ON FUZZY TOPOLOGICAL GROUPS
AND FUZZY CONTINUOUS FUNCTIONS

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Abstract
Function spaces play an important role in complex analysis, in the
theory of differential equations, in functional analysis and in almost
every other branch of modern mathematics. Let $F_C(Y, Z)$ be the set
of all fuzzy continuous functions from a fuzzy topological space $Y$ into a
fuzzy topological space $Z$. Our aim in this paper is to study the notion
of group, fuzzy group, topological group, and fuzzy topological group
on the (fuzzy) function space $F_C(Y, Z)$.

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space.

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1. Introduction

The definitions and results for fuzzy topological spaces $(X, \tau)$ and fuzzy topological
groups $(X, \cdot, \tau)$ which are used in this paper, have already been standardized. Some
definitions and results of [1], [4], [7], [9], [10], [13], and [14], which will be needed in the
sequent are recalled here.

Throughout this paper, the symbol $I$ will denote the unit interval $[0, 1]$.

Let $X$ be a nonempty set. A fuzzy set in $X$ is a function with domain $X$ and values
in $I$, that is, an element of $I^X$.

Let $A, B \in I^X$. We define the following fuzzy sets (see [14]):

1. $A \land B \in I^X$ by $(A \land B)(x) = \min\{A(x), B(x)\}$ for every $x \in X$ (intersection).

2. $A \lor B \in I^X$ by $(A \lor B)(x) = \max\{A(x), B(x)\}$ for every $x \in X$ (union).

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Let $X$, $Y$ be two nonempty sets, $f : X \to Y$, $A \in \mathcal{F}(X)$, and $B \in \mathcal{F}(Y)$. Then, $f(A)$ is the fuzzy set in $Y$ defined by

$$f(A)(y) = \begin{cases} \sup\{A(x) : x \in f^{-1}(y)\}, & \text{if } f^{-1}(y) \neq \emptyset, \\ 0, & \text{if } f^{-1}(y) = \emptyset, \end{cases} \quad y \in Y,$$

and $f^{-1}(B)$ is the fuzzy set in $X$ defined by

$$f^{-1}(B)(x) = B(f(x)), \quad x \in X.$$

A fuzzy set in $X$ is called a fuzzy point if and only if it takes the value 0 for all $y \in X$ except one, say $x \in X$. If its value at $x$ is $r$ ($0 < r \leq 1$) we denote the fuzzy point by $p^r_x$, where the point $x$ is called its support and is denoted by supp $p^r_x$ (see, for example [9], [10] and [13]).

The fuzzy point $p^r_x$ is said to be contained in a fuzzy set $A$, or to belong to $A$, denoted by $p^r_x \in A$, if and only if $r \leq A(x)$. Evidently, every fuzzy set $A$ can be expressed as the union of all the fuzzy points which belongs to $A$ (see [9]).

A fuzzy set $A$ in a fuzzy topological space $(X, \tau)$ is called a neighbourhood of a fuzzy point $p^r_x$ if and only if there exists a $V \in \tau$ such that $p^r_x \in V \leq A$ (see [9]). A neighbourhood $A$ is said to be open if and only if $A$ is open.

A fuzzy point $p^r_x$ is said to be quasi-coincident with $A$, denoted by $p^r_x q A$, if and only if $r > A(x)$, or equivalently $r + A(x) > 1$ (see [9]).

A fuzzy set $A$ is said to be quasi-coincident with $B$, denoted by $A q B$, if and only if there exists $x \in X$ such that $A(x) > B(x)$, that is $A(x) + B(x) > 1$ (see [9]). If $A$ is not quasi-coincident with $B$, then we write $A \not q B$.

A fuzzy set $A$ in a fuzzy topological space $(X, \tau)$ is called a $Q$-neighbourhood of $p^r_x$ if and only if there exists $B \in \tau$ such that $p^r_x q B$ and $B \leq A$. The family of all $Q$-neighbourhoods of $p^r_x$ is called the system of $Q$-neighbourhoods of $p^r_x$ (see [9]). A $Q$-neighbourhood of a fuzzy point generally does not contain the point itself.

Let $f$ be a function from $X$ to $Y$. Then (see for example [7], [9], [10] and [13]):

1. Let $p$ be a fuzzy point of $X$, $A$ be a fuzzy set in $X$ and $B$ be a fuzzy set in $Y$. Then, we have:
   - If $f(p) q B$, then $p q f^{-1}(B)$.
   - If $p q A$, then $f(p) q f(A)$.

2. Let $A$ and $B$ be fuzzy sets in $X$ and $Y$, respectively. Let $p$ be a fuzzy point in $X$. Then we have:
   - $p \in f^{-1}(B)$ if $f(p) \in B$.
   - $f(p) \in f(A)$ if $p \in A$.

Let $f$ be a function from a fuzzy topological space $(X, \tau_1)$ into a fuzzy topological space $(Y, \tau_2)$. The map $f$ is said to be fuzzy continuous if for every $U \in \tau_2$, $f^{-1}(U) \in \tau_1$ (see [10]).

Let $f$ be a function from a fuzzy topological space $(X, \tau_1)$ into a fuzzy topological space $(Y, \tau_2)$. Then the following are equivalent (see Theorem 1.1 of [9]):

1. $f$ is fuzzy continuous,
2. for each fuzzy point $p$ in $X$ and each neighbourhood $V$ of $f(p)$ in $Y$, there exists a neighbourhood of $p$ in $X$ such that $f(U) \leq V$.
3. for each fuzzy point $p$ in $X$ and each fuzzy open $Q$-neighbourhood $V$ of $f(p)$ in $Y$, there exists a fuzzy open $Q$-neighbourhood $U$ of $p$ in $X$ such that $f(U) \leq V$.

Let $FC(Y, Z)$ be the set of all fuzzy continuous functions from a fuzzy topological space $Y$ into a fuzzy topological space $Z$. Our aim in this paper is to study the notion of group,
fuzzy group, topological group, and fuzzy topological group on the (fuzzy) function space $F(Y, Z)$.

2. Topological and fuzzy topological groups

2.1. Notation. Let $(X, \cdot)$ be a group, $A, B \in I^X$ and $C, D \subseteq X$. We define $A \cdot B \in I^X$, $A^{-1} \in I^X$, $C \cdot D \subseteq X$ and $C^{-1} \subseteq X$ by the respective formulas:

$$(A \cdot B)(x) = \sup \{\min \{A(x_1), B(x_2)\} : x_1 \cdot x_2 = x\}$$

and

$$A^{-1}(x) = A(x^{-1}),$$

for every $x \in X$. Also

$$C \cdot D = \{c \cdot d : c \in C \text{ and } d \in D\}$$

and

$$C^{-1} = \{e^{-1} : e \in C\}.$$

2.2. Definition. (see [8]) Let $X$ be a set of elements. A triad $(X, \cdot, \tau_X)$ is called a fuzzy topological group if

(i) $(X, \cdot)$ is a group.

(ii) $(X, \tau_X)$ is a fuzzy topological space.

(iii) For all $x, y \in X$ and any fuzzy open $Q$-neighbourhood $W$ of the fuzzy point $p^x_w$, there are fuzzy open $Q$-neighbourhoods $U$ and $V$ of $p^x_u$ and $p^y_v$, respectively such that:

$$U \cdot V \subseteq W.$$

(iv) For all $x \in X$ and any fuzzy open $Q$-neighbourhood $V$ of $p^x_{-1}$, there exists a fuzzy open $Q$-neighbourhood $U$ of $p^x_y$ such that:

$$U^{-1} \subseteq V.$$

2.3. Theorem. Let $(Y, \tau_Y)$ be a fuzzy topological space, $(Z, \cdot, \tau_Z)$ a fuzzy topological group, and $f, g \in FC(Y, Z)$. Then, the maps $f \ast g$ and $f^{-1}$ from the fuzzy topological space $Y$ into the fuzzy topological space $Z$ with the types:

$$(f \ast g)(y) = f(y) \cdot g(y)$$

and

$$f^{-1}(y) = (f(y))^{-1},$$

for every $y \in Y$, are fuzzy continuous.

Proof. Let $p^y_v$, where $r \in (0, 1]$ and $y \in Y$, be a fuzzy point of $Y$ and $W$ a fuzzy open $Q$-neighbourhood of $(f \ast g)(p^y_v) = p^y_{f \ast g(y)} = p^y_{f(y) \cdot g(y)}$ in $Z$. Since $(Z, \cdot, \tau_Z)$ is a fuzzy topological group there exist $Q$-neighbourhoods $U$ and $V$ of $p^y_{f(y)}$ and $p^y_{g(y)}$, respectively such that

$$U \ast V \subseteq W.$$ 

Now, since the maps $f$ and $g$ are fuzzy continuous, there exist fuzzy open $Q$-neighbourhoods $U_1$ and $V_1$ of the fuzzy point $p^y_r$ in $Y$ such that

$$f(U_1) \subseteq U \text{ and } g(V_1) \subseteq V.$$ 

Clearly, the fuzzy set $U_1 \land V_1 \in I^Y$ is a fuzzy open $Q$-neighbourhood of $p^y_r$ in $Y$. We prove that

$$(f \ast g)(U_1 \land V_1) \subseteq W.$$
Indeed, let $p^*_y \in U_1 \land V_1$. We prove that
\[(f \ast g)(p^*_y) = p^*_{f(y_1) \ast g(y_1)} = p^*_{g(y_1)} \in W.\]
We have $p^*_y \in U_1$ and $p^*_y \in V_1$. Therefore, $f(p^*_y) = p^*_{f(y_1)} \in U$ and $g(p^*_y) = p^*_{g(y_1)} \in V$. Hence $r_1 \leq U(f(y_1))$ and $r_1 \leq V(g(y_1))$.

Also, clearly
\[(U \ast V)(f(y_1) \cdot g(y_1)) = \sup\{\min\{U(z_1), V(z_2)\} : z_1 \cdot z_2 = f(y_1) \cdot g(y_1)\} \geq r_1.
\]
Thus, by relation (2.1) we have:
\[r_1 \leq (U \ast V)(f(y_1) \cdot g(y_1)) \leq W(f(y_1) \cdot g(y_1)),\]
and therefore,
\[p^*_{(f(y_1) \cdot g(y_1))} = p^*_{g(y_1)} \in W.\]
So, $(f \ast g)(U_1 \land V_1) \subseteq W$ and, therefore, the map $f \ast g$ is fuzzy continuous.

Finally, we prove that the map $f^{-1}$ is fuzzy continuous. Let $p^*_y$ be a fuzzy point of $Y$ and $W$ a fuzzy open $Q$-neighbourhood of $f^{-1}(p^*_y) = p^*_{f^{-1}(y)} = p^*_{g(\psi^{-1}(y))}$. Since $(Z, , \tau_Z)$ is a fuzzy topological group there exists a fuzzy open $Q$-neighbourhood $U$ of $p^*_{f(y)}$ in $Z$ such that
\[(2.2) \quad U^{-1} \subseteq W.\]
Now, since the map $f$ is fuzzy continuous at the fuzzy point $p^*_y$ in $Y$, there exists a fuzzy open $Q$-neighbourhood $U_1$ of $p^*_y$ such that
\[f(U_1) \subseteq U.\]
For the fuzzy open $Q$-neighbourhood $U_1$ of $p^*_y$ in $Y$ we have:
\[f^{-1}(U_1) \subseteq W.\]
Indeed, let $p^*_{f(y)} \in U_1$. It is sufficient to prove that $f^{-1}(p^*_{f(y)}) \subseteq W$, that is $r_1 \leq W(f^{-1}(y_1)) = W((f(y_1))^{-1})$.

We have $f(p^*_{f(y)}) = p^*_{f(y_1)} \in U$. Thus
\[r_1 \leq U(f(y_1)) = U^{-1}(f(y_1))^{-1} \leq W((f(y_1))^{-1}) \text{ (see relation (2.2))},\]
and therefore, the map $f^{-1}$ is fuzzy continuous. 

\[\square\]

2.4. Definition. (see, for example, [8]) Let $X$ be a set and $r \in [0, 1]$. By $r^* \in \mathcal{I}^X$ we denote the fuzzy set of $X$ for which $r^*(x) = r$, for every $x \in X$. Also, a fuzzy topological space $(X, \tau)$ is called fully stratified if $r^* \in \tau$, for every $r \in [0, 1]$

2.5. Theorem. Let $(Y, \gamma_Y)$ be a fully stratified fuzzy topological space, $(Z, , \tau_Z)$ a fuzzy topological group, and $e$ the identity element of the group $(Z, )$. Then, the map $e'$ from the fuzzy topological space $Y$ into the fuzzy topological space $Z$ with the type:
\[e'(y) = e,\]
for every $y \in Y$, is fuzzy continuous.

Proof. Let $U \in \gamma_Y$. It suffices to prove that
\[(e')^{-1}(U) \in \gamma_Y.\]
We have
\[((e')^{-1}(U))(y) = U(e'(y)) = U(e),\]
for every $y \in Y$. This means that the fuzzy set $(e')^{-1}(U)$ is constant. Therefore, since the fuzzy space $(Y, \gamma_Y)$ is fully stratified, we have $(e')^{-1}(U) \in \gamma_Y$. 

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Thus, the map $e'$ is fuzzy continuous. 

2.6. Theorem. Let $(Y, \tau_Y)$ be a fully stratified fuzzy topological space and $(Z, \cdot, \tau_Z)$ a fuzzy topological group. Then, the pair $(FC(Y, Z), \cdot)$ is a group.

Proof. Let $f, g, h \in FC(Y, Z)$. Then, we have:

$$(f \cdot g) \cdot h(y) = (f \cdot g)(y) \cdot h(y) = (f(y) \cdot g(y)) \cdot h(y) = f(y) \cdot (g \cdot h)(y) = (f \cdot (g \cdot h))(y)$$

for every $y \in Y$, and therefore $(f \cdot g) \cdot h = f \cdot (g \cdot h)$.

Also, let $f \in FC(Y, Z)$. Then, for the map $e' \in FC(Y, Z)$ we have:

$$f \cdot e' = e' \cdot f = f.$$ 

Thus, the map $e' \in FC(Y, Z)$ is the identity element.

Finally, for every $f \in FC(Y, Z)$ there exists the map $f^{-1} \in FC(Y, Z)$ such that

$$f \cdot f^{-1} = f^{-1} \cdot f = e'.$$

Thus, the map $f^{-1} \in FC(Y, Z)$ is the inverse element of $f$.

By the above the pair $(FC(Y, Z), \cdot)$ is a group. 

2.7. Theorem. Let $(Y, \tau_Y)$ be a fully stratified fuzzy topological space, and $(Z, \cdot, \tau_Z)$ a fuzzy topological group. If $(Z, \cdot)$ is an abelian group, then the pair $(FC(Y, Z), \cdot)$ is an abelian group.

Proof. By Theorem 2.6 the pair $(FC(Y, Z), \cdot)$ is a group. Also, for every $f, g \in FC(Y, Z)$ we have:

$$(f \cdot g)(y) = f(y) \cdot g(y) = g(y) \cdot f(y) = (g \cdot f)(y),$$

for every $y \in Y$. Thus $f \cdot g = g \cdot f$, for every $f, g \in FC(Y, Z)$, and, therefore, the pair $(FC(Y, Z), \cdot)$ is an abelian group. 

2.8. Definition. (see [12]) Let $(X, \cdot)$ be a group and $G \in I^X$. Then $G$ is a fuzzy group in $X$ if the following conditions are satisfied:

(i) $G(x \cdot y) \geq \min\{G(x), G(y)\}$, for all $x, y \in X$,
(ii) $G(x^{-1}) \geq G(x)$, for all $x \in X$.

2.9. Theorem. Let $(Y, \tau_Y)$ be a fully stratified fuzzy topological space, $(Z, \cdot, \tau_Z)$ a fuzzy topological group, and $Z_1 \in I^Z$ a fuzzy group. Then, the fuzzy set $G \in I^{FC(Y, Z)}$ for which

$$G(f) = \inf\{Z_1(f(y)) : y \in Y\}, \ f \in FC(Y, Z)$$

is a fuzzy group.

Proof. By Theorem 2.6, the pair $(FC(Y, Z), \cdot)$ is a group.
Now, let \( f, g \in FC(Y, Z) \). Then, we have:
\[
G(f \ast g) = \inf\{Z_1((f \ast g)(y)) : y \in Y\} \\
= \inf\{Z_1(f(y) \cdot g(y)) : y \in Y\} \\
\geq \inf\{\min\{Z_1(f(y)), Z_1(g(y))\} : y \in Y\} \\
\geq \min\{\{Z_1(f(y)) : y \in Y\}, \{Z_1(g(y)) : y \in Y\}\} \\
= \min\{G(f), G(g)\}
\]
Also, for the map \( f^{-1} \in FC(Y, Z) \) we have:
\[
G(f^{-1}) = \inf\{Z_1(f^{-1}(y)) : y \in Y\} \\
= \inf\{Z_1((f(y))^{-1}) : y \in Y\} \\
\geq \inf\{Z_1(f(y)) : y \in Y\} \\
= G(f).
\]
Thus, \( G \in FC(Y, Z) \) is a fuzzy group. \( \square \)

2.10. Definition. (see [3]) Let \( U \in I^Z \) be a fuzzy open set of \( Z \) and \( p^*_y \), where \( r \in (0, 1) \) and \( y \in Y \), a fuzzy point of \( Y \). Then by \([p^*_y; U]\) we denote the following subset of \( FC(Y, Z) \)
\[
[p^*_y; U] = \{f \in FC(Y, Z) : f(p^*_y) < U\}.
\]
The \( F \)-point open topology \( \tau_{F-p-o} \) on \( FC(Y, Z) \) is the topology which has as a subbase the family \( B = \{[p^*_y; U] : U \in I^Z \} \) is a fuzzy open set of \( Z \) and \( p^*_y \) a fuzzy point of \( Y \) \( \cup \{FC(Y, Z)\} \).

2.11. Theorem. Let \((Y, \tau_Y)\) be a fully stratified fuzzy topological space, and \((Z, \cdot, \tau_Z)\) a fuzzy topological group. Then, the triad \((FC(Y, Z), \ast, \tau_{F-p-o})\) is a topological group.

Proof. Clearly, we have:
(i) The pair \((FC(Y, Z), \ast)\) is a group (see Theorem 2.6), and
(ii) The pair \((FC(Y, Z), \tau_{F-p-o})\) is a topological space.

Now, let \( f, g \in FC(Y, Z) \) and let \([p^*_y; W]\) be a subbasic neighbourhood of \( f \ast g \) in \( \tau_{F-p-o} \).
We prove that there exist subbasic neighbourhoods \([p^*_y; U]\) and \([p^*_{y'}; V]\) of \( f \) and \( g \), respectively, such that
\[
[p^*_y; U] \ast [p^*_{y'}; V] \subseteq [p^*_y; W].
\]
Since \( f \ast g \in [p^*_y; W] \) we have \((f \ast g)(p^*_y) < W\) and, therefore,
\[
r < W((f \ast g)(y)),
\]
that is
\[
r < W(f(y) \cdot g(y)).
\]
Let us suppose that \( r = 1 - m \). Then the fuzzy set \( W \) is a fuzzy open \( Q \)-neighbourhood of \( p^m_{m(y); g(y)} \). Indeed, we have
\[
r < W(f(y) \cdot g(y)),
\]
that is
\[
1 - m < W(f(y) \cdot g(y)).
\]
Thus \( 1 < m + W(f(y) \cdot g(y)) \) and therefore \( p^m_{m(y); g(y)} \not\subseteq W \).
Now, since \((Z, \cdot, \tau_Z)\) is a fuzzy topological group, there exist fuzzy open \(Q\)-neighbourhoods \(U\) and \(V\) of \(p^{m}_{f(y)}\) and \(p^{m}_{g(y)}\), respectively such that: 

\[ U \circ V \subseteq W. \]

We consider the subsets: 

\[ [p^{\cdot m}_{y} - m; U] \text{ and } [p^{\cdot m}_{y} - m; V] \]

of \(FC(Y, Z)\). Clearly, the above subsets are subbasic neighbourhoods of \(f\) and \(g\), respectively. We prove that: 

\[ [p^{\cdot m}_{y} - m; U] \ast [p^{\cdot m}_{y} - m; V] \subseteq [p^{\cdot m}_{y} - m; W]. \]

Let \(f_{1} \ast g_{1} \in [p^{\cdot m}_{y} - m; U] \ast [p^{\cdot m}_{y} - m; V]\). Then, we have: 

\[ 1 - m < U(f_{1}(y)) \text{ and } 1 - m < V(g_{1}(y)). \]

Thus, 

\[
W((f_{1} \ast g_{1})(y)) = W(f_{1}(y) \cdot g_{1}(y)) \\
\geq (U \circ V)(f_{1}(y) \cdot g_{1}(y)) \\
= \sup\{\min\{U(z_{1}), V(z_{2})\} : z_{1} \cdot z_{2} = f_{1}(y) \cdot g_{1}(y)\} \\
\geq \min\{U(f_{1}(y)), V(g_{1}(y))\} \\
> 1 - m.
\]

Hence \((f_{1} \ast g_{1})(p^{\cdot m}_{y}) = p^{\cdot m}_{f_{1} \ast g_{1} (y)} < W\) and therefore \(f_{1} \ast g_{1} \in [p^{\cdot m}_{y} - m; W]\).

Finally, let \(f \in [p^{\cdot m}_{y}; W]\). We prove that there exists a subbasic neighbourhood \([p^{\cdot m}_{y} - m; U]\) of \(f^{-1}\) such that: 

\[ [p^{\cdot m}_{y} - m; U]^{-1} \subseteq [p^{\cdot m}_{y}; W]. \]

Since \(f \in [p^{\cdot m}_{y}; W]\) we have \(r < W(f(y))\). Let \(r = 1 - m\). Then the fuzzy set \(W\) is a fuzzy open \(Q\)-neighbourhood of \(p^{m}_{f(y)}\) in \(Z\). Since \((Z, \cdot, \tau_Z)\) is a fuzzy topological group there exists a fuzzy open \(Q\)-neighbourhood \(U\) of \(p^{m}_{f(y)}\) and \(U^{-1}\) such that: 

\[ U^{-1} \subseteq W. \]

Clearly, the subset \([p^{\cdot m}_{y} - m; U]^{-1}\) is a neighbourhood of \(f^{-1}\). We prove that 

\[ [p^{\cdot m}_{y} - m; U]^{-1} \subseteq [p^{\cdot m}_{y}; W]. \]

Let \(f_{1} \in [p^{\cdot m}_{y} - m; U]^{-1}\). Then \(f_{1}^{-1} \in [p^{\cdot m}_{y} - m; U]\) and therefore \(1 - m < U((f_{1}(y))^{-1})\). Thus 

\[
W(f_{1}(y)) \geq U^{-1}((f_{1}(y))^{-1}) \geq U((f_{1}(y))^{-1}) > 1 - m \text{ and therefore:} \\
[p^{\cdot m}_{y} - m; U]^{-1} \subseteq W,
\]

as required. \(\square\)

\textbf{2.12. Definition.} (see [2], and for a similar definition see [4], [5], [6], and [11]) Let \(Y, Z\) be two fixed fuzzy topological spaces, \(U \in I^{Z}\) a fuzzy open set of \(Z\), and \(y \in Y\). Then, by \((y; U) \in FC(Y, Z)\) we denote the fuzzy set for which 

\[ (y; U)(f) = U(f(y)), \]

for every \(f \in FC(Y, Z)\).

The fuzzy point-open topology \(\tau_{fp}\) on \(FC(Y, Z)\) is generated by fuzzy sets of the form: 

\[ (y; U), \]

where \(y \in Y\) and \(U \in I^{Z}\) is a fuzzy open set of \(Z\).

\textbf{2.13. Theorem.} Let \((Y, \tau_Y)\) be a fully stratified fuzzy topological space, and \((Z, \cdot, \tau_Z)\) a fuzzy topological group. Then, the triad \((FC(Y, Z), *, \tau_{fp})\) is a fuzzy topological group.
We consider the fuzzy sets \((p, Q)\) and so that is also not difficult to prove that for every \(p\) topological group, and \(Q\). Since \(p\), we prove that there exist fuzzy subbasic \(Q\)-neighbourhoods \((y_1; U)\) and \((y_2; V)\) of \(p_f^r\) and \(p_{g}^r\), respectively such that
\[
(y_1; U) \bullet (y_2; V) \leq (y; W).
\]
Since \(p^r_{f \ast g} q (y; W)\) we have:
\[
r + W((f \ast g)(y)) > 1,
\]
that is
\[
r + W(f(y) \cdot g(y)) > 1
\]
and so
\[
p^r_{f(y) \cdot g(y)} q W.
\]
Thus, the fuzzy set \(W\) is a fuzzy open \(Q\)-neighbourhood of \(p^r_{f(y) \cdot g(y)}\). Now, since \((Z, \cdot, \tau_Z)\) is a fuzzy topological group there exist fuzzy open \(Q\)-neighbourhoods \(U\) and \(V\) of \(p^r_{f(y)}\) and \(p_{g(y)}^r\) such that
\[
U \bullet V \leq W.
\]
We consider the fuzzy sets \((y; U)\) and \((y; V)\). Clearly, the fuzzy sets \((y; U)\) and \((y; V)\) are \(Q\)-neighbourhoods of \(p^r_f\) and \(p_{g}^r\), respectively. We prove that:
\[
(y; U) \bullet (y; V) \leq (y; W).
\]
Indeed, let \(f \in FC(Y, Z)\). Then, we have:
\[
(y; U) \bullet (y; V)(f) = \sup\{\min\{(y; U)(f_1), (y; V)(f_2)\} : f_1 \ast f_2 = f\} = \sup\{\min\{U(f_1(y)), V(f_2(y))\} : f_1 \ast f_2 = f\} \leq \sup\{\min\{U(z_1), V(z_2)\} : z_1 \cdot z_2 = f(y)\} = (U \bullet V)(f(y)) \leq W(f(y)) = (y; W)(f).
\]
Also, it is not difficult to prove that for every \(f \in FC(Y, Z)\) and for every fuzzy open \(Q\)-neighbourhood \((y; V)\) of \(p^r_f\), there exists a fuzzy open \(Q\)-neighbourhood \((y_1; U)\) of \(p^r_f\) such that
\[
(y_1; U)^{-1} \leq (y; V).
\]
Thus, the triad \((FC(Y, Z), \ast, \tau_{fp})\) is a fuzzy topological group.

**2.14. Theorem.** Let \((Y, \tau_Y)\) be a fully stratified fuzzy topological space, \((Z, \cdot, \tau_Z)\) a fuzzy topological group, and \((Z, \tau_Z)\) a fully stratified space. Then, the triad \((FC(Y, Z), \ast, \tau_{fp})\) is a fuzzy topological group and the pair \((FC(Y, Z), \tau_{fp})\) is a fully stratified space.

**Proof.** By Theorem 2.13 the triad \((FC(Y, Z), \ast, \tau_{fp})\) is a fuzzy topological group.

Now, let \(r \in [0, 1]\) and denote by \(r^\ast\) the constant fuzzy subset of \(Z\) with value \(r\). If \(y\) is a fixed but arbitrary element of \(Y\) then for any \(f \in FC(Y, Z)\) we have
\[
(y; r^\ast)(f) = r^\ast(f(y)) = r,
\]
and so \((y; f^\ast)\) is the constant fuzzy subset of \(FC(Y, Z)\) with value \(r\). Since \((y; r^\ast) \in \tau_{fp}\) it follows that \((FC(Y, Z), \tau_{fp})\) is fully stratified.
2.15. Theorem. Let \((Y, \tau_Y)\) be a fully stratified fuzzy topological space, \((Z, \cdot, \tau_Z)\) a fuzzy topological group, and \((Z, \tau_Z)\) a fully stratified space. Then the mappings \(F(f) = f \cdot a\), \(G(f) = a \cdot f\), and \(H(f) = f^{-1}\) are all homeomorphic mappings of \(FC(Y, Z)\) onto itself, where \(a \in FC(Y, Z)\) is a definite point.

Proof. By Theorem 2.14 the pair \((FC(Y, Z), \tau_{\mu})\) is a fully stratified space. Thus, by Proposition 2.4 of [8] the maps \(F\), \(G\), and \(H\) are all homeomorphic mappings of \(FC(Y, Z)\) onto itself. \(\Box\)

2.16. Theorem. Let \((Y, \tau_Y)\) be a fully stratified fuzzy topological space, \((Z, \cdot, \tau_Z)\) a fuzzy topological group, and \((Z, \tau_Z)\) a fully stratified space. Then for every fuzzy points \(p^*_r f\) and \(p^*_r g\) of \(FC(Y, Z)\), there exists a homeomorphic mapping \(F\) of \(FC(Y, Z)\) onto itself such that \(F(p^*_r f) = p^*_r g\). This property is called the homogeneity of the fuzzy topological group \((FC(Y, Z), \cdot, \tau_{\mu})\).

Proof. By Theorem 2.14 the pair \((FC(Y, Z), \tau_{\mu})\) is a fully stratified space. Thus, by Propositions 2.4 and 2.6 of [8], for all fuzzy points \(p^*_r f\) and \(p^*_r g\) of \(FC(Y, Z)\), there exists a homeomorphic mapping \(F\) of \(FC(Y, Z)\) onto itself such that \(F(p^*_r f) = p^*_r g\). \(\Box\)

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References