Incomplete generalized Fibonacci and Lucas polynomials

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Abstract

In this paper, we define the incomplete $h(x)$-Fibonacci and $h(x)$-Lucas polynomials, we study the recurrence relations, some properties of these polynomials and the generating function of the incomplete Fibonacci and Lucas polynomials.

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1. Introduction

Fibonacci numbers and their generalizations have many interesting properties and applications in many fields of science and art (see, e.g., [7]). The Fibonacci numbers $F_n$ are defined by the recurrence relation

$$F_0 = 0, \quad F_1 = 1, \quad F_n = F_{n-1} + F_{n-2}, \quad n \geq 1.$$ 

The incomplete Fibonacci and Lucas numbers were introduced by Filipponi [6]. The incomplete Fibonacci numbers $F_n(k)$ and the incomplete Lucas numbers $L_n(k)$ are defined by

$$F_n(k) = \sum_{j=0}^{k} \binom{n-1-j}{j} \quad (n = 1, 2, 3, \ldots; 0 \leq k \leq \lfloor \frac{n-1}{2} \rfloor),$$

and

$$L_n(k) = \sum_{j=0}^{k} \frac{n}{n-j} \binom{n-j}{j} \quad (n = 1, 2, 3, \ldots; 0 \leq k \leq \lfloor \frac{n}{2} \rfloor).$$

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Is is easily seen that [7]
\[ F_n \left( \left\lfloor \frac{n - 1}{2} \right\rfloor \right) = F_n \quad \text{and} \quad L_n \left( \left\lfloor \frac{n}{2} \right\rfloor \right) = L_n. \]


A large classes of polynomials can also be defined by Fibonacci-like recurrence relations such yield Fibonacci numbers. Such polynomials are called Fibonacci polynomials [7]. They were studied in 1883 by Catalan and Jacobsthal. The polynomials \( F_n(x) \) studied by Catalan are defined by the recurrence relation
\[ F_0(x) = 0, \quad F_1(x) = 1, \quad F_{n+1}(x) = xF_n(x) + F_{n-1}(x), \quad n \geq 1. \]

The Fibonacci polynomials studied by Jacobsthal are defined by
\[ J_0(x) = 1, \quad J_1(x) = 1, \quad J_{n+1}(x) = J_n(x) + xJ_{n-1}(x), \quad n \geq 1. \]

The Lucas polynomials \( L_n(x) \), originally studied in 1970 by Bicknell, are defined by
\[ L_0(x) = 2, \quad L_1(x) = x, \quad L_{n+1}(x) = xL_n(x) + L_{n-1}(x), \quad n \geq 1. \]

Nalli and Haukkanen [8] introduced the \( h(x) \)-Fibonacci polynomials that generalize Catalan's Fibonacci polynomials \( F_n(x) \) and the \( k \)-Fibonacci numbers \( F_{k,n} \) [5]. Let \( h(x) \) be a polynomial with real coefficients. The \( h(x) \)-Fibonacci polynomials \( \{F_{h,n}(x)\}_{n \in \mathbb{N}} \) are defined by the recurrence relation
\[ F_{h,0}(x) = 0, \quad F_{h,1}(x) = 1, \quad F_{h,n+1}(x) = h(x)F_{h,n}(x) + F_{h,n-1}(x), \quad n \geq 1. \]

For \( h(x) = x \) we obtain Catalan's Fibonacci polynomials, and for \( h(x) = k \) we obtain \( k \)-Fibonacci numbers. For \( k = 1 \) and \( k = 2 \) we obtain the usual Fibonacci numbers and the Pell numbers.

Let \( h(x) \) be a polynomial with real coefficients. The \( h(x) \)-Lucas polynomials \( \{L_{h,n}(x)\}_{n \in \mathbb{N}} \) are defined by the recurrence relation
\[ L_{h,0}(x) = 2, \quad L_{h,1}(x) = h(x), \quad L_{h,n+1}(x) = h(x)L_{h,n}(x) + L_{h,n-1}(x), \quad n \geq 1. \]

For \( h(x) = x \) we obtain the Lucas polynomials, and for \( h(x) = k \) we have the \( k \)-Lucas numbers [3]. For \( k = 1 \) we obtain the usual Lucas numbers. Nalli and Haukkanen [8] obtained some relations for these polynomials sequences. In particular, they found an explicit formula to \( h(x) \)-Fibonacci polynomials and \( h(x) \)-Lucas polynomials respectively
\[ F_{h,n}(x) = \sum_{i=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n-i-1}{i} x^{n-2i-1}(x), \]
\[ L_{h,n}(x) = \sum_{i=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n-i}{i} x^{n-2i}(x). \]

From Equations (1.2) and (1.3), we introduce the incomplete \( h(x) \)-Fibonacci and \( h(x) \)-Lucas polynomials and we obtain new recurrence relations, new identities and the generating function of the incomplete \( h(x) \)-Fibonacci and \( h(x) \)-Lucas polynomials.
2. Some Properties of $h(x)$-Fibonacci and $h(x)$-Lucas Polynomials

The characteristic equation associated with the recurrence relation (1.1) is $v^2 = h(x)v + 1$. The roots of this equation are

$$
\alpha(x) = \frac{h(x) + \sqrt{h(x)^2 + 4}}{2}, \quad \beta(x) = \frac{h(x) - \sqrt{h(x)^2 + 4}}{2}.
$$

Then we have the following basic identities:

$$
\alpha(x) + \beta(x) = h(x), \quad \alpha(x) - \beta(x) = \sqrt{h(x)^2 + 4}, \quad \alpha(x)\beta(x) = -1.
$$

The $h(x)$-Fibonacci polynomials and the $h(x)$-Lucas numbers verify the following properties (see [8] for the proofs).

- Binet formula: 
  $$
  F_{h,n}(x) = \frac{\alpha(x)^n - \beta(x)^n}{\alpha(x) - \beta(x)}, \quad L_{h,n}(x) = \alpha(x)^n + \beta(x)^n.
  $$

- Generating function: 
  $$
  g_f(t) = \frac{t}{1 - h(x) t - t^2}.
  $$

- Relation with $h(x)$-Fibonacci polynomials:
  $$
  L_{h,n}(x) = F_{h,n-1}(x) + F_{h,n+1}(x), \quad n \geq 1.
  $$

3. The incomplete $h(x)$-Fibonacci Polynomials

3.1. Definition. The incomplete $h(x)$-Fibonacci polynomials are defined by

$$
F_{h,n}^l(x) = \sum_{i=0}^{l} \binom{n-1-i}{i} h^{n-2i-1}(x), \quad 0 \leq l \leq \left\lfloor \frac{n-1}{2} \right\rfloor.
$$

In Table 1, some polynomials of incomplete $h(x)$-Fibonacci polynomials are provided.

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<td>$h^6 + 5h^4 + 6h^2$</td>
<td>$h^6 + 5h^4 + 6h^2 + 1$</td>
</tr>
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</table>

Table 1. The polynomials $F_{h,n}^l(x)$, for $1 \leq n \leq 7$.

Note that

$$
F_{h,n}^\left\lfloor \frac{n+1}{2} \right\rfloor(x) = F_n.
$$

For $h(x) = 1$, we get incomplete Fibonacci numbers [6]. If $h(x) = k$ we obtained incomplete $k$-Fibonacci numbers [11].
Some special cases of (3.1) are

\[ F^0_{h,n}(x) = h^{n-1}(x), \quad (n \geq 1); \]
\[ F^1_{h,n}(x) = h^{n-1}(x) + (n-2)h^{n-3}(x), \quad (n \geq 3); \]
\[ F^2_{h,n}(x) = h^{n-1}(x) + (n-2)h^{n-3}(x) + \frac{(n-4)(n-3)}{2}h^{n-5}(x), \quad (n \geq 5); \]
\[ F^\lfloor \frac{n-1}{2} \rfloor_{h,n}(x) = F_{h,n}(x), \quad (n \geq 1); \]
\[ F^\lfloor \frac{n-3}{2} \rfloor_{h,n}(x) = \begin{cases} F_{h,n}(x) - \frac{nh(x)}{2}, & \text{if } n \geq 3 \text{ and even;} \\ F_{h,n}(x) - 1, & \text{if } n \geq 3 \text{ and odd.} \end{cases} \]

3.2. Proposition. The recurrence relation of the incomplete \( h(x) \)-Fibonacci polynomials \( F^l_{h,n}(x) \) is

\[ F^{l+1}_{h,n+2}(x) = h(x)F^{l+1}_{h,n+1}(x) + F^l_{h,n}(x), \quad 0 \leq l \leq \left\lfloor \frac{n-2}{2} \right\rfloor. \]  

The relation (3.2) can be transformed into the non-homogeneous recurrence relation

\[ F^l_{h,n+2}(x) = h(x)F^l_{h,n+1}(x) + F^l_{h,n}(x) - \binom{n-1-l}{l}h^{n-2l}(x). \]

Proof. From Definition 3.1 we get

\[
h(x)F^{l+1}_{h,n+1}(x) + F^l_{h,n}(x) \\
= h(x) \sum_{i=0}^{l+1} \binom{n-i}{i}h^{n-2i}(x) + \sum_{i=0}^{l} \binom{n-i-1}{i}h^{n-2i-1}(x) \\
= \sum_{i=0}^{l+1} \binom{n-i}{i}h^{n-2i+1}(x) + \sum_{i=1}^{l+1} \binom{n-i}{i-1}h^{n-2i+1}(x) \\
= h^{n-2l}(x) \left( \sum_{i=0}^{l+1} \left[ \binom{n-i}{i} + \binom{n-i}{i-1} \right] \right) - h^{n+1}(x) \left( \frac{n}{1} \right) \\
= \sum_{i=0}^{l+1} \binom{n-i+1}{i}h^{n-2i+1}(x) - 0 \\
= F^l_{h,n+2}(x). \]

3.3. Proposition. The following equality holds:

\[ \sum_{i=0}^{s} \binom{s}{i} F^{l+i}_{h,n+1}(x)h^i(x) = F^{l+s}_{h,n+2s}(x), \quad 0 \leq l \leq \frac{n-s-1}{2}. \]
Proof. We proceed by induction on \( s \). The sum (3.4) clearly holds for \( s = 0 \) and \( s = 1 \); see (3.2). Now suppose that the result is true for all \( j < s + 1 \). We prove it for \( s + 1 \):

\[
\sum_{i=0}^{s+1} \binom{s+1}{i} F_{h,n+i}(x) h^i(x) = \sum_{i=0}^{s+1} \left( \begin{array}{c} s+1 \\ i \end{array} \right) F_{h,n+i}(x) h^i(x)
\]

\[
= \sum_{i=0}^{s+1} \binom{s+1}{i} F_{h,n+i}(x) h^i(x) + \sum_{i=0}^{s+1} \binom{s}{i-1} F_{h,n+i}(x) h^i(x)
\]

\[
= F_{h,n+2s+2}(x) + \binom{s+1}{1} F_{h,n+s+1}(x) h^{s+1}(x) + \binom{s+1}{2} F_{h,n+s+2}(x) h^{s+1}(x)
\]

\[
= F_{h,n+2s+2}(x) + h(x) \sum_{i=0}^{s} \binom{s}{i} F_{h,n+i+1}(x) h^i(x) + 0
\]

\[
= F_{h,n+2s+2}(x) + h(x) F_{h,n+s+2}(x) = F_{h,n+2s+2}(x).
\]

\[\square\]

3.4. Proposition. For \( n \geq 2l + 2 \),

\[
\sum_{i=0}^{s-1} F_{h,n+i}(x) h^{s-1-i}(x) = F_{h,n+s+1}(x) - h^s(x) F_{h,n+1}(x)
\]

Proof. We proceed by induction on \( s \). The sum (3.5) clearly holds for \( s = 1 \); see (3.2). Now suppose that the result is true for all \( j < s \). We prove it for \( s \):

\[
\sum_{i=0}^{s} F_{h,n+i}(x) h^{s-i}(x) = h(x) \sum_{i=0}^{s} F_{h,n+i}(x) h^{s-i-1}(x) + F_{h,n+s}(x)
\]

\[
= h(x) \left( F_{h,n+s+1}(x) - h^s(x) F_{h,n+1}(x) \right) + F_{h,n+s}(x)
\]

\[
= \left( h(x) F_{h,n+s+1}(x) + F_{h,n+s}(x) \right) - h^{s+1}(x) F_{h,n+1}(x)
\]

\[
= F_{h,n+s+2}(x) - h^{s+1}(x) F_{h,n+1}(x).
\]

\[\square\]

3.5. Lemma. The following equality holds:

\[
F'_{h,n}(x) = h'(x) \left( \frac{nL_{h,n}(x) - h(x) F_{h,n}(x)}{h^2(x) + 4} \right)
\]

Proof. By deriving into the Binet’s formula it is obtained:

\[
F'_{h,n}(x) = n \left[ \alpha^{n-1}(x) - (-\alpha(x))^{-n-1} \right] \alpha'(x)
\]

\[
= \frac{\alpha^n(x) - (-\alpha(x))^{-n}}{\alpha(x) + \alpha^{-1}(x)^{-1}} - \frac{\alpha^n(x) - (-\alpha(x))^{-n}}{[\alpha(x) + \alpha^{-1}(x)]^2}.
\]
where \( \alpha(x) = (h(x) + \sqrt{h^2(x) + 4})/2 \). Then \( \alpha'(x) = (h'(x)\alpha(x))/\alpha(x) + \alpha^{-1}(x) \), 1 - \( \alpha^{-2}(x) = h(x)/\alpha(x) \). Therefore

\[
F'_{h,n}(x) = \frac{n [\alpha^n(x) + (-\alpha(x))^{-n}] h'(x)}{[\alpha(x) + \alpha^{-1}(x)]^2} - \frac{[\alpha^n(x) - (-\alpha(x))^{-n}]}{\alpha(x) + \alpha^{-1}(x)} \cdot \frac{h(x)h'(x)}{[\alpha(x) + \alpha^{-1}(x)]^2}.
\]

On the other hand, \( F_{h,n+1}(x) + F_{h,n-1}(x) = \alpha^n(x) + \beta^n(x) = \alpha^n(x) + (-\alpha(x))^{-n} = L_{h,n}(x) \).

From where, after some algebra Equation (3.6) is obtained.

Lemma 3.5 generalizes Proposition 13 of [4].

3.6. Lemma. The following equality holds:

\[
\sum_{i=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} i \left( n - 1 - i \right) h^{n-2i}(x) = \frac{(h(x)^2 + 4)n - 4)F_{h,n}(x) - nh(x)L_{h,n}(x)}{2(h^2(x) + 4)}.
\]

Proof. From Equation (1.2) we have

\[
h(x)F_{h,n}(x) = \sum_{i=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \left( n - 1 - i \right) h^{n-2i}(x).
\]

By deriving into the above equation:

\[
h'(x)F_{h,n}(x) + h(x)F'_{h,n}(x) = \sum_{i=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} (n - 2i) \left( n - 1 - i \right) h^{n-2i-1}(x)h'(x) = nF_{h,n}(x)h'(x) - 2 \sum_{i=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} i \left( n - 1 - i \right) h^{n-2i-1}(x)h'(x).
\]

From Lemma 3.5

\[
h'(x)F_{h,n}(x) + h(x)h'(x) \left( \frac{nL_{h,n}(x) - h(x)F_{h,n}(x)}{h^2(x) + 4} \right) = nF_{h,n}(x)h'(x) - 2 \sum_{i=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} i \left( n - 1 - i \right) h^{n-2i-1}(x)h'(x).
\]

From where, after some algebra Equation (3.7) is obtained.

3.7. Proposition. The following equality holds:

\[
\sum_{i=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} F'_{h,n}(x) = \begin{cases} 
\frac{4F_{h,n}(x) + nh(x)L_{h,n}(x)}{2(h^2(x) + 4)}, & \text{if } n \text{ is even;} \\
\frac{(h^2(x) + 8)F_{h,n}(x) + nh(x)L_{h,n}(x)}{2(h^2(x) + 4)}, & \text{if } n \text{ is odd.}
\end{cases}
\]
Proof. We have

\[
F_{h,n}(x) = \binom{n-1}{0} h^{n-1}(x) + \binom{n-1}{1} h^{n-3}(x) + \cdots + \binom{n-1}{\frac{n-1}{2}} h^{n-2\lfloor \frac{n-1}{2} \rfloor}(x)
\]

From Lemma 3.6 the Equation (3.8) is obtained. \qed

4. The incomplete \(h(x)\)-Lucas Polynomials

4.1. Definition. The incomplete \(h(x)\)-Lucas polynomials are defined by

\[
L_{l,h,n}(x) = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1}{i} h^{n-2i}(x), \quad 0 \leq l \leq \lfloor \frac{n}{2} \rfloor.
\]

In Table 2, some polynomials of incomplete \(h(x)\)-Lucas polynomials are provided. Note that

\[
L_{l,h,n}(x) = L_n.
\]
The following equality holds:

\[ \text{Proposition.} \]

The relation (4.3) can be transformed into the non-homogeneous recurrence relation

\[ \text{(4.3)} \]

Applying Definition 3.1 to the right-hand side (RHS) of (4.2) results

\[ \text{Proof.} \]

Some special cases of (4.1) are

\[ L_{h,n}^0(x) = h^n(x), \quad (n \geq 1); \]
\[ L_{h,n}^1(x) = h^n(x) + nh^{n-2}(x), \quad (n \geq 2); \]
\[ L_{h,n}^2(x) = h^n(x) + nh^{n-2}(x) + \frac{n(n-3)}{2} h^{n-4}(x), \quad (n \geq 4); \]
\[ L_{h,n}^{\left\lfloor \frac{n-2}{2} \right\rfloor}(x) = L_{h,n}(x), \quad (n \geq 1); \]
\[ L_{h,n}^{\left\lceil \frac{n-2}{2} \right\rceil}(x) = \begin{cases} L_{h,n}(x) - 2, & \text{if } n \geq 2 \text{ and even;} \\ L_{h,n}(x) - nh(x), & \text{if } n \geq 2 \text{ and odd.} \end{cases} \]

4.2. Proposition. The following equality holds:

\[ (4.2) \quad L_{h,n}^l(x) = F_{h,n-1}^l(x) + F_{h,n+1}^l(x); \quad 0 \leq l \leq \left\lfloor \frac{n}{2} \right\rfloor. \]

Proof. Applying Definition 3.1 to the right-hand side (RHS) of (4.2) results

\[ (RHS) = \sum_{i=0}^{l-1} \binom{n-2-i}{i} h^{n-2i}(x) + \sum_{i=0}^{l} \binom{n-i}{i} h^{n-2i}(x) \]
\[ = \sum_{i=1}^{l} \left( \binom{n-1-i}{i-1} h^{n-2i}(x) + \sum_{i=0}^{l} \binom{n-i}{i} h^{n-2i}(x) \right) \]
\[ = \sum_{i=0}^{l} \left[ \frac{(n-1-i)}{i-1} + \binom{n-i}{i} \right] h^{n-2i}(x) - \binom{n-1}{-1} \]
\[ = \sum_{i=0}^{l} \frac{n}{n-i} \binom{n-i}{i} h^{n-2i}(x) + 0 = L_{h,n}^l(x). \]

\[ \square \]

4.3. Proposition. The recurrence relation of the incomplete \( h(x) \)-Lucas polynomials \( L_{h,n}^l(x) \) is

\[ (4.3) \quad L_{h,n+2}^{l+1}(x) = h(x) L_{h,n+1}^{l+1}(x) + L_{h,n}^l(x), \quad 0 \leq l \leq \left\lfloor \frac{n}{2} \right\rfloor. \]

The relation (4.3) can be transformed into the non-homogeneous recurrence relation

\[ (4.4) \quad L_{h,n+2}^l(x) = h(x) L_{h,n+1}^l(x) + L_{h,n}^l(x) - \frac{n}{n-l} \binom{n-l}{l} h^{n-2l}(x). \]

Proof. It is clear from (4.2) and (3.2).
4.4. Proposition. The following equality holds:

\[ h(x)L^l_{h,n}(x) = F^l_{h,n+2}(x) - F^{l-2}_{h,n-2}(x), \quad 0 \leq l \leq \left\lfloor \frac{n-1}{2} \right\rfloor. \]

Proof. By (4.2),

\[ F^l_{h,n+2}(x) = L^l_{h,n+1}(x) - F^{l-1}_{h,n}(x) \quad \text{and} \quad F^{l-2}_{h,n-2}(x) = L^{l-1}_{h,n-1}(x) - F^{l-1}_{h,n}(x), \]

whence, from (4.3)

\[ F^l_{h,n+2}(x) - F^{l-2}_{h,n-2}(x) = L^l_{h,n+1}(x) - L^{l-1}_{h,n-1}(x) = h(x)L^l_{h,n}(x). \]

\[ \square \]

4.5. Proposition. The following equality holds:

\[ \sum_{i=0}^{s} \binom{s}{i} L^{i+1}_{h,n+i}(x)h^i(x) = L^{i+s}_{h,n+2s}(x), \quad 0 \leq l \leq \frac{n-s}{2}. \]

Proof. Using (4.2) and (3.4), we get

\begin{align*}
\sum_{i=0}^{s} \binom{s}{i} L^{i+1}_{h,n+i}(x)h^i(x) &= \sum_{i=0}^{s} \binom{s}{i} \left[ F^{i+1}_{h,n+i-1}(x) + F^{i+1}_{h,n+i+1}(x) \right] h^i(x) \\
&= \sum_{i=0}^{s} \binom{s}{i} F^{i+1}_{h,n+i+1}(x)h^i(x) + \sum_{i=0}^{s} \binom{s}{i} F^{i+1}_{h,n+i-1}(x)h^i(x) \\
&= F^{i+s}_{h,n+1+2s}(x) + F^{i+s}_{h,n+1+2s}(x) = L^{i+s}_{h,n+2s}(x).
\end{align*}

\[ \square \]

4.6. Proposition. For \( n \geq 2l + 1, \)

\[ \sum_{i=0}^{s} L^{i+1}_{h,n+i}(x)h^{s-1-i}(x) = L^{i+1}_{h,n+s+1}(x) - h^s(x)L^{i+1}_{h,n+1}(x). \]

The proof can be done by using (4.3) and induction on \( s. \)

4.7. Lemma. The following equality holds:

\[ \sum_{i=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n}{i} \frac{n-i}{n} h^{n-2i}(x) = \frac{n}{2} \left[ L_{h,n}(x) - h(x)F_{h,n}(x) \right]. \]

The proof is similar to Lemma 3.6.

4.8. Proposition. The following equality holds:

\[ \sum_{l=0}^{\left\lfloor \frac{n}{2} \right\rfloor} L^l_{h,n}(x) = \begin{cases} 
L_{h,n}(x) + \frac{n h(x)}{2} F_{h,n}(x), & \text{if } n \text{ is even}; \\
\frac{1}{2} \left( L_{h,n}(x) + nh(x) F_{h,n}(x) \right), & \text{if } n \text{ is odd}.
\end{cases} \]

Proof. An argument analogous to that of the proof of Proposition 3.7 yields

\[ \sum_{l=0}^{\left\lfloor \frac{n}{2} \right\rfloor} L^l_{h,n}(x) = \left( \left\lfloor \frac{n}{2} \right\rfloor + 1 \right) L_{h,n}(x) - \sum_{i=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{n}{n-i} \binom{n}{i} h^{n-2i}(x). \]

From Lemma 4.7 the Equation (4.5) is obtained. \[ \square \]
5. Generating functions of the incomplete \( h(x) \)-Fibonacci and \( h(x) \)-Lucas polynomials

In this section, we give the generating functions of incomplete \( h(x) \)-Fibonacci and \( h(x) \)-Lucas polynomials.

5.1. Lemma. (See [9], p. 592). Let \( \{s_n\}_{n=0}^{\infty} \) be a complex sequence satisfying the followin non-homogeneous recurrence relation:

\[
s_n = as_{n-1} + bs_{n-2} + r_n, \quad n > 1,
\]

where \( a \) and \( b \) are complex numbers and \( \{r_n\} \) is a given complex sequence. Then the generating function \( U(t) \) of the sequence \( \{s_n\} \) is

\[
U(t) = \frac{G(t) + s_0 - r_0 + (s_1 - sa - r_1)t}{1 - at - bt^2},
\]

where \( G(t) \) denotes the generating function of \( \{r_n\} \).

5.2. Theorem. The generating function of the incomplete \( h(x) \)-Fibonacci polynomials \( F_{h,n}^l(x) \) is given by

\[
R_{h,l}(x) = \sum_{i=0}^{\infty} F_{h,i}^l(x)t^i
\]

\[
= t^{2l+1} [F_{h,2l+1}(x) + (F_{h,2l+2}(x) - h(x)F_{h,2l+1}(x)) t
\]

\[- \frac{t^2}{(1 - h(x)t)^{l+1}}] [1 - h(x)t - t^2]^{-1}.
\]

Proof. Let \( l \) be a fixed positive integer. From (3.1) and (3.3), \( F_{h,n}^l(x) = 0 \) for \( 0 \leq n < 2l+1 \), \( F_{h,2l+1}^l(x) = F_{h,2l+1}(x) \), and \( F_{h,2l+2}^l(x) = F_{h,2l+2}(x) \), and that

\[
F_{h,n}^l(x) = h(x)F_{h,n-1}^l(x) + F_{h,n-2}^l(x) - \binom{n-3-l}{l} h^{n-3-2l}(x).
\]

Now let

\[
s_0 = F_{h,2l+1}^l(x), \quad s_1 = F_{h,2l+2}^l(x), \quad \text{and} \quad s_n = F_{h,n+2l+1}^l(x).
\]

Also let \( r_0 = r_1 = 0 \), and

\[
r_n = \binom{n+l-1}{n-2} h^{n-2}(x).
\]

The generating function of the sequence \( \{r_n\} \) is \( G(t) = t^{2l}/(1 - h(x)t)^{l+1} \); see [13, p. 355].

Thus, from Lemma 5.1, we get the generating function \( R_{h,l}(x) \) of sequence \( \{s_n\} \).

5.3. Theorem. The generating function of the incomplete \( h(x) \)-Lucas polynomials \( L_{h,n}^l(x) \) is given by

\[
S_{h,l}(x) = \sum_{i=0}^{\infty} L_{h,i}^l(x)t^i
\]

\[
= t^{2l} [L_{h,2l}(x) + (L_{h,2l+1}(x) - h(x)L_{h,2l}(x)) t
\]

\[- \frac{t^2(2 - t)}{(1 - h(x)t)^{l+1}}] [1 - h(x)t - t^2]^{-1}.
\]
Proof. The proof is similar to the proof of Theorem 5.2. Let \( l \) be a fixed positive integer. From (4.1) and (4.4),
\[
 L^l_{h,n}(x) = 0 \quad \text{for} \quad 0 \leq n < 2l, \quad L^l_{h,2l}(x) = L_h(2l)(x), \quad \text{and} \quad L^l_{h,2l+1}(x) = L_h(2l+1)(x),
\]
and that
\[
 L^l_{h,n}(x) = h(x)L^l_{h,n-2}(x) + \frac{n-2l}{n-2} \left( \frac{n-2-l}{n-2-2l} \right) h^{n-2-2l}(x).
\]

Now let
\[
s_0 = L^l_{h,2l}(x), \quad s_1 = L^l_{h,2l+1}(x), \quad \text{and} \quad s_n = L^l_{h,n+2l}(x).
\]
Also let \( r_0 = r_1 = 0 \), and
\[
r_n = \left( \frac{n+2l-2}{n+l-2} \right) h^{n+2l-2}(x).
\]
The generating function of the sequence \( \{r_n\} \) is \( G(t) = t^2(2-t)/(1-h(x)t)^{l+1} \); see [13, p. 355]. Thus, from Lemma 5.1, we get the generating function \( S_{h,l}(x) \) of sequence \( \{s_n\} \).

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References
