(bf(r_1, \ldots, r_n) - qf(r_1, \ldots, r_n)q^{-1}c)f(r_1, \ldots, r_n) \in C$, for any $r_1, \ldots, r_n \in I \otimes E$ (Lemma 2 in [24] and Proposition in [29]). Therefore we may assume that $R \cong M_t(E)$ and $I = eR = (e_1R + \cdots + e_0R)$, where $t \geq 2$ and $l \leq t$.

Suppose that $t \geq 2$, otherwise we are done and denote $q = \sum_{r,s} q_{rs}e_{rs}$ and $q^{-1}c = \sum_{r,s} c_{rs}e_{rs}$, for $q_{rs}, c_{rs} \in E$. As in Lemma 3 we write

$$f(x_1, \ldots, x_n) = \sum t_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)x_i$$

and there exists some $t_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)x_i$ which is not an identity for $I$. In particular $qt_i(x_{1-1}, x_{i+1}, \ldots, x_n)x_i$ is not an identity for $R$, because $q$ is invertible. Hence, again for

$$f(ex_1, \ldots, ex_{i-1}, ex_i(1-e), ex_{i+1}, \ldots, ex_n) = t_i(ex_1, \ldots, ex_{i-1}, ex_{i+1}, \ldots, ex_n)ex_i(1-e)$$

and by our hypothesis, we have that

$$qt_i(ex_1, \ldots, ex_{i-1}, ex_{i+1}, \ldots, ex_n)ex_i(1-e)q^{-1}d_i(ex_1, \ldots, ex_{i-1}, ex_{i+1}, \ldots, ex_n)ex_i(1-e)$$

is an identity for $R$, and by the primeness of $R$ it follows that

$$(1-e)q^{-1}d_i(ex_1, \ldots, ex_{i-1}, ex_{i+1}, \ldots, ex_n)e$$

is an identity for $R$. By [16] and since $t_i(ex_1, \ldots, ex_{i-1}, ex_{i+1}, \ldots, ex_n)ex_i$ is not an identity for $R$, the previous identity says that $(1-e)q^{-1}ce = 0$. Thus $q^{-1}ce \leq I$.

In case $[f(x_1, \ldots, x_n), x_{n+2}]x_{n+2}$ is an identity for $I$, then by our assumption we get $(b - c)f(r_1, \ldots, r_n)^2 \in C$ for all $r_1, \ldots, r_n \in I$. In view of Fact 3.1 either $(b - c)I = (0)$ and we are done, or $b - c \in C$ and $f(x_1, \ldots, x_n)^2$ is central valued on $R$, provided that $b \neq c$.

Consider finally the case $[f(x_1, \ldots, x_n), x_{n+1}]x_{n+2}$ is not an identity for $I$. By Lemma 3 in [6], for any $i \leq l$, $j \neq i$, the element $e_{ij}$ falls in the additive subgroup of $RC$ generated by all valuations of $f(x_1, \ldots, x_n)$ in $I$. Since the matrix $(be_{ij} - q^{-1}c)e_{ij}$ has rank at most 1, then it is not central. Therefore $qe_{ij}q^{-1}ce_{ij} = 0$, i.e. $q_{ki}(q^{-1}c)_{ji} = 0$ for all $k$ and all $j \neq i$. Since $q$ is invertible, there exists some $q_{ki} \neq 0$, therefore $(q^{-1}c)_{ji} = 0$ for all $j \neq i$.

Consider the following automorphism of $R$:

$$\lambda(x) = (1 + e_{ij})x(1 - e_{ij}) = x + e_{ij}x - e_{ij}e_{ij}$$

for any $i, j \leq l$, and note that $\lambda(I) \subseteq I$ is a right ideal of $R$ satisfying

$$\left[(\lambda(b)f(x_1, \ldots, x_n) - \lambda(q)f(x_1, \ldots, x_n)\lambda(q^{-1}c))f(x_1, \ldots, x_n), x_{n+1}\right].$$

If we denote $\lambda(q^{-1}c) = \sum_{rs} c_{rs}e_{rs}$, the above argument says that $c_{rs} = 0$ for all $s \leq l$ and $r \neq s$. In particular the $(i, j)$-entry of $\lambda(q^{-1}c)$ is zero. This implies that $e_{ii} = c_{ij} = \alpha$, for all $i, j \leq l$. Therefore $q^{-1}ce = \alpha x$ for all $x \in I$. This leads to $(b - c)f(r_1, \ldots, r_n)^2 \in C$ for all $r_1, \ldots, r_n \in I$ and we conclude by the same argument above.

For the sake of completeness, we would like to conclude this paper by showing the explicit meaning of the conclusion $F(I)I = (0)$, more precisely we state the following:
3.8. Remark. Let \( R \) be a prime ring, \( I \) be a non-zero right ideal of \( R \) and \( F : R \to R \) be a non-zero generalized skew derivation of \( R \). If \( F(I)I = (0) \) then there exist \( a, b \in Q \) and \( \alpha \in \text{Aut}(R) \) such that \( F(x) = (a + b)x - \alpha(x)b \) for all \( x \in R \), \( aI = (0) \) and one of the following holds:

(i) \( bI = (0) \);

(ii) there exist \( \lambda \in C \) and an invertible element \( q \in Q \) such that \( \alpha(x) = qxq^{-1} \), for all \( x \in R \), and \( q^{-1}by = \lambda y \), for all \( y \in I \).

Proof. As previously remarked we can write \( F(x) = ax + d(x) \) for all \( x \in R \), where \( a \in Q \) and \( d \) is a skew derivation of \( R \) (see \([3]\)). Let \( \alpha \in \text{Aut}(R) \) be the automorphism associated with \( d \), in the sense that \( d(xy) = d(x)y + \alpha(x)d(y) \), for all \( x, y \in R \). Thus, by the hypothesis, for all \( x, y \in I \),

\[
(ax + d(x))y = 0.
\]

For all \( x, y, z \in I \) we have:

\[
0 = F(xz)y = (ax + d(x))zy + \alpha(x)d(z)y
\]

and by (3.23) we obtain \( \alpha(x)d(z)y = 0 \) for all \( x, y, z \in I \). Moreover \( \alpha(I) \) is a non-zero right ideal of \( R \), so that it follows

\[
d(z)y = 0
\]

for all \( y \in I \). Once again by (3.23) we get \( axy = 0 \) for all \( z, y \in I \), that is \( aI = (0) \).

Finally in (3.24) replace \( z \) with \( xs \), for any \( x \in I \) and \( s \in R \), then:

\[
0 = d(xs)y = d(x)sy + \alpha(x)d(s)y
\]

for all \( x, y \in I \), \( s \in R \). In case \( d \) is \( X \)-outer, it follows that \( d(x)sy + \alpha(x)ty = 0 \), for all \( x, y \in I \) and \( s, t \in R \) (Theorem 1 in \([15]\)). In particular \( \alpha(x)ty = 0 \), which implies the contradiction \( \alpha(x) = 0 \) for all \( x \in I \). Therefore we may assume that \( d \) is \( X \)-inner, that is there exists \( b \in Q \) such that \( d(r) = br - \alpha(r)b \), for all \( r \in R \) and by (3.24)

\[
(bx - \alpha(x)b)y = 0
\]

for all \( x, y \in I \). Consider first the case \( \alpha \) is \( X \)-outer and replace \( x \) with \( xr \), for any \( r \in R \). Then \( (brx - \alpha(x)br)y = 0 \) and, by Theorem 3 in \([14]\), \( (brx - \alpha(x)bx)y = 0 \) for all \( x, y \in I \) and \( r, s \in R \). In particular \( bIRI = (0) \), which implies \( bI = (0) \) and we are done.

On the other hand, if there exists an invertible element \( q \in Q \) such that \( \alpha(r) = qrq^{-1} \), for all \( r \in R \), from (3.26) we have \( brx - qrq^{-1}b) = 0 \), for all \( x, y \in I \). Left multiplying by \( q^{-1} \), it follows \( \lambda[outer] = 0 \), and by Lemma in \([4]\) there exists \( \lambda \in C \) such that \( qrq^{-1}b = \lambda x \) for all \( x \in I \).

References