An inequality for warped product submanifolds of a locally product Riemannian manifold

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Abstract

Recently, Sahin studied the warped product semi-slant submanifolds of locally product Riemannian manifolds. In this paper, we obtain some geometric properties of such submanifolds with an example. Also, we establish a sharp relationship between the squared norm of the second fundamental form and the warping function in terms of the slant angle. The equality case is also considered.

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1. Introduction

The idea of slant immersions is an increasing development in differential geometry which is given by Chen [6] in almost Hermitian setting. Recently, Sahin defined and studied these immersions for locally product Riemannian manifolds [10]. In [9], Papaghuic has extended this idea to semi-slant submanifolds of almost Hermitian manifolds which are the generalization of holomorphic, totally real and slant submanifolds. Recently, these submanifolds of locally product Riemannian manifolds were studied by Li and Li [7].

On the other hand, the idea of warped product manifolds is given by Bishop and O’Neill in [5]. They defined these manifolds as follows: Let $M_1$ and $M_2$ be two Riemannian manifolds with Riemannian metrics $g_1$ and $g_2$, respectively, and a positive differentiable function $f$ on $M_1$. Consider the product manifold $M_1 \times M_2$ with its projections $\pi_1 : M_1 \times M_2 \to M_1$ and $\pi_2 : M_1 \times M_2 \to M_2$. Then their warped product manifold $M = M_1 \times_f M_2$ is the Riemannian manifold $M_1 \times M_2 = (M_1 \times M_2, g)$ equipped with the Riemannian structure such that

$$g(X, Y) = g_1(\pi_1 \ast X, \pi_1 \ast Y) + (f \circ \pi_1)^2 g_2(\pi_2 \ast X, \pi_2 \ast Y)$$

for any vector field $X, Y$ tangent to $M$, where $\ast$ is the symbol for the tangent maps. A warped product manifold $M = M_1 \times_f M_2$ is said to be trivial or simply Riemannian product manifold if the warping function $f$ is constant. Let $X$ be an unit vector field
tangent to $M_1$ and $U$ be another unit vector field on $M_2$, then from Lemma 7.3 of [5], we have
\[ \nabla_X U = \nabla_U X = (X \tilde{f}) U, \]  
where $\nabla$ is the Levi-Civita connection on $M$. If $M = M_1 \times_f M_2$ be a warped product manifold then $M_1$ is a totally geodesic submanifold of $M$ and $M_2$ is a totally umbilical submanifold of $M$ [5].

Recently, warped product submanifolds of locally product Riemannian manifolds were studied in [1–3, 11, 12, 15]. Furthermore, the warped products of almost contact metric and almost Hermitian manifolds were appeared in [13, 14, 16, 17]. In this paper, we study warped product semi-slant submanifolds of a locally product Riemannian manifold. We discuss some geometric properties of such submanifolds and give an example, our example fulfil the definition of a proper semi-slant submanifold i.e., the slant angle lies in the first quadrant. Also, we establish an inequality for the squared norm of second fundamental form in terms of the warping function and the slant angle. The equality case is also discussed.

2. Preliminaries

Let $\tilde{M}$ be a $m$-dimensional differentiable manifold with a tensor field $F$ of type $(1,1)$ such that $F^2 = I$ and $F \neq \pm I$, then we say that $\tilde{M}$ is an almost product manifold with almost product structure $F$. If we set
\[ P = \frac{1}{2}(I + F), \quad Q = \frac{1}{2}(I - F) \]
then we can easily see that
\[ P + Q = I, \quad P^2 = P, \quad Q^2 = Q, \quad PQ = QP = 0, \quad \text{and} \quad P - Q = F. \]
Thus $P$ and $Q$ define two orthogonal complementary distributions $\mathcal{D}_1$ and $\mathcal{D}_2$ on $\tilde{M}$. If $\tilde{M}$ admits a Riemannian metric $g$ such that
\[ g(FX, FY) = g(X, Y) \]  
for any $X, Y \in \Gamma(T\tilde{M})$, then $\tilde{M}$ is called an almost product Riemannian manifold [18], where $\Gamma(T\tilde{M})$ denotes the set all vector fields of $\tilde{M}$. Since $F^2 = I$, we can easily see that the eigenvalues of $F$ are 1 or $-1$. An eigenvector corresponding to the eigenvalue 1 associates with $P$ and the eigenvector corresponding to the eigenvalue $-1$ is associated with $Q$. Let $\tilde{\nabla}$ denotes the Levi-Civita connection on $\tilde{M}$ with respect to the Riemannian metric $g$. Then the covariant derivative of $F$ is defined by
\[ (\tilde{\nabla}_X F)Y = \tilde{\nabla}_X FY - F\tilde{\nabla}_X Y \]
for any $X, Y \in \Gamma(T\tilde{M})$. If $(\tilde{\nabla}_X F)Y = 0$, the almost product Riemannian manifold $\tilde{M}$ is said to be a locally product Riemannian manifold.

Let $M$ be a Riemannian manifold isometrically immersed in $\tilde{M}$ and we denote by the same symbol $g$ the Riemannian metric induced on $M$. Let $\Gamma(TM)$ be the Lie algebra of vector fields in $M$ and $\Gamma(T^\perp M)$, the set of all vector fields normal to $M$. Let $\nabla$ be the Levi-Civita connection on $M$. Then the Gauss and Weingarten formulas are respectively given by
\[ \tilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y), \quad \tilde{\nabla}_X N = -A_N X + \nabla_X^N N \]  
for any $X, Y \in \Gamma(TM)$ and $N \in \Gamma(T^\perp M)$, where $\nabla^\perp$ is the normal connection in the normal bundle $\Gamma(T^\perp M)$ and $A_N$ is the shape operator of $M$ with respect to $N$. Moreover, $\sigma : TM \times TM \rightarrow T^\perp M$ is the second fundamental form of $M$ in $\tilde{M}$. Furthermore, $A_N$ and $\sigma$ are related by
\[ g(\sigma(X, Y), N) = g(A_N X, Y) \]
for any $X, Y \in \Gamma(TM)$ and $N \in \Gamma(T^\perp M)$.

For any $X$ tangent to $M$, we write

$$FX = TX + \omega X,$$

(2.4)

where $TX$ (resp. $\omega X$) denotes the tangential (resp. normal) component of $FX$. Then $T$ is an endomorphism of tangent bundle $TM$ and $\omega$ is a normal bundle valued 1-form on $TM$.

A submanifold $M$ is said to be $F$-invariant if $\omega$ is identically zero, i.e., $FX \in \Gamma(TM)$, for any $X \in \Gamma(TM)$. On the other hand, $M$ is said to be $F$-anti-invariant if $T$ is identically zero i.e., $FX \in \Gamma(T^\perp M)$, for any $X \in \Gamma(TM)$. Moreover, from (2.1) and (2.4), we have $g(TX, TY) = g(X, TY)$, for any $X, Y \in \Gamma(TM)$.

A submanifold $M$ of a locally product Riemannian manifold $M$ is said to be totally umbilical submanifold if $\sigma(X, Y) = g(X, Y)H$, for any $X, Y \in \Gamma(TM)$, where $H = \frac{1}{n} \sum_{i=1}^{n} \sigma(e_i, e_i)$, the mean curvature vector of $M$. A submanifold $M$ is said to be totally geodesic if $\sigma(X, Y) = 0$. Also, we set

$$\sigma_{ij} = g(\sigma(e_i, e_j), e_r), \quad i, j = 1, \ldots, n; \quad r = n + 1, \ldots, m,$

and

$$\|\sigma\|^2 = \sum_{i,j=1}^{n} g(\sigma(e_i, e_j), \sigma(e_i, e_j))$$

(2.5)

where $\{e_1, \ldots, e_n\}$ is an orthonormal basis of the tangent space $T_pM$, for any $p \in M$.

For a differentiable function $f$ on an $m$-dimensional manifold $M$, the gradient $\nabla f$ of $f$ is defined as $g(\nabla f, X) = Xf$, for any $X$ tangent to $M$. As a consequence, we have

$$\|\nabla f\|^2 = \sum_{i=1}^{m} (e_i(f))^2$$

(2.6)

for an orthonormal frame $\{e_1, \ldots, e_m\}$ on $M$.

By the analogy with submanifolds in a Kaehler manifold, different classes of submanifolds in a locally product Riemannian manifold were considered.

Let $\bar{M}$ be a locally product Riemannian manifold and $M$ a submanifold of $\bar{M}$, then $M$ is said to be semi-invariant submanifold of $\bar{M}$ [4] if there exist a differentiable distribution $D : p \rightarrow D_p \subset T_pM$ such that $D$ is invariant with respect to $F$ and the complementary distribution $D^\perp$ is anti-invariant with respect to $F$ [8].

Let $M$ be a Riemannian submanifold of a locally product Riemannian manifold $\bar{M}$. For each nonzero vector field $X \in \Gamma(TM)$ at $p \in M$, the angle $\theta(X)$, $0 \leq \theta(X) \leq \frac{\pi}{2}$ between $FX$ and $T_pM$ is called the Wirtinger angle of $X$. If the angle $\theta(X)$ is constant, which is independent of the choice $p \in M$ and $X \in \Gamma(TM)$, then $M$ is called a slant submanifold of $\bar{M}$ and the angle $\theta$ is called the slant angle of the immersion. Thus, $F$-invariant immersion and $F$-anti-invariant immersion are slant immersions with slant angle $\theta = 0$ and $\theta = \frac{\pi}{2}$, respectively. A slant immersion which is neither $F$-invariant nor $F$-anti-invariant is called proper slant immersion. It is easy to see that $M$ is slant submanifold of a locally product Riemannian manifold $\bar{M}$ if and only if

$$T^2 = M$$

(2.7)

for some real number $\lambda \in [0, 1]$ [10], where $I$ denotes the identity transformation of the tangent bundle $TM$ of the submanifold $M$. Moreover, if $M$ is a slant submanifold and $\theta$ is the slant angle of $M$, then $\lambda = \cos^2 \theta$. The following relations are consequences of (2.7)

$$g(TX, TY) = \cos^2 \theta g(X, Y)$$

$$g(\omega X, \omega Y) = \sin^2 \theta g(X, Y)$$

for any $X, Y \in \Gamma(TM)$. 
Moreover, we say that $M$ is a semi-slant submanifold of $\tilde{M}$, if there exist two orthogonal distributions $\mathcal{D}$ and $\mathcal{D}^\theta$ such that

(i) $TM = \mathcal{D} \oplus \mathcal{D}^\theta$,
(ii) the distribution $\mathcal{D}$ is invariant, i.e. $F(\mathcal{D}) = \mathcal{D}$,
(iii) the distribution $\mathcal{D}^\theta$ is slant with slant angle $\theta \neq 0$.

We will call $\mathcal{D}^\theta$ as a slant distribution. In particular, if $\theta = \frac{\pi}{2}$, then the semi-slant submanifold is a semi-invariant submanifold [3]. On the other hand, If we denote the dimension of $\mathcal{D}$ and $\mathcal{D}^\theta$ by $d_1$ and $d_2$, respectively. It is clear that if $d_1 = 0$ and $\theta \neq \frac{\pi}{2}$, then $M$ is a proper slant submanifold with slant angle $\theta$. Also, if $d_2 = 0$ then $M$ is $F$-invariant submanifold and if $d_1 = 0$ and $\theta = \frac{\pi}{2}$ then $M$ is $F$-anti-invariant submanifold. Furthermore, if neither $d_1 = 0$ nor $\theta = \frac{\pi}{2}$, then $M$ is a proper semi-slant submanifold.

3. Warped product semi-slant submanifolds

Warped product semi-slant submanifolds of locally product Riemannian manifolds were studied by Atceken and Sahin in [1] and [12]. They proved that the warped products of the type $M_T \times_f M_\theta$ do not exist. On the other hand, they provided some examples and a characterization on the existence of warped products $M_\theta \times_f M_T$, where $M_T$ and $M_\theta$ are invariant and proper slant submanifolds of a locally product Riemannian manifold $M$, respectively. In this section, we study the warped product semi-slant submanifolds of type $M_\theta \times_f M_T$ of locally product Riemannian manifolds which have not been considered in earlier studies. First, we prove the following results for later use.

**Lemma 3.1.** Let $M = M_\theta \times_f M_T$ be a warped product semi-slant submanifold of a locally product Riemannian manifold $\tilde{M}$. Then

(i) $g(\sigma(X,U), \omega V) = -g(\sigma(X,V), \omega U)$;
(ii) $g(\sigma(X,Y), \omega U) = -(U \ln f)g(X, FY) + (TU \ln f)g(X,Y)$;
(iii) $g(\sigma(X,Y), \omega TU) = -(TU \ln f)g(X, FY) + \cos^2 \theta(U \ln f)g(X,Y)$

for any $X, Y \in \Gamma(TM_T)$ and $U, V \in \Gamma(TM_\theta)$.

**Proof.** For any $X \in \Gamma(TM_T)$ and $U, V \in \Gamma(TM_\theta)$, we have

$$g(\sigma(X,U), \omega V) = g(\tilde{\nabla}_X U, FV) - g(\tilde{\nabla}_X U, TV)$$

$$= g((\tilde{\nabla}_X F)U, V) - g(\tilde{\nabla}_X FU, V) - (U \ln f)g(X, TV).$$

First and last terms in the right hand side are identically zero by using the structure of a locally product Riemannian manifold and the orthogonality of vector fields, then from (2.4), we obtain

$$g(\sigma(X,U), \omega V) = g(\tilde{\nabla}_X TU, V) + g(\tilde{\nabla}_X \omega U, V).$$

By using (2.2)-(2.3) and (1.1) in above relation, we get (i). For the second part, consider $X, Y \in \Gamma(TM_T)$ and $U \in \Gamma(TM_\theta)$, then we have

$$g(\sigma(X,Y), \omega U) = g(\tilde{\nabla}_X Y, \omega U)$$

$$= g(\tilde{\nabla}_X Y, FU) - g(\tilde{\nabla}_X Y, TU)$$

$$= g((F \tilde{\nabla}_X Y, U) + g(Y, \tilde{\nabla}_X TU)$$

$$= g(\tilde{\nabla}_X FY, U) + (TU \ln f)g(X,Y)$$

$$= -g(FY, \tilde{\nabla}_X U) + (TU \ln f)g(X,Y).$$

Using (2.2) and (1.1), we obtain

$$g(\sigma(X,Y), \omega U) = -(U \ln f)g(X, FY) + (TU \ln f)g(X,Y),$$

which is (ii). If we replace $U$ by $TU$ in (ii), and then using (2.7), we get (iii), which proves the lemma completely. □
From the above lemma we can easily find the following relations by interchanging $X$ by $FX$ and $Y$ by $FY$, for any $X, Y \in \Gamma(TM_T)$ in Lemma 3.1 (ii)-(iii)
\begin{align}
g(\sigma(X, FY), \omega U) &= -(U \ln f)g(X, Y) + (TU \ln f)g(X, FY), \quad (3.1) \\
g(\sigma(FX, Y), \omega U) &= -(U \ln f)g(X, Y) + (TU \ln f)g(FX, Y) \quad (3.2)
\end{align}
and
\begin{align}
g(\sigma(FX, Y), \omega TU) &= -(TU \ln f)g(X, Y) + \cos^2 \theta(U \ln f)g(FX, Y) \quad (3.3) \\
g(\sigma(X, FY), \omega TU) &= -(TU \ln f)g(X, Y) + \cos^2 \theta(U \ln f)g(X, FY). \quad (3.4)
\end{align}
Then using (2.1) in (3.1) and (3.2), we find
\begin{align}
g(\sigma(X, FY), \omega U) = g(\sigma(FX, Y), \omega U). \quad (3.5)
\end{align}
Similarly, from (3.3) and (3.4), we obtain
\begin{align}
g(\sigma(FX, Y), \omega TU) = g(\sigma(X, FY), \omega TU). \quad (3.6)
\end{align}
Also, if we interchange $X$ by $FX$ in (3.1) and $Y$ by $FY$ in (3.3), we arrive at
\begin{align}
g(\sigma(FX, FY), \omega U) &= -(U \ln f)g(FX, Y) + (TU \ln f)g(X, Y), \quad (3.7) \\
g(\sigma(FX, FY), \omega TU) &= -(TU \ln f)g(FX, Y) + \cos^2 \theta(U \ln f)g(X, Y). \quad (3.8)
\end{align}
Then from (3.7) and Lemma 3.1 (ii), we get
\begin{align}
g(\sigma(FX, FY), \omega U) = g(\sigma(X, Y), \omega U) \quad (3.9)
\end{align}
and by Lemma 3.1 (iii) and (3.8), we derive
\begin{align}
g(\sigma(FX, FY), \omega TU) = g(\sigma(X, Y), \omega TU). \quad (3.10)
\end{align}
Now, we construct the following example of warped product semi-slant submanifolds of a locally product Riemannian manifold.

**Example 3.2.** Let us consider the almost product manifold $\mathbb{R}^7 = \mathbb{R}^3 \times \mathbb{R}^4$ with coordinates $(x_1, x_2, x_3, y_1, y_2, y_3, y_4)$ and the product structure
\begin{align}
F(\frac{\partial}{\partial x_i}) = -\frac{\partial}{\partial x_i}, \quad F(\frac{\partial}{\partial y_j}) = \frac{\partial}{\partial y_j}, \quad i = 1, 2, 3 \quad \text{and} \quad j = 1, 2, 3, 4.
\end{align}
Let $M$ be a submanifold of $\mathbb{R}^7$ given by
\begin{align}
f(\theta, \varphi, v, u) = (u \cos \theta, \ u \sin \theta, \ u + v, \ \sqrt{3}u - v, \ v, \ u \sin \varphi, \ u \cos \varphi)
\end{align}
with $u \neq 0, \ v \neq 0$ and $\theta, \ \varphi \in \left(0, \frac{\pi}{2}\right)$.
Then the tangent space $TM$ of $M$ is spanned by the following vector fields
\begin{align}
Z_1 &= -u \sin \theta \frac{\partial}{\partial x_1} + u \cos \theta \frac{\partial}{\partial x_2}, \quad Z_2 = u \cos \varphi \frac{\partial}{\partial y_3} - u \sin \varphi \frac{\partial}{\partial y_4}, \\
Z_3 &= \frac{\partial}{\partial x_3} - \frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_2} \\
Z_4 &= \cos \theta \frac{\partial}{\partial x_1} + \sin \theta \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3} + \sqrt{3} \frac{\partial}{\partial y_1} + \sin \varphi \frac{\partial}{\partial y_3} + \cos \varphi \frac{\partial}{\partial y_4}.
\end{align}
Then with respect to the Riemannian product structure $F$, we get
\begin{align}
FZ_1 &= -Z_1, \quad FZ_2 = Z_2, \quad FZ_3 = -\frac{\partial}{\partial x_3} - \frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_2} \\
FZ_4 &= -\cos \theta \frac{\partial}{\partial x_1} - \sin \theta \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_3} + \sqrt{3} \frac{\partial}{\partial y_1} + \sin \varphi \frac{\partial}{\partial y_3} + \cos \varphi \frac{\partial}{\partial y_4}.
\end{align}
Then, it is easy to see that the invariant and slant distributions are spanned by
\begin{align}
\mathcal{D} = \text{span}\{Z_1, Z_2\} \quad \text{and} \quad \mathcal{D}^{\theta_1} = \text{span}\{Z_3, Z_4\}.
\end{align}
with the slant angle say $\theta_1$. Then

$$\theta_1 = \arccos \left( \frac{g(FZ_3, Z_3)}{\|FZ_3\| \|Z_3\|} \right) = \arccos \left( \frac{g(FZ_4, Z_4)}{\|FZ_4\| \|Z_4\|} \right) = \arccos \left( \frac{1}{3} \right) = 70^\circ31'. $$

Thus, $M$ is a proper semi-slant submanifold of $\mathbb{R}^7$. It is also easy to check that $\mathcal{D}$ and $\mathcal{D}^{\theta_1}$ are integrable. If we denote the integral manifolds of $\mathcal{D}$ and $\mathcal{D}^{\theta_1}$ by $M_T$ and $M_{\theta_1}$, respectively. Then the induced metric tensor on $M$ is given by

$$g = \left( 6du^2 + 3dv^2 \right) + u^2 \left( d\theta^2 + d\varphi^2 \right) = g_{M_{\theta_1}} + u^2 g_{M_T}. $$

Thus $M$ is a warped product submanifold of the form $M = M_{\theta_1} \times_f M_T$ with the warping function $f_1 = u$.

If $M$ is a warped product semi-slant submanifold of the form $M = M_\theta \times_f M_T$ of a locally product Riemannian manifold $\tilde{M}$ and if there is no $p$-components in the normal bundle of $M$, then $M$ is mixed totally geodesic (Proposition 4.1 [12]) i.e., $\sigma(X, U) = 0$, for any $X \in \Gamma(TM_T)$ and $U \in \Gamma(TM_\theta)$.

Now, we construct the following frame field for an $n$-dimensional warped product semi-slant submanifold $M = M_\theta \times_f M_T$ of a $m$-dimensional locally product Riemannian manifold $\tilde{M}$. Let us denote by $\mathcal{D}$ and $\mathcal{D}^\theta$ the tangent bundles of $M_T$ and $M_\theta$, respectively instead of $TM_T$ and $TM_\theta$. Also, we consider the $\dim(M_T) = t$ and $\dim(M_\theta) = q$, then the orthonormal frames of $\mathcal{D}$ and $\mathcal{D}^\theta$, respectively are given by $\{e_1 = F\epsilon_1, \ldots, e_k = F\epsilon_k, e_{k+1} = -F\epsilon_{k+1}, \ldots, e_l = -F\epsilon_l\}$ and $\{e_{l+1} = e_1^* = \sec \theta T\epsilon_1^*, \ldots, e_{l+q} = e_q^* = \sec \theta T\epsilon_q^*\}$. Then the orthonormal frame fields of the normal subbundles of $\omega \mathcal{D}_\theta$ and $\mu$, respectively are $\{e_{n+1} = \tilde{\epsilon}_1 = \csc \theta \omega \epsilon_1^*, \ldots, e_{n+q} = \tilde{\epsilon}_q = \csc \theta \omega \epsilon_q^*\}$ and $\{e_{n+q+1} = \tilde{\epsilon}_{q+1}, \ldots, e_m = \tilde{\epsilon}_{m-n-q}\}$. Now, we are able to construct the following inequality with the help of the above constructed frame fields and some previous formulas which we have obtained for warped product semi-slant submanifolds of a locally product Riemannian manifold.

**Theorem 3.3.** Let $M = M_\theta \times_f M_T$ be a proper warped product semi-slant submanifold of a locally product Riemannian manifold $\tilde{M}$, where $M_T$ and $M_\theta$ are invariant and proper slant submanifolds of $\tilde{M}$, respectively. Then

(i) The squared norm of the second fundamental form of the warped product immersion satisfies

$$\|\sigma\|^2 \geq t(csc \theta - \cot \theta)^2 \|\nabla^\theta \ln f\|^2$$

where $t = \dim(M_T)$ and $\nabla^\theta \ln f$ is gradient of the function $\ln f$ along $M_\theta$.

(ii) If equality sign in (i) holds identically, then $M_\theta$ is totally geodesic in $\tilde{M}$ and $M_T$ is a totally umbilical submanifold of $M$. Furthermore, $M_\theta \times_f M_T$ is a mixed totally geodesic submanifold of $\tilde{M}$.

**Proof.** From (2.5), we have

$$\|\sigma\|^2 = \sum_{i,j=1}^n g(\sigma(e_i, e_j), \sigma(e_i, e_j)) = \sum_{r=n+1}^m \sum_{i,j=1}^n g(\sigma(e_i, e_j), e_r)^2. $$

Then from the assumed frame fields of $\mathcal{D}$ and $\mathcal{D}^\theta$, we derive

$$\|\sigma\|^2 = \sum_{r=n+1}^m \sum_{i,j=1}^t g(\sigma(e_i, e_j), e_r)^2 + 2 \sum_{r=n+1}^m \sum_{i=1}^t \sum_{j=t+1}^n g(\sigma(e_i, e_j), e_r)^2$$

$$+ \sum_{r=n+1}^m \sum_{i,j=t+1}^n g(\sigma(e_i, e_j), e_r)^2. $$

(3.11)
After leaving the second and third term in the right hand side of (3.11) and using the constructed frame fields, we find
\[
\|\sigma\|^2 \geq \sum_{r=1}^{q} \sum_{i,j=1}^{t} g(\sigma(e_i, e_j), \csc \theta \omega e^*_r)^2 + \sum_{r=q+1}^{m-n-q} \sum_{i,j=1}^{t} g(\sigma(e_i, e_j), \tilde{e}_r)^2
\]
(3.12)
The second term in the right hand side of the above expression has the \(\mu\)-components, therefore we shall leave this term and hence using Lemma 3.1, we get
\[
\|\sigma\|^2 \geq \csc^2 \theta \sum_{r=1}^{q} \sum_{i,j=1}^{t} [(e^*_r \ln f)^2 g(e_i, Fe_j)^2 + (Te^*_r \ln f)^2 g(e_i, e_j)^2
\]
(3.13)
Now, since \(\nabla^\theta \ln f \in \Gamma(TM\theta)\) and \(e^*_r = \sec \theta Te^*_r\), then from (2.6) we have
\[
\sum_{r=1}^{q} (Te^*_r \ln f)(e^*_r \ln f) = \sum_{r=1}^{q} \cos \theta (e^*_r \ln f)^2 = \cos \theta \|\nabla^\theta \ln f\|^2
\]
(3.14)
and
\[
\sum_{r=1}^{q} (Te^*_r \ln f)^2 = \sum_{r=1}^{q} g(Te^*_r, \nabla^\theta \ln f)^2 = \cos^2 \theta \|\nabla^\theta \ln f\|^2.
\]
(3.15)
Then, from (3.13), (3.14) and (3.15), we find
\[
\|\sigma\|^2 \geq t \csc^2 \theta (1 - \cos \theta)^2 \|\nabla^\theta \ln f\|^2,
\]
which is inequality (i). For the equality, from the remaining terms of (3.11), we find
\[
\sigma(D, D^\theta) = 0, \quad \text{and} \quad \sigma(D^\theta, D^\theta) = 0.
\]
(3.16)
Also, from the remaining second term in the right hand side of (3.12), we observe that
\[
\sigma(D, D) \perp \mu \quad \Rightarrow \quad \sigma(D, D) \in \Gamma(\omega D^\theta)
\]
(3.17)
The second condition of (3.16) implies that \(M_\theta\) is totally geodesic in \(\tilde{M}\) due to \(M_\theta\) being totally geodesic in \(M\) [5]. On the other hand, (3.17) implies that \(M_T\) is totally umbilical in \(\tilde{M}\) with the fact that \(M_T\) is totally umbilical in \(M\) [5]. Moreover, all conditions of (3.16) imply that \(M\) is a mixed totally geodesic submanifold of \(\tilde{M}\). Hence, the proof is complete.

From the above theorem, we have the following remark.

**Remark 3.4.** In Theorem 3.3, if we assume \(\theta = \frac{\pi}{2}\), then the warped product becomes \(M = M_\perp \times_T M_T\) in a locally product Riemannian manifold \(\tilde{M}\), where \(M_T\) and \(M_\perp\) are invariant and anti-invariant submanifolds of \(\tilde{M}\), respectively, which is a case of warped product semi-nivariant submanifolds which have been discussed in ([3], [11]). Thus, Theorem 4.2 of [11] and Theorem 4.1 of [3] are the special cases of Theorem 3.3.

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References

Characterizing local rings via complete intersection homological dimensions

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Abstract
Let \((R, m)\) be a commutative Noetherian local ring. It is known that \(R\) is Cohen-Macaulay if there exists either a nonzero Cohen-Macaulay \(R\)-module of finite projective dimension or a nonzero finitely generated \(R\)-module of finite injective dimension. In this article, we will prove the complete intersection analogues of these facts. Also, by using complete intersection homological dimensions, we will characterize local rings which are either regular, complete intersection or Gorenstein.

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1. Introduction and prerequisites
Throughout this paper, \((R, m)\) is a local ring and all rings are commutative and Noetherian with identity. The projective dimension is a familiar and famous numerical invariant in classical homological algebra. One of the fascinating theorems which is related to this dimension, is Auslander-Buchsbaum-Serre Theorem [1,15] which asserts that \(R\) is a regular ring if every finitely generated \(R\)-module has finite projective dimension. Motivated by this, Auslander and Bridger [2], introduced the Gorenstein dimension (abbr. G-dimension) for any finitely generated \(R\)-module and they proved that \(R\) is Gorenstein when every finitely generated \(R\)-module has finite G-dimension. The G-dimension has a very essential role for studying Gorenstein homological algebra and it was studied in more details in [2,7]. Let us recall the definition of G-dimension. Let \(M\) be a nonzero finitely generated \(R\)-module. The G-dimension of \(M\) is zero, \(G\text{-dim}_R M = 0\), if and only if the natural homomorphism \(M \rightarrow \text{Hom}_R(\text{Hom}_R(M, R), R)\) is an isomorphism, and \(\text{Ext}^i_R(M, R) = 0 = \text{Ext}^i_R(\text{Hom}_R(M, R), R)\) for any \(i > 0\). We set \(G\text{-dim}_R 0 = -\infty\). Also, for an integer \(n\), \(G\text{-dim}_R M \leq n\) if and only if there exists an exact sequence

\[ 0 \rightarrow X_n \rightarrow X_{n-1} \rightarrow \ldots \rightarrow X_1 \rightarrow X_0 \rightarrow M \rightarrow 0 \]

of \(R\)-modules such that \(G\text{-dim}_R X_i = 0\) for any \(0 \leq i \leq n\). More recently, Avramov, Gasharov and Peeva [3] introduced the concept of complete intersection dimension for finitely generated \(R\)-modules as a generalization of projective dimension. They proved that \(R\) is complete intersection when every finitely generated \(R\)-module has finite complete intersection dimension. For defining complete intersection dimension, we need the
definition of quasi-deformation of \( R \). A quasi-deformation of \( R \) is a diagram of local ring homomorphisms \( R \rightarrow R' \leftarrow Q \) such that \( R \rightarrow R' \) is faithfully flat and \( R' \leftarrow Q \) is surjective with the kernel which is generated by a \( Q \)-regular sequence. The complete intersection dimension of a finitely generated \( R \)-module \( M \), \( \text{CI-dim}_R M \) is defined as follows:

\[
\text{CI-dim}_R M := \inf\{\text{pd}_Q(M \otimes_R R') - \text{pd}_Q R' | R \rightarrow R' \leftarrow Q \text{ is a quasi-deformation}\}.
\]

These homological dimensions satisfy in the following inequalities

\[
G \text{-dim}_R M \leq \text{CI-dim}_R M \leq \text{pd}_R M, \quad (1)
\]

with equality to the left of any finite quantity, see [3].

Wagstaff [20] extended the definition of complete intersection dimension for any \( R \)-module \( N \). He defined the complete intersection projective dimension of \( N \), \( \text{CI-pd}_R N \), the complete intersection flat dimension of \( N \), \( \text{CI-fd}_R N \), and the complete intersection injective dimension of \( N \), \( \text{CI-id}_R N \), as follows:

\[
\begin{align*}
\text{CI-pd}_R N &:= \inf\{\text{pd}_Q(N \otimes_R R') - \text{pd}_Q R' | R \rightarrow R' \leftarrow Q \text{ is a quasi-deformation}\}, \\
\text{CI-fd}_R N &:= \inf\{\text{id}_Q(N \otimes_R R') - \text{id}_Q R' | R \rightarrow R' \leftarrow Q \text{ is a quasi-deformation}\}, \\
\text{CI-id}_R N &:= \inf\{\text{id}_Q(N \otimes_R R') - \text{pd}_Q R' | R \rightarrow R' \leftarrow Q \text{ is a quasi-deformation}\}.
\end{align*}
\]

One can see that if \( N \) is finitely generated, then \( \text{CI-pd}_R N = \text{CI-dim}_R N \).

We denote the category of finitely generated \( R \)-modules by \( \text{mod}(R) \), the subcategory of \( \text{mod}(R) \) consisting of all free \( R \)-modules by \( F(R) \), the subcategory of \( \text{mod}(R) \) consisting of zero module and all \( R \)-modules \( M \) such that \( G \text{-dim}_R M = 0 \) (resp. \( \text{CI-dim}_R M = 0 \)) by \( G(R) \) (resp. \( CI(R) \)). By using (1), we have the following inclusion relations between the subcategories of \( \text{mod}(R) \),

\[
F(R) \subseteq CI(R) \subseteq G(R).
\]

Takahashi [18] defined \( R \) to be G-regular if \( G(R) = F(R) \). We define \( R \) to be CI-regular if \( CI(R) = F(R) \). The first goal of this paper is a characterization of local rings by using complete intersection homological dimensions. We prove that \( R \) is regular if and only if \( R \) is complete intersection and CI-regular. Also, we prove that \( R \) is complete intersection if and only if \( R \) is Gorenstein and \( CI(R) = G(R) \). Let \( M \) be an \( R \)-module of finite depth, that is, \( \text{Ext}^i_R(k, M) \neq 0 \) for some \( i \in \mathbb{Z} \). In Theorem 2.5 below, we show that if either

i) \( \text{CI-pd}_R M < \infty \) and \( \text{id}_R M < \infty \) or

ii) \( \text{CI-id}_R M < \infty \) and \( \text{fd}_R M < \infty \),

then \( R \) is Gorenstein.

In the classical homological algebra, there exist two celebrated and important facts which are obtained by virtue of the (Peskine-Szpiro) intersection theorem [12] as follow:

i) If there exists a nonzero Cohen-Macaulay \( R \)-module of finite projective dimension, then \( R \) is Cohen-Macaulay,

ii) If there exists a nonzero \( R \)-module of finite injective dimension, then \( R \) is Cohen-Macaulay,

where the second assertion is known as Bass’s Theorem. Now, it is natural to ask the following questions:

**Question 1.1.** If there exists a nonzero Cohen-Macaulay \( R \)-module with finite complete intersection dimension, is then \( R \) Cohen-Macaulay?

**Question 1.2.** If there exists a nonzero finitely generated \( R \)-module with finite complete intersection injective dimension, is then \( R \) Cohen-Macaulay?

As a second goal of this article, we give the positive answers to the above questions, see Theorem 2.7. Also, for the interested reader, we recall that the Gorenstein analogues of i) and ii) are still open questions and many people have tried to prove them; see [8, 16, 17, 21].
2. The results

We start this section by the following definition.

Definition 2.1. We say that a local ring \((R, m, k)\) is CI-regular if \(CI(R)\) coincides with \(F(R)\).

Proposition 2.2. Let \(R\) be a local ring.

i) \(R\) is CI-regular if and only if \(CI\)-dim\(M\) = \(pd_R M\) for any finitely generated \(R\)-module \(M\).

ii) A normal local ring \(R\) is CI-regular if and only if \(CI\)-dim\(R/I\) = \(pd_R R/I\) for every ideal \(I\) of \(R\).

Proof. i) \(\Rightarrow\) Let \(M\) be a finitely generated \(R\)-module. It suffices to show that \(pd_R M \leq CI\)-dim\(M\). Without loss of generality, we assume that \(CI\)-dim\(M\) is finite. Set \(n := CI\)-dim\(M\). By [19, Corollary 3.9], there exists an exact sequence

\[
0 \rightarrow F_n \rightarrow F_{n-1} \rightarrow \ldots \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0
\]

such that \(CI\)-dim\(F_i\) = 0 for any \(0 \leq i \leq n\), and so by the assumption, \(F_i\) is projective for any \(0 \leq i \leq n\) which implies that \(pd_R M \leq n\).

\(\Leftarrow\) \(\Rightarrow\) implies from i).

\(\Leftarrow\) Let \(M\) be a finitely generated \(R\)-module such that \(CI\)-dim\(M\) = 0. Then \(G\)-dim\(M\) = 0 and so \(M\) is torsion-free \(R\)-module by [7, Lemma 1.1.8]. By [4, Theorem 6 in Chapter VII §4], there exists an exact sequence \(0 \rightarrow R^n \rightarrow M \rightarrow I \rightarrow 0\), where \(I\) is an ideal of \(R\). By [19, Lemma 3.6], one can see that \(CI\)-dim\(R I\) is finite and so using again of [19, Lemma 3.6] for the exact sequence \(0 \rightarrow I \rightarrow R \rightarrow R/I \rightarrow 0\), implies that \(CI\)-dim\(R I\) is finite. So by the assumption, \(pd_R R/I\) is finite and then from the exact sequence

\[
0 \rightarrow R^n \rightarrow M \rightarrow R \rightarrow R/I \rightarrow 0
\]

one can deduce that \(pd_R M\) is finite. Hence \(pd_R M = CI\)-dim\(M\) = 0, and so \(M\) is free.

Next, we present a criterion for specification regular local rings.

Proposition 2.3. Let \((R, m, k)\) be a local ring. The following are equivalent:

i) \(R\) is regular,

ii) \(R\) is complete intersection and CI-regular.

Proof. i)\(\Rightarrow\) ii) By [5, Proposition 3.1.20 and Theorem 2.2.7], \(R\) is complete intersection and for any finitely generated \(R\)-module \(M\), \(pd_R M\) is finite, and so by [3, Theorem 1.4], \(pd_R M = CI\)-dim\(M\) which implies that \(R\) is CI-regular by Proposition 2.2 i).

ii)\(\Rightarrow\) i) By [3, Theorem 1.3] and Proposition 2.2 i), \(pd_R k\) is finite, and so \(R\) is regular by [5, Theorem 2.2.7].

Theorem 1.3 in [3] characterizes complete intersection local rings with complete intersection dimension. In the following, we will characterize these rings with complete intersection homological dimensions. Before doing this, we recall that an \(R\)-module \(N\) is said to be Gorenstein injective if there exists an exact complex \(I\) of injective \(R\)-modules such that \(N \cong im(I_1 \rightarrow I_0)\) and \(Hom_R(E, I)\) is exact for all injective \(R\)-modules \(E\). Any injective \(R\)-module is Gorenstein injective. Let \(N\) be an \(R\)-module. We say that Gorenstein injective dimension of \(N\), \(Gid_R N\), is finite if it has a finite Gorenstein injective resolution. It is easy to see that when the usual injective dimension of \(N\), \(id_R N\), is finite, then \(Gid_R N\) is also finite.

Proposition 2.4. Let \((R, m, k)\) be a local ring. The following are equivalent:
There exists a nonzero $\text{CI–id}_R M < \infty$ for any $R$-module $M$,

iii) $\text{CI–id}_R k < \infty$,

iv) $\text{CI–dim}_R k < \infty$,

v) $R$ is Gorenstein and $\text{CI}(R) = G(R)$.

**Proof.** i)$\Rightarrow$ ii) Since $R$ is complete intersection, then there exists a regular ring $Q$ such that $\Lambda^m(R) \cong Q/xQ$, where $x = x_1, x_2, ..., x_n$ is a $Q$-regular sequence and $\Lambda^m(R)$ is an $m$-adic completion of $R$. Consider a quasi-deformation $R \rightarrow R' \leftarrow Q$. Then $\text{id}_Q(M \otimes_R \Lambda^m(R)) < \infty$ for any $R$-module $M$ and this completes the proof.

ii)$\Rightarrow$ iii) is clear.

iii)$\Rightarrow$ iv) There exists a quasi-deformation $R \rightarrow R' \leftarrow Q$ such that $R' \cong Q/I$ for some ideal $I$ of $Q$ and $\text{id}_Q(k \otimes_R R') = \text{id}_Q(R'/mR') < \infty$. Then, we can conclude that for some ideal $J$ of $Q$, we have $\text{id}_Q(Q/J) < \infty$ and so $\text{Gid}_Q(Q/J) < \infty$ which implies that $Q$ should be a Gorenstein ring by [9, Theorem 4.5]. So $\text{pd}_Q(k \otimes_R R') < \infty$ by [5, Exercise 3.1.25] which implies that $\text{CI–dim}_R k < \infty$.

iv)$\Rightarrow$ v) $R$ is a Gorenstein ring by [5, Proposition 3.1.20], and also for any finitely generated $R$-module $M$, we have $\text{CI–dim}_R M = \text{G–dim}_R M$ by [3, Theorems 1.3 and 1.4].

v)$\Rightarrow$ i) Since $R$ is Gorenstein, then $\text{G–dim}_R k$ is finite by [7, Theorem 1.4.9]. Set $n := \text{G–dim}_R k$. Let $P$ be a projective resolution of $k$, as follow:

$\cdots \rightarrow P_n \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow k \rightarrow 0$.  

Then $\ker(P_{n-1} \rightarrow P_{n-2}) \in G(R)$ by [7, Theorem 2.3.16], and so by the assumption, $\ker(P_{n-1} \rightarrow P_{n-2}) \in CI(R)$. Hence, $\text{CI–dim}_R k$ is finite by [19, Corollary 3.8] and the assertion follows from [3, Theorem 1.3].

We recall that for any $R$-module $M$, $\text{depth}_R M$ is defined as follow:

$\text{depth}_R M := \inf\{i \in \mathbb{Z} | \text{Ext}_R^i(k, M) \neq 0\}$.

If $M$ is a non-zero finitely generated $R$-module, then $M$ has finite depth.

**Theorem 2.5.** Let $(R, m, k)$ be a local ring. The following are equivalent:

i) $R$ is Gorenstein,

ii) There exists a nonzero $R$-module $M$ with finite depth such that $\text{CI–pd}_R M < \infty$ and $\text{id}_R M < \infty$,

iii) There exists a nonzero $R$-module $M$ with finite depth such that $\text{CI–fd}_R M < \infty$ and $\text{id}_R M < \infty$,

iv) There exists a nonzero $R$-module $M$ with finite depth such that $\text{CI–id}_R M < \infty$ and $\text{fd}_R M < \infty$.

**Proof.** i)$\Rightarrow$ iii) and i)$\Rightarrow$ iv) Set $M := R$.

ii)$\iff$ iii) For any $R$-module $M$, by [20, Remark 2.5] one has that $\text{CI–pd}_R M < \infty$ if and only if $\text{CI–fd}_R M < \infty$.

iii)$\Rightarrow$ i) Since $\text{CI–fd}_R M < \infty$, [14, Theorem 4.5] yields that $\text{CI–fd}_R M = \text{Gfd}_R M$ and so, the assertion follows from [10, Corollary 3.3].

iv)$\Rightarrow$ i) By the definition of $\text{CI–id}_R M$, there exists a quasi-deformation $R \rightarrow R' \leftarrow Q$ such that $\text{id}_Q(M \otimes_R R') < \infty$ and so $\text{Gid}_Q(M \otimes_R R') < \infty$. Assume that $x = x_1, x_2, ..., x_n$ is a $Q$-regular sequence such that $R' \cong Q/xQ$. Then $\text{Gid}_Q(M \otimes_R R') = \text{Gid}_Q(M \otimes_R R) + n$ by [6, Theorem 4.2], and so we have $\text{Gid}_Q(M \otimes_R R') < \infty$. Since $R \rightarrow R'$ is a flat extension and $\text{depth}_R M < \infty$, we deduce that $\text{depth}_R (M \otimes_R R') < \infty$ by [11, Corollary 2.6]. Also, we have $\text{fd}_R(M \otimes_R R') < \infty$. Hence, [10, Corollary 3.3] yields that $R'$ is Gorenstein, and so $R$ is Gorenstein by [5, Corollary 3.3.15].
Finally, we characterize Cohen-Macaulay local rings with complete intersection homological dimensions which also gives positive answers to questions 1.1 and 1.2. Before doing that, we mention the following theorem which is obtained by virtue of the (Peskine-Szpiro) intersection theorem. The implication iii) ⇒ i) in Theorem 2.6, is called Bass’ conjecture which was proven by Peskine-Szpiro [12, Theorem 5.1] by using the theorem of intersection [12, Theorem 2.1] for any Noetherian local ring of characteristic \( p > 0 \). After some years, Roberts [13] proved that the theorem of intersection holds for every Noetherian local ring and he called it "New intersection theorem", see [13, Theorem 13.4.1].

**Theorem 2.6.** Let \((R, \mathfrak{m}, k)\) be a local ring. The following are equivalent:

i) \( R \) is Cohen-Macaulay,

ii) There exists a nonzero finite length \( R \)-module of finite projective dimension,

iii) There exists a nonzero finitely generated \( R \)-module of finite injective dimension.

**Proof.** i) ⇒ ii) Assume that \( n := \dim R \). Since \( R \) is Cohen-Macaulay, there is a maximal \( R \)-regular sequence such as \( x = x_1, x_2, ..., x_n \). Then \( R/(x) \) is a nonzero finite length \( R \)-module and by [5, Exercise 1.3.6], its projective dimension is finite.

ii) ⇒ iii) Assume that \( M \) is a nonzero finite length \( R \)-module of finite projective dimension. Set \( N := \text{Hom}_R(M, E_R(k)) \), where \( E_R(k) \) is an injective envelope of \( k \). By [5, Proposition 3.2.12 b)], \( N \) is a finite length \( R \)-module. Also, it is easy to see that for any projective \( R \)-module \( Q \), \( R \)-module \( \text{Hom}_R(Q, E_R(k)) \) is an injective \( R \)-module. By using this and our assumption, we conclude that injective dimension of \( N \) is finite.

iii) ⇒ i) The same proof of [12, Theorem 5.1] works by using [13, Theorem 13.4.1] instead of the theorem of intersection [12, Theorem 2.1]. \( \square \)

**Theorem 2.7.** Let \((R, \mathfrak{m}, k)\) be a local ring. The following are equivalent:

i) \( R \) is Cohen-Macaulay,

ii) There exists a nonzero Cohen-Macaulay \( R \)-module \( M \) such that \( \text{CI-dim}_R M < \infty \),

iii) There exists a nonzero finitely generated \( R \)-module \( M \) such that \( \text{CI-id}_R M < \infty \).

**Proof.** i) ⇒ ii) Set \( M := R \).

ii) ⇒ i) By [20, Remark 2.5 and Theorem 3.2], there exists a quasi-deformation \( R \to R' \leftarrow Q \) such that \( Q \) is complete, the closed fiber \( R'/\mathfrak{m}R' \) is Artinian and Gorenstein and \( \text{fd}_Q(M \otimes_R R') \) is finite. So \( \text{pd}_Q(M \otimes_R R') \) is finite, too. [5, Theorem 2.1.7] yields that \( M \otimes_R R' \) is a Cohen-Macaulay \( R' \)-module. Since \( \text{depth}_Q(M \otimes_R R') = \text{depth}_{R'}(M \otimes_R R') \) by [5, Exercise 1.2.26 b)] and \( \dim_Q(M \otimes_R R') = \dim_{R'}(M \otimes_R R') \), then \( M \otimes_R R' \) is a Cohen-Macaulay \( Q \)-module. Hence, \( Q \) is a Cohen-Macaulay ring, and so by [5, Theorem 2.1.3 a)], \( R \) is a Cohen-Macaulay ring which implies that \( R \) should be a Cohen-Macaulay ring by [5, Exercise 2.1.23].

i) ⇒ iii) The assertion follows from Theorem 2.6, i) ⇒ iii). Note that \( \text{CI-id}_R M \leq \text{id}_R M \) for any \( R \)-module \( M \).

iii) ⇒ i) Since \( \text{CI-id}_R M \) is finite, there exists a quasi-deformation \( R \to R' \leftarrow Q \) such that the finitely generated \( Q \)-module \( M \otimes_R R' \) has finite injective dimension. Hence, \( Q \) is a Cohen-Macaulay ring by Theorem 2.6 and so \( R' \) is a Cohen-Macaulay ring by [5, Theorem 2.1.3 a)] which implies that \( R \) should be a Cohen-Macaulay ring by [5, Exercise 2.1.23]. \( \square \)

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**References**


Coefficient estimates for \( m \)-fold symmetric bi-subordinate functions

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Abstract
A function is said to be bi-univalent in the open unit disk \( \mathbb{U} \) if both the function and its inverse map are univalent in \( \mathbb{U} \). By the same token, a function is said to be bi-subordinate in \( \mathbb{U} \) if both the function and its inverse map are subordinate to a given function in \( \mathbb{U} \). In this paper, we consider the \( m \)-fold symmetric transform of such functions and use their Faber polynomial expansions to find upper bounds for their \( n \)-th \( (n \geq 3) \) coefficients subject to a given gap series condition. We also determine bounds for the first two coefficients of such functions with no restrictions imposed.

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1. Introduction
Let \( \mathcal{A} \) be the class of analytic functions in the open unit disk \( \mathbb{U} := \{ z \in \mathbb{C} : |z| < 1 \} \) and let \( \mathcal{S} \) be the class of functions \( f \) that are analytic and univalent in \( \mathbb{U} \) and are of the form

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}
\]

For \( f(z) \) and \( F(z) \) analytic in \( \mathbb{U} \), we say that \( f(z) \) is subordinate to \( F(z) \), written \( f \prec F \), if there exists a Schwarz function \( w(z) \) with \( w(0) = 0 \) and \( |w(z)| < 1 \) in \( \mathbb{U} \) such that \( f(z) = F(w(z)) \). We note that \( f(\mathbb{U}) \subset F(\mathbb{U}) \) if both \( f \) and \( F \) are in \( \mathcal{S} \). Moreover, for the Schwarz function \( w(z) = \sum_{n=1}^{\infty} w_n z^n \) we have \( |w_n| \leq 1 \) (e.g. see [3]).

For each function \( f \in \mathcal{S} \), the \( m \)-fold symmetric function given by

\[
f_m(z) = \sqrt[m]{f(z^m)} = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1}, \quad (z \in \mathbb{U}, m \in \mathbb{N}),
\]
is univalent in the unit disk \( \mathbb{U} \) (e.g. see [3]). We denote the class of such functions by \( \mathcal{S}_m \).

The functions in the class \( \mathcal{S}_1 = \mathcal{S} \) are univalent one-fold symmetric.

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Since the functions in $S$ are one-to-one, they are invertible and their inverse maps need not be defined on the entire unit disk $U$. In fact, the Koebe one-quarter theorem (e.g. see [3]) ensures that every univalent function $f \in S$ contains a disk of radius $1/4$. Thus every function $f \in S$ has an inverse map $f^{-1}$, which is defined by $f^{-1}(f(z)) = z$ and $f(f^{-1}(w)) = w$ where $z \in U$ and $|w| < r_0(f) \geq 1/4$.

It is easy to verify that for $f \in S_1 = S$ of the form (1.1), the inverse function $g = f^{-1}$ is given by

$$g(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2a_3 + a_4\right) w^4 + \cdots.$$  

(1.2)

Similarly, for the m-fold symmetric function $f_m \in S_m$, its inverse function $g_m = f_m^{-1}$ is of the form

$$g_m(w) = w - a_{m+1} w^{m+1} + [(m + 1)a_{m+1} - a_{2m+1}] w^{2m+1}$$

$$- \frac{1}{2} (m + 1)(3m + 2)a_{m+1}^3 - (3m + 2)a_{m+1}a_{2m+1} + a_{3m+1}] w^{3m+1} + \cdots.$$  

(1.3)

A function $f \in A$ is said to be bi-univalent in $U$ if both $f$ and its inverse map $g = f^{-1}$ are univalent in $U$. Similarly, a function $f_m \in A$ is said to be m-fold symmetric bi-univalent in $U$ if both $f_m$ and its inverse map $g_m = f_m^{-1}$ are univalent in $U$. We let $\Sigma_m$ be the class of all m-fold symmetric bi-univalent functions in $U$. Obviously, for $m = 1$, the formula (1.2) coincides with the formula (1.1) of the class $\Sigma_1 = \Sigma_1$. For a brief history of functions in the class $\Sigma$, see the work of Srivastava et al. [9] and the references cited therein. The concept of m-fold symmetric bi-univalent functions has been introduced concurrently by Hamidi and Jahangiri [5] and Srivastava et al. [10]. Not much was known about the bounds of the general coefficients $a_n$ $(n \geq 4)$ of subclasses of bi-univalent functions up until the publication of the article [7] by Jahangiri and Hamidi who used the Faber polynomial series expansions to obtain bounds for the n-th coefficients $a_n$ $(n \geq 3)$ of certain subclasses of the normalized bi-univalent functions subject to a given gap series condition. Here we consider the m-fold symmetric transformation of a subordination version of a class of functions considered in [7] and obtain the upper bounds for the general coefficients $|a_{m(n-1)+1}|$ of such functions subject to a given gap series condition. We also determine the upper bounds for their first two coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ as well as bounds for their Feket-Szego coefficient body $|a_{2m+1} - \frac{n+1}{2} a_{m+1}^2|$. In general, our results are new on their own rights and in particular improve a few of the previously known results.

2. Main results

Let the function $\varphi \in A$ have positive real part in $U$ so that $\varphi$ maps the unit disk $U$ onto a region starlike with respect to 1, symmetric with respect to the real axis, $\varphi(0) = 1$ and $\varphi'(0) > 0$ (e.g. see [8]). Here we use the m-fold symmetric transformation of the function $\varphi \in A$, denoted by $\varphi_m \in A$. Obviously, by the properties of m-fold symmetric analytic functions (e.g. see [3]), $\varphi_m$ is an analytic function with positive real part in the unit disk $U$, satisfying $\varphi_m(0) = 1$, $\varphi_m'(0) > 0$ and symmetric with respect to the real axis having the power series expansion

$$\varphi_m(z) = 1 + B_m z^m + B_{2m} z^{2m} + B_{3m} z^{3m} + \cdots \quad (B_m > 0).$$

Using the above definition of functions $\varphi_m \in A$ we introduce the following

**Definition 2.1.** A function $f_m \in \Sigma_m$ is said to be in the class $\Sigma_m(\lambda; \varphi_m)$ if

$$(1 - \lambda) \frac{f_m(z)}{z} + \lambda f'_m(z) \prec \varphi_m(z) \quad (z \in U),$$

where $\lambda \in \mathbb{C}$ is a fixed constant.
and

\[(1 - \lambda) \frac{g_m(w)}{w} + \lambda g_m'(w) < \varphi_m(w) \quad (w \in U),\]

where \(\lambda \geq 0, \ m \in \mathbb{N}\) and \(g_m\) is given by (1.3).

In order to prove our theorems in this section, we need to use the Faber polynomial expansions of inverse functions. For the function \(f \in S\) of the form (1.1), the coefficients of its inverse map \(g = f^{-1}\) may be expressed (e.g. see [1] and [2]) by

\[g(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, \cdots) w^n,\]

where

\[K_{n-1}^{-n} = \frac{(-n)!}{(-2n+1)!(n-1)!} a_2^{n-1} + \frac{(-n)!}{2(-n+1)!(n-3)!} a_2^{n-3} a_3 \]

\[+ \frac{(-n)!}{(2n+3)!(n-4)!} a_2^{n-4} a_4 + \frac{(-n)!}{(2n+2)!(n-5)!} a_2^{n-5} \]

\[\times [a_5 + (n-2)a_2^2] + \frac{(-n)!}{(-2n+5)!(n-6)!} a_2^{n-6} [a_6 + (2n-5)a_3 a_4] \]

\[+ \sum_{j \geq 7} a_2^{n-j} V_j,\]

such that \(V_j\) with \(7 \leq j \leq n\) is a homogeneous polynomial in the variables \(a_2, a_3, \cdots, a_n\).

In particular, the first three terms of \(K_{n-1}^{-n}\) are

\[
\frac{1}{2} K_2^{-2} = -a_2, \quad \frac{1}{3} K_2^{-3} = 2a_2^2 - a_3, \quad \frac{1}{4} K_3^{-4} = -(5a_2^3 - 5a_2 a_3 + a_4).
\]

In general, for \(n \geq 1\) and real values of \(p\), an expansion of \(K_n^{-p}\) is (see [1, 12] or [2, page 349])

\[K_n^{-p} = p a_n + \frac{p(p-1)}{2} D_n^2 + \frac{p^3}{(p-3)!} D_n^3 + \cdots + \frac{p^n}{(p-n+1)!(n-1)!} D_n^{n-1},\]

where \(D_n^{-p} = D_n^{-p}(a_2, a_3, \cdots, a_n)\) are homogeneous polynomials explicating in

\[D_n^{-p}(a_2, a_3, \cdots, a_n) = \sum_{n=2}^{\infty} \frac{m!(a_2)^{\mu_1} \cdots (a_n)^{\mu_{n-1}}}{\mu_1! \cdots \mu_{n-1}!} \quad \text{for} \quad p \leq n - 1,\]

and the sum is taken over all nonnegative integers \(\mu_1, \ldots, \mu_{n-1}\) satisfying

\[
\begin{cases}
\mu_1 + \mu_2 + \cdots + \mu_{n-1} = p, \\
\mu_1 + 2\mu_2 + \cdots + (n-1)\mu_{n-1} = n-1.
\end{cases}
\]

It is clear that \(D_n^{-1}(a_2, a_3, \cdots, a_n) = a_2^{n-1}\).

Now we are ready to state and prove our first theorem which provides an upper bound for the general coefficients of functions in \(\Sigma_m(\lambda, \varphi_m)\) subject to a given gap series condition.

**Theorem 2.2.** For \(\lambda \geq 0, \ m \in \mathbb{N}\), let the function \(f_m \in \Sigma_m(\lambda, \varphi_m)\) be given by (1.3). If \(a_k = 0\) for \(m + 1 \leq k \leq (n-2)m + 1\), then

\[|a_{(n-1)m+1}| \leq \frac{B_m}{1 + (n-1)m\lambda} \quad n \geq 3.
\]

**Proof.** By definition, for function \(f_m \in \Sigma_m(\lambda, \varphi_m)\), we have

\[(1 - \lambda) \frac{f_m(z)}{z} + \lambda f_m'(z) = 1 + \sum_{n=2}^{\infty} [1 + (n-1)m\lambda] a_{(n-1)m+1} z^{(n-1)m}, \quad (2.1)
\]
and for its inverse map, \( g_m = f_m^{-1} \), we obtain
\[
(1 - \lambda) \frac{g_m(w)}{w} + \lambda g_m'(w) = 1 + \sum_{n=2}^{\infty} [1 + (n - 1)m\lambda] b_{(n-1)m+1} w^{(n-1)m}.
\]
Comparing the corresponding coefficients of (2.1) and (2.3), we obtain
\[
[1 + (n - 1)m\lambda] a_{(n-1)m+1} = \sum_{k=1}^{n-1} B_{km} D_{n-1}^k(p_{m}, p_{2m}, \ldots, p_{nm} \cdot z^{nm}).
\]

Similarly, by comparing the corresponding coefficients of (2.2) and (2.4), we obtain
\[
[1 + (n - 1)m\lambda] \frac{1}{n} K_{n-1}^{-n}(a_{m+1}, a_{2m+1}, \ldots, a_{(n-1)m+1}) = \sum_{k=1}^{n-1} B_{km} D_{n-1}^k(q_{m}, q_{2m}, \ldots, q_{(n-1)m}).
\]
Letting \( a_k = 0 \) for \( m + 1 \leq k \leq (n - 2)m + 1 \) yields \( b_{(n-1)m+1} = -a_{(n-1)m+1} \) and hence
\[
[1 + (n - 1)m\lambda] a_{(n-1)m+1} = B_m p_{(n-1)m},
\]
and
\[
- [1 + (n - 1)m\lambda] a_{(n-1)m+1} = B_m q_{(n-1)m}.
\]
Now taking the absolute values of either of the above two equations and using the facts that \( |p_{(n-1)m}| \leq 1 \) and \( |q_{(n-1)m}| \leq 1 \), we obtain
\[
|a_{(n-1)m+1}| \leq \frac{B_m |p_{(n-1)m}|}{1 + (n - 1)m\lambda} \leq \frac{B_m |q_{(n-1)m}|}{1 + (n - 1)m\lambda} \leq \frac{B_m}{1 + (n - 1)m\lambda}.
\]
Our next two theorems provide bounds for the first two coefficients of certain subclasses of $\Sigma_m(\lambda; \varphi_m)$ with no gap series restrictions imposed.

**Theorem 2.3.** For $\lambda \geq 0$, $m \in \mathbb{N}$ and $0 \leq \beta < 1$ let $f_m \in \Sigma_m \left( \lambda; \frac{1+(1-2\beta)z^m}{1-z^m} \right)$. Then

$$|a_{m+1}| \leq \min \left\{ \frac{2(1-\beta)}{1+m\lambda}, \frac{\sqrt{4(1-\beta)}}{(1+2m\lambda)(m+1)} \right\}$$

and

$$|a_{2m+1}| \leq \frac{2(1-\beta)}{1+2m\lambda},$$

and

$$\left| a_{2m+1} - \frac{m+1}{2} a_{m+1} \right| \leq \frac{2(1-\beta)}{1+2m\lambda}.$$  

**Proof.** The equations (2.5) and (2.6) for $n = 2$ and $n = 3$, respectively, imply

$$\begin{align*}
(1 + m\lambda) a_{m+1} &= 2(1-\beta) p_m, \quad (1 + 2m\lambda) a_{2m+1} = 2(1-\beta) p_{2m} + 2(1-\beta) p_m^2, \quad (1 + m\lambda) a_{m+1} = 2(1-\beta) q_m, \\
(1 + 2m\lambda) [(m+1) a_{m+1}^2 - a_{2m+1}^2] &= 2(1-\beta) q_{2m} + 2(1-\beta) q_m^2.
\end{align*}$$

Taking absolute values of (2.7) or (2.9), we get

$$|a_{m+1}| \leq \frac{2(1-\beta)}{1+m\lambda}.$$  

Also by adding (2.8) and (2.10), we have

$$(1 + 2m\lambda)(m+1) a_{m+1}^2 = 2(1-\beta) \left[ (p_{2m} + p_m^2) + (q_{2m} + q_m^2) \right].$$

Taking the absolute values of the above equation yields

$$(1 + 2m\lambda)(m+1)|a_{m+1}|^2 \leq 2(1-\beta) \left[ |p_{2m} + p_m^2| + |q_{2m} + q_m^2| \right].$$

Now by using [6, Corollary 2.3], we have

$$(1 + 2m\lambda)(m+1)|a_{m+1}|^2 \leq 2(1-\beta) \left[ 1 + (1-1)|p_m|^2 + 1 + (1-1)|q_m|^2 \right].$$

Therefore,

$$|a_{m+1}| \leq \frac{\sqrt{4(1-\beta)}}{(1+2m\lambda)(m+1)}.$$  

Next, by solving (2.8) for $a_{2m+1}$, taking the absolute values and using [6, Corollary 2.3] we get

$$|a_{2m+1}| \leq \frac{2(1-\beta)}{1+2m\lambda} \left[ 1 + (1-1)|p_m|^2 \right] = \frac{2(1-\beta)}{1+2m\lambda}.$$

Finally, subtracting (2.10) from (2.8) and considering the fact that $p_m^2 = q_m^2$ we obtain

$$2(1 + 2m\lambda) \left( a_{2m+1} - \frac{m+1}{2} a_{m+1}^2 \right) = 2(1-\beta) (p_{2m} - q_{2m}).$$

Taking the absolute values of both sides and using the fact that $|p_{2m}| \leq 1$ and $|q_{2m}| \leq 1$ we obtain

$$\left| a_{2m+1} - \frac{m+1}{2} a_{m+1}^2 \right| \leq \frac{2(1-\beta)}{1+2m\lambda}.$$  

This completes the proof.  

$\Box$
**Theorem 2.4.** For \( \lambda \geq 0, m \in \mathbb{N} \) and \( 0 < \alpha \leq 1 \) let \( f_m \in \Sigma_m \left( \lambda, \left( \frac{1 + z^m}{1 - z^m} \right) \right) \). Then

\[
|a_{m+1}| \leq \min \left\{ \frac{2\alpha}{1 + m\lambda}, \frac{2\alpha}{\sqrt{(1 + m\lambda)^2 + am(1 + 2m\lambda - m\lambda^2)}} \right\} 
\]  
(2.11)

\[
|a_{2m+1}| \leq \frac{2\alpha}{1 + 2m\lambda},
\]

and

\[
|a_{2m+1} - \frac{m+1}{2} a_{m+1}| \leq \frac{2\alpha}{1 + 2m\lambda}.
\]

**Proof.** The equations (2.5) and (2.6) for \( n = 2 \) and \( n = 3 \), respectively, imply

\[
(1 + m\lambda)a_{m+1} = 2\alpha p_m,
\]
(2.12)

\[
(1 + 2m\lambda)a_{2m+1} = 2\alpha p_{2m} + 2\alpha^2 p_m^2,
\]
(2.13)

\[-(1 + m\lambda)a_{m+1} = 2\alpha q_m,
\]
(2.14)

\[(1 + 2m\lambda)[(m+1) a_{m+1}^2 - a_{2m+1}] = 2\alpha q_{2m} + 2\alpha^2 q_m^2.
\]
(2.15)

Taking the absolute values of (2.12) or (2.14), we get

\[
|a_{m+1}| \leq \frac{2\alpha}{1 + m\lambda}.
\]
(2.16)

Also by adding (2.13) and (2.15), we have

\[(1 + 2m\lambda)(m+1) a_{m+1}^2 = 2\alpha \left[ (p_{2m} + \alpha p_m^2) + (q_{2m} + \alpha q_m^2) \right].
\]

Taking the absolute values of the above equation yields

\[(1 + 2m\lambda)(m+1) |a_{m+1}|^2 \leq 2\alpha \left[ |p_{2m} + \alpha p_m^2| + |q_{2m} + \alpha q_m^2| \right].
\]

Now, for \( 0 < \alpha \leq 1 \) we use [6, Corollary 2.3], to obtain

\[(1 + 2m\lambda)(m+1) |a_{m+1}|^2 \leq 2\alpha \left[ 1 + (\alpha - 1)|p_m|^2 + 1 + (\alpha - 1)|q_m|^2 \right].
\]

Solve the above equation for \( |a_{m+1}| \) and apply the fact that \( |p_m|^2 = |q_m|^2 = \frac{(1+m\lambda)^2 |a_{m+1}|^2}{4\alpha^2} \) to obtain

\[
|a_{m+1}| \leq \frac{2\alpha}{\sqrt{(1 + m\lambda)^2 + am(1 + 2m\lambda - m\lambda^2)}}.
\]
(2.17)

So, (2.16) in conjunction with (2.17) yield (2.11).

Next, we solve (2.13) for \( a_{2m+1} \), take the absolute values and apply [6, Corollary 2.3] to obtain

\[
|a_{2m+1}| \leq \frac{2\alpha}{1 + 2m\lambda} \left[ 1 + (\alpha - 1)|p_m|^2 \right] \leq \frac{2\alpha}{1 + 2m\lambda}.
\]

Finally, subtracting (2.15) from (2.13) and considering the fact that \( p_m^2 = q_m^2 \) we obtain

\[
2(1 + 2m\lambda) \left( a_{2m+1} - \frac{m+1}{2} a_{m+1}^2 \right) = 2\alpha (p_{2m} - q_{2m}).
\]

Taking the absolute values of both sides and using the fact that \( |p_{2m}| \leq 1 \) and \( |q_{2m}| \leq 1 \) we obtain

\[
|a_{2m+1} - \frac{m+1}{2} a_{m+1}^2| \leq \frac{2\alpha}{1 + 2m\lambda}.
\]

This completes the proof. \( \square \)

**Remark 2.5.** Theorem 2.2 for \( m = 1 \) and \( \varphi_1(z) = \frac{1+(1-2\beta)z}{1-z} \) yields the estimates obtained by Jahangiri and Hamidi [7, Theorem 1].

**Remark 2.6.** Theorems 2.3 and 2.4 are improvements of the estimates obtained by Sümer Eker [11, Theorems 2 and 1], respectively.
Coefficient estimates for $m$-fold symmetric bi-subordinate functions

**Remark 2.7.** Theorems 2.3 and 2.4 for $m = 1$ are improvements of the estimates obtained by Frasin and Aouf [4, Theorems 3.2 and 2.2], respectively.

**Remark 2.8.** Theorems 2.3 and 2.4 for $\lambda = 1$ are improvements of the estimates obtained by Srivastava et al. [10, Theorems 3 and 2], respectively.

**Remark 2.9.** Letting $\lambda = 1$ in Theorem 2.3 yields the following bounds for $|a_2|$ and $|a_3|$ which are improvements of the estimates obtained by Srivastava et al. [9, Theorem 2]

\[
|a_2| \leq \begin{cases} 1 - \beta, & \frac{1}{3} \leq \beta < 1, \\ \frac{2(1 - \beta)}{3}, & 0 \leq \beta < \frac{1}{3}. \end{cases}
\]

and

\[
|a_3| \leq \frac{2(1 - \beta)}{3}.
\]

**Remark 2.10.** Letting $\lambda = 1$ in Theorem 2.4 we obtain $|a_3| \leq (2\alpha/3)$ which is an improvement of the estimate obtained by Srivastava et al. [9, Theorem 1].

**References**


Study of velocity and shear stress for unsteady flow of incompressible Oldroyd-B fluid between two concentric rotating circular cylinders

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Abstract

This investigation deals with the study of unsteady flow of incompressible Oldroyd-B fluid between two rotating circular cylinders, both cylinders are rotating around their common axis \((r = 0)\). The governing differential equations are formulated with appropriate boundary conditions and then solved by means of Laplace and Hankel transforms to obtain velocity and shear stress for unsteady flow of Oldroyd-B fluid between two infinite concentric rotating circular cylinders. The obtained solutions can easily be reduced to equivalent solutions for Maxwell and classical Newtonian fluids. Finally, the influence of different physical parameters on the fluid velocity and shear stress is graphically underlined and discussed.

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1. Introduction

Over the years, non-Newtonian fluids have been considered due to their practical importance and huge-unfold applications in various branches of engineering, science and technology: particularly in drilling operations, material processing, oil exploitation, polymer chemical industries, and bioengineering. A number of industrially important fluids including exotic lubricants, extrusions of polymers, food stuffs, drilling mud, slurry type fuels, suspension and colloidal mixtures display non-Newtonian characteristics. In literature, for non-Newtonian fluids, a wide range of models are offered to explore their behaviors and properties \([1, 2, 23, 25]\), because a particular model cannot define all the multifaceted properties of non-Newtonian fluids. Amongst these, Oldroyd-B fluid model is an important non-Newtonian viscoelastic model, which has attained much attention of the researchers \([4, 5, 12, 16, 19, 24]\) because of its wide spread industrial applications. With the recent advances of complex and viscoelastic materials; applications of Oldroyd-B fluid have increased. Both theoretically and practically, the flow analysis of such fluids is very vital. Hayat et al. presented a detailed analysis of some simple flows of Oldroyd-B fluid \([7]\). Fetc locals and Fetc locals investigated the flow characteristics of Oldroyd-B fluids that flow
unsteadily in a rectangular channel [3]. Tanveer et al. discussed magneto-hydrodynamic flow of generalized Oldroyd-B fluid over an infinite oscillating plate with slip condition using Fox H-function [22].

Motion of the fluid under translating or rotating cylinders is of great importance to both practical and theoretical fields. The study of viscoelastic fluid flows in the region of rotating circular cylinders is of vital significance as this types of fluid flows have many uses in several industries, like food and petroleum industries, chemical engineering, medicines, and bioengineering. Moreover, such flows have wide coverage on the development of energy generation and in astrophysical and geophysical fluid dynamics. The academic workers and engineers are very much interested in the geometry of such types of flows [6,17]. The literature about motion under translating or rotating cylinders for non-Newtonian fluids is not so well organized, but some interesting studies of such types of fluid flows are given by Jamil et al. [8,9], Kamran et al. [10,11], and Mahmood et al. [14,15].

The motivation of this study is to examine the flow of Oldroyd-B fluid between two coaxially rotating cylinders. At time $t = 0$, the fluid is at rest. Due to rotational shear stress which is time-dependent, the inner cylinder starts rotation about its own axis and the outer cylinder is rotating around its axis at time $t = 0^+$ through the angular velocity $R_2 \omega t$. The flow of Oldroyd-B fluid is then generated by the rotation of two cylinders which at time $t = 0^+$ begin to rotate around their common axis. Closed form solutions for velocity and shear stress for the flow of Oldroyd-B fluid between two rotating cylinders are derived under series form in terms of generalized $G$ functions with the help of Laplace and Hankel transforms. These solutions, which are new in the literature, give the complete pattern of flow field and have widespread applications in many industrial fields. Moreover, the derived expressions for velocity and shear stress are in the most simplified form, and the point worth mentioning is that these expressions are free from convolution product and integral of the product of generalized $G$ function. Furthermore, the effects of various physical parameters on velocity field and shear stress are examined and illustrated graphically.

2. Basic equations

We write down the basic equations governing the motion of an incompressible non-Newtonian fluid. These are the equation of continuity

$$\text{div} \mathbf{u} = 0, \quad (2.1)$$

and the linear momentum equation (in absence of body forces)

$$\text{div} \mathbf{T} = \rho \frac{d \mathbf{u}}{dt}, \quad (2.2)$$

where $\mathbf{u}$ is the velocity field, $\mathbf{T}$ is the Cauchy stress tensor, $\rho$ is the constant density, and $d/dt = \partial_t + \mathbf{u} \cdot \nabla$ is the material time derivative.

The constitutive equations of an incompressible Oldroyd-B fluid are given by

$$\mathbf{T} = -\rho \mathbf{I} + \mathbf{S}; \quad (\lambda_1 \partial_t + 1) \mathbf{S} = \mu (\lambda_2 \partial_t + 1) \mathbf{A}, \quad (2.3)$$

where $-\rho \mathbf{I}$ is the spherical stress due to the constraint of incompressibility, $\mathbf{S}$ is the extra stress tensor, $\mu$ is the dynamic viscosity, $\lambda_1$ is the relaxation time, $\lambda_2$ is the retardation time, and $\mathbf{A}$ is the first Rivlin-Ericksen tensor defined as [20]

$$\mathbf{A} = \mathbf{L} + \mathbf{L}^\top,$$

where $\mathbf{L}$ is the velocity gradient and the superscript $\top$ denotes the transpose operator.

For the problem under consideration, let us take velocity field and extra-stress of the following form [4,21]

$$\mathbf{u} = \mathbf{u}(r, t) = q(r, t)\mathbf{e}_\theta; \quad \mathbf{S} = \mathbf{S}(r, t), \quad (2.4)$$
where \( \mathbf{e}_\theta \) is the transverse unit vector of cylindrical coordinates. Moreover, initial conditions, when the fluid is at rest, are

\[
\mathbf{u}(r, 0) = \mathbf{0}; \quad \mathbf{S}(r, 0) = \mathbf{0}.
\]  

(2.5)

The governing equations related to such type of flow of Oldroyd-B fluid in the absence of pressure gradient in axial direction are given by \([4,19]\)

\[
(\lambda_1 \partial_t + 1) \partial_t q(r, t) = \nu (\lambda_2 \partial_t + 1) \left[ \frac{\partial^2}{r^2} + \frac{1}{r} \partial_r - \frac{1}{r^2} \right] q(r, t); \quad (2.6)
\]

\[
(\lambda_1 \partial_t + 1) \sigma(r, t) = \mu (\lambda_2 \partial_t + 1) \left[ \partial_r - \frac{1}{r} \right] q(r, t), \quad (2.7)
\]

where \( \nu \) represents the kinematic viscosity and \( \sigma(r, t) = S_{r\theta}(r, t) \) is the shear stress which is different from zero.

3. Formulation and solutions of the problem

We consider an annular region between two straight infinite circular cylinders of radii \( R_1 \) and \( R_2(> R_1) \), filled with incompressible Oldroyd-B fluid under the assumption to be at rest initially, as shown in Fig.1. At time \( t = 0^+ \), both cylinders begin to rotate about their common axis. The inner cylinder starts rotation because a shear stress given in equation (3.1) is applied on its boundary \([11]\), and the outer cylinder is rotating around its axis through the angular velocity \( R_2 \omega t \).

\[
\sigma(R_1, t) = g \lambda_1^{-1} \left( \frac{R_1}{r} \right)^2 M_{1,-1}(\lambda_1^{-1}, t), \quad (3.1)
\]

where \( g \) is a constant, and \( M \) represents the generalized functions defined by \([13]\)

\[
M_{x,y}(b, t) = L^{-1} \left\{ \frac{s^y}{s^2 - b} \right\} = \sum_{j=0}^{\infty} \frac{(b)^j}{\Gamma(j + 1)} t^{y-j} x^{-y-j}; \quad (3.2)
\]

\[
\text{Re}(x - y) > 0; \quad \text{Re}(s) > 0; \quad \left| \frac{b}{s^2} \right| < 1,
\]

where \( \Gamma(\bullet) \) is the Gamma function.

![Figure 1. Geometry of the problem](image)

Owing to the shear, the fluid between two rotating cylinders gradually starts moving and its velocity in cylindrical coordinates \((r, \theta, z)\) is given in equation (2.4)\textsubscript{1}.
Based on the above suppositions, the governing equations of incompressible Oldroyd-B fluid, corresponding to this motion are given by equations (2.6), (2.7). The corresponding initial and boundary conditions are

\[ q(r, 0) = \partial_t q(r, 0) = 0 ; \quad \sigma(r, 0) = 0 ; \quad r \in [R_1, R_2] , \]  

(3.3)

and

\[ (\lambda_1 \partial_t + 1) \sigma(r, t) \bigg |_{r=R_1} = \mu (\lambda_2 \partial_t + 1) \left[ \partial_r - \frac{1}{r} \right] q(r, t) \bigg |_{r=R_1} = g ; \]

\[ q(r, t) \bigg |_{r=R_2} = R_2 \omega t ; \quad t > 0 , \]  

(3.4)

where \( \omega \) represents the angular acceleration of outer cylinder.

### 3.1. Velocity field

Application of Laplace transform to equation (2.6), taking into account the initial and boundary conditions given in equations (3.3), (3.4) gives

\[ \overline{q}(r, s) \left[ (\lambda_2 s + 1) \left\{ \partial_r^2 + \frac{1}{r} \partial_r - \frac{1}{r^2} \right\} - \frac{s(\lambda_1 s + 1)}{\nu} \right] = 0 , \]  

(3.5)

where \( \overline{q}(r, s) = \mathcal{L}\{q(r, t)\} \), \( \mathcal{L} \) denotes the Laplace transform operator, and

\[ \left[ \partial_r - \frac{1}{r} \right] \overline{q}(r, s) \bigg |_{r=R_1} = \frac{g}{\mu s (\lambda_2 s + 1)} ; \quad \overline{q}(r, s) \bigg |_{r=R_2} = \frac{R_2 \omega}{s^2} , \]  

(3.6)

where \( s \) is the transform parameter.

Let us denote finite Hankel transform of the function \( \overline{q}(r, s) \) by [18, 23]

\[ H\{\overline{q}(r, s)\} = \overline{q}_\mu (r, s) = \int_{R_1}^{R_2} r B_1(r b_n) \overline{q}(r, s) \, dr ; \quad n = 1, 2, 3, \ldots , \]  

(3.7)

where

\[ B_1(r b_n) = J_1(r b_n) Y_1(R_2 b_n) - J_1(R_2 b_n) Y_1(r b_n) , \]  

(3.8)

where \( J_k(\bullet) \) and \( Y_k(\bullet) \) are Bessel functions of order \( k \) of the first and second kind respectively, and \( b_n \) are the positive roots of \( B_1(r b_n) = 0 \).

Applying Hankel transform to equation (3.5), taking into account the conditions given in equation (3.6) and using the following relation

\[ \int_{R_1}^{R_2} r B_1(r b_n) \left\{ \partial_r^2 + \frac{1}{r} \partial_r - \frac{1}{r^2} \right\} \overline{q}(r, s) \, dr = -b_n^2 \overline{q}_\mu (r, s) \]

\[ + \frac{2g}{\pi \mu b_n s (\lambda_2 s + 1)} + \frac{b_n R_2^2 \omega}{s^2} B_2(R_2 b_n) , \]

(3.9)

where

\[ B_2(R_2 b_n) = J_2(R_2 b_n) Y_2(R_1 b_n) - J_2(R_1 b_n) Y_2(R_2 b_n) , \]

we have

\[ \overline{q}_\mu (r, s) = \frac{2g}{\pi \mu b_n^3} \left[ \frac{1}{s} - \frac{\lambda_2 \nu b_n^2 + \lambda_1 s + 1}{s (\lambda_2 \nu b_n^2 + \lambda_1 s + 1) + \nu b_n^2} \right] + \frac{R_2^2 \omega B_2(R_2 b_n)}{b_n} \]

\[ \times \left\{ \frac{1}{s} \left( 1 - \frac{1}{\nu b_n^2} \right) - \frac{1}{s (\lambda_2 \nu b_n^2 + \lambda_1 s + 1) + \nu b_n^2} \left( \lambda_1 - \frac{\lambda_2 \nu b_n^2 + \lambda_1 s + 1}{\nu b_n^2} \right) \right\} \]  

(3.10)
Now applying inverse Hankel transform formula of $\mathfrak{q}_H (r, s)$ defined as [18, 23]

$$
\mathfrak{q}(r, s) = \frac{\pi^2}{2} \sum_{n=1}^{\infty} \frac{b_n^2 J'_1((R_2 b_n)/R_1 b_n) B_1(r b_n)}{J'_1(R_1 b_n) - J'_1(R_2 b_n)} \mathfrak{q}_H (r, s), \tag{3.11}
$$

to equation (3.10), and using the following relation [11]

$$
\int_{R_1}^{R_2} (r^2 - R^2) B_1(r b_n) dr = \frac{4}{\pi b_n^2} \left[ \frac{R_2}{R_1} \right]^2, \tag{3.12}
$$

we arrive at

$$
\mathfrak{q}(r, s) = \frac{g}{2 \mu s} \left[ \frac{R_1}{R_2} \right]^2 \left\{ \frac{r^2 - R^2}{r} \right\} - \frac{\pi g}{\mu} \sum_{n=1}^{\infty} b_n [J'_1(R_1 b_n) - J'_1(R_2 b_n)]
\times \frac{J'_1(R_2 b_n) B_1(r b_n)}{s (\lambda_2 \nu b_n^2 + \lambda_1 s + 1) \nu b_n^2}
\times \left[ \frac{1}{s} - \frac{1}{\nu b_n} \right] - \frac{1}{s (\lambda_2 \nu b_n^2 + \lambda_1 s + 1) \nu b_n^2} \left\{ \lambda_1 b_n + \lambda_2 \nu b_n^2 + \lambda_1 s + 1 \right\}, \tag{3.13}
$$

Using the following relation

$$
\frac{\lambda_1}{s (\lambda_2 \nu b_n^2 + \lambda_1 s + 1) \nu b_n^2} = \sum_{h=0}^{\infty} \sum_{l=0}^{h} \frac{h! \lambda_2^l}{l! (h - l)!} \frac{\nu b_n^2}{s + \frac{1}{\lambda_1}}^{h+1} \left[ \frac{-\nu b_n^2}{\lambda_1} \right]^h, \tag{3.14}
$$
in equation (3.13), we get

$$
\mathfrak{q}(r, s) = \frac{g}{2 \mu s} \left[ \frac{R_1}{R_2} \right]^2 \left\{ \frac{r^2 - R^2}{r} \right\} - \frac{\pi g}{\mu \lambda_1} \sum_{n=1}^{\infty} b_n [J'_1(R_1 b_n) - J'_1(R_2 b_n)]
\times \sum_{h=0}^{\infty} \sum_{l=0}^{h} \frac{h! \lambda_2^l}{l! (h - l)!} \frac{\lambda_2 \nu b_n^2 + \lambda_1 s + 1}{s + \frac{1}{\lambda_1}}^{h+1} \left[ \frac{-\nu b_n^2}{\lambda_1} \right]^h
\times \frac{J'_1(R_2 b_n) B_1(r b_n)}{s (\lambda_2 \nu b_n^2 + \lambda_1 s + 1) \nu b_n^2}
\times \left[ \frac{1}{s} - \frac{1}{\nu b_n} \right] - \frac{1}{s (\lambda_2 \nu b_n^2 + \lambda_1 s + 1) \nu b_n^2} \left\{ \lambda_1 b_n + \lambda_2 \nu b_n^2 + \lambda_1 s + 1 \right\}, \tag{3.15}
$$

In order to avoid exhausting and lengthy computations of residues and contour integrals, the discrete inverse Laplace transform is utilized in equation (3.15), taking into account the following relation [13]

$$
G_{x, y, z} (b, t) = \mathcal{L}^{-1} \left\{ \frac{s^y}{(s^z - b)^z} \right\} = \sum_{j=0}^{\infty} \frac{(b)^j \Gamma(j + z) t^{(j+z)x-y-1}}{\Gamma(z) \Gamma(j + 1) \Gamma[(j + z)x - y]}; \tag{3.16}
$$

$\text{Re}(xz - y) > 0$ ; $\left| \frac{b}{s^z} \right| < 1$.\]
to get the velocity field as

$$q(r, t) = \frac{g}{2\mu} \left[ \frac{R_1}{R_2} \right]^2 \left\{ \frac{r^2 - R_2^2}{r} \right\} - \frac{\pi g}{\mu \lambda_1} \sum_{n=1}^{\infty} \frac{J_1^2(R_2 b_n) B_1(r b_n)}{b_n} \left[ J_1^2(R_1 b_n) - J_1^2(R_2 b_n) \right]$$

$$\times \sum_{h=0}^{\infty} \sum_{l=0}^{h} \frac{h! \lambda_2 l!}{(h-l)!} \left[ \frac{-\nu b_n}{\lambda_1} \right]^h \left\{ \lambda_1 G_{1, l-h, h+1}(-\lambda_1^{-1}, t) + \left( 1 + \lambda_2 \nu b_n^2 \right) \right\} \times G_{1, l-h-1, h+1}(-\lambda_1^{-1}, t)$$

$$\times \left[ \frac{1}{\nu b_n} + (\lambda_2 - \lambda_1) b_n \right] G_{1, l-h-1, h+1}(-\lambda_1^{-1}, t) \right\}$$

(3.17)

3.2. Shear stress

By implementing Laplace transform to equation (2.7), we get

$$(\lambda_1 s + 1) \sigma(r, s) = \mu (\lambda_2 s + 1) \left[ \frac{\nu b_n^2}{s} \right] \bar{q}(r, s)$$

(3.18)

For finding shear stress $\sigma(r, t)$, we write equation (3.10) in the following form

$$\bar{q}_H(r, s) = \frac{1}{(\lambda_2 s + 1)} \left[ \frac{2g}{\pi \mu b_n^4} \left\{ \frac{1}{s} - \frac{\lambda_1 s + 1}{s (\lambda_2 \nu b_n^2 + \lambda_1 s + 1) + \nu b_n^2} \right\} \right]$$

$$+ \frac{R_2^2 \omega B_2(R_2 b_n)}{b_n s} \left\{ \frac{1}{s} - \frac{\lambda_2 \nu b_n^2 (\lambda_2 + 1) - \lambda_1 s - 1}{s (\lambda_2 \nu b_n^2 + \lambda_1 s + 1) + \nu b_n^2} \right\}$$

(3.19)

Application of inverse Hankel transform to equation (3.19) and utilizing relation (3.12), we have

$$\bar{q}(r, s) = \frac{g}{\mu (\lambda_2 s + 1)} \left[ \frac{R_1^2}{2 s R_2} \left\{ \frac{r^2 - R_2^2}{r} \right\} - \frac{\pi}{\mu \lambda_1} \sum_{n=1}^{\infty} \frac{J_1^2(R_2 b_n) B_1(r b_n)}{b_n} \right]$$

$$\times [\frac{1}{s (\lambda_2 s + 1)} \left\{ \frac{1}{s} + \frac{\lambda_2 \nu b_n^2 (\lambda_2 + 1) - \lambda_1 s - 1}{s \lambda_2 \nu b_n^2 + \lambda_1 s + 1 + \nu b_n^2} \right\}]$$

(3.20)

where

$$\bar{q}(r, s) = \frac{g}{\mu \lambda_2 s + 1} \left[ \frac{R_1^2}{s r^2} \right] + \frac{\pi}{\mu \lambda_1} \sum_{n=1}^{\infty} \frac{J_1^2(R_2 b_n) B_1(r b_n)}{b_n} \left[ J_1^2(R_1 b_n) - J_1^2(R_2 b_n) \right]$$

$$\times [\frac{1}{s \lambda_2 s + 1} \left\{ \frac{1}{s} + \frac{\lambda_2 \nu b_n^2 (\lambda_2 + 1) - \lambda_1 s - 1}{s \lambda_2 \nu b_n^2 + \lambda_1 s + 1 + \nu b_n^2} \right\}]$$

(3.21)
Substituting equation (3.21) into equation (3.18), yields

\[
\sigma(r, s) = \frac{g}{s(\lambda_1 s + 1)} \left[ \frac{R_1}{r} \right]^2 + \frac{\pi g}{\lambda_1} \sum_{n=1}^{\infty} \frac{J_n^2(R_2 b_n) \tilde{B}_1(r b_n)}{J_n^2(R_1 b_n) - J_n^2(R_2 b_n)} \times \frac{1}{s(\lambda_2 \nu b_n + \lambda_1 s + 1) + \nu b_n^2} - \frac{\mu R_2^2 \omega \pi^2}{2} \sum_{n=1}^{\infty} \frac{b_n^2 J_n^2(R_2 b_n) \tilde{B}_1(r b_n) B_2(R_2 b_n)}{J_n^2(R_1 b_n) - J_n^2(R_2 b_n)} \times \left[ \frac{1}{s(\lambda_1 s + 1)} \left\{ \frac{\lambda_2 \nu b_n^2(\lambda_2 + 1) - \lambda_1 s - 1}{s(\lambda_2 \nu b_n + \lambda_1 s + 1) + \nu b_n^2} \right\} \right] (3.22)
\]

Utilizing equation (3.14) into equation (3.22), we have

\[
\sigma(r, s) = \frac{g}{s(\lambda_1 s + 1)} \left[ \frac{R_1}{r} \right]^2 + \frac{\pi g}{\lambda_1} \sum_{n=1}^{\infty} \frac{J_n^2(R_2 b_n) \tilde{B}_1(r b_n)}{J_n^2(R_1 b_n) - J_n^2(R_2 b_n)} \times \sum_{h=0}^{\infty} \sum_{l=0}^{h} \frac{h! \lambda_2^l}{l! (h - l)!} \left[ \frac{-\nu b_n^2}{\lambda_1} \right]^h \left[ \frac{\mu R_2^2 \omega \pi^2}{2} \sum_{n=1}^{\infty} \frac{b_n^2 J_n^2(R_2 b_n) \tilde{B}_1(r b_n) B_2(R_2 b_n)}{J_n^2(R_1 b_n) - J_n^2(R_2 b_n)} \times \left[ \frac{1}{s(\lambda_1 s + 1)} \left\{ \frac{1}{s + \frac{1}{\lambda_1}} \right\}^{h+1} \right] \right] (3.23)
\]

By taking inverse Laplace transform and utilizing equations (3.2), (3.16), the shear stress can be acquired as

\[
\sigma(r, t) = \frac{g}{\lambda_1} \left[ \frac{R_1}{r} \right]^2 M_{1,-1}(-\lambda_1^{-1}, t) + \frac{\pi g}{\lambda_1} \sum_{n=1}^{\infty} \frac{J_n^2(R_2 b_n) \tilde{B}_1(r b_n)}{J_n^2(R_1 b_n) - J_n^2(R_2 b_n)} \times \sum_{h=0}^{\infty} \sum_{l=0}^{h} \frac{h! \lambda_2^l}{l! (h - l)!} \left[ \frac{-\nu b_n^2}{\lambda_1} \right]^h G_{1,l-h-1,h+1}(-\lambda_1^{-1}, t) - \frac{\mu R_2^2 \omega \pi^2}{2 \lambda_1} \times \sum_{n=1}^{\infty} \frac{b_n^2 J_n^2(R_2 b_n) \tilde{B}_1(r b_n) B_2(R_2 b_n)}{J_n^2(R_1 b_n) - J_n^2(R_2 b_n)} \times \left[ M_{1,-2}(-\lambda_1^{-1}, t) + \frac{1}{\lambda_1} \right] (3.24)
\]

Now taking into account the following results

\[
\frac{1}{\lambda_1} M_{1,-1}(-\lambda_1^{-1}, t) = 1 - e^{-t/\lambda_1} \quad \frac{1}{\lambda_1} M_{1,-2}(-\lambda_1^{-1}, t) = t + \lambda_1 + \lambda_1 e^{-t/\lambda_1},
\]
equation (3.24) yields
\[
\sigma(r, t) = g \left[ \frac{R_1}{r} \right]^2 \left\{ 1 - e^{-t/\lambda_1} \right\} + \frac{\pi g}{\lambda_1} \sum_{n=1}^{\infty} \frac{J''_1(R_2 b_n) \tilde{B}_1(r b_n)}{J'_1(R_1 b_n) - J'_1(R_2 b_n)}
\]
\[
\times \sum_{h=0}^{\infty} \sum_{l=0}^{h} \frac{h! \lambda_2^h}{l! (h-l)!} \left[ \frac{-\nu b_n^2}{\lambda_1} \right]^h G_{1,l-h-1,h+1}(-\lambda_1^{-1}, t) - \frac{\mu R_2^2 \omega \pi^2}{2}
\]
\[
\times \sum_{n=1}^{\infty} b_n^2 \frac{J''_1(R_2 b_n) \tilde{B}_1(r b_n) B_2(R_2 b_n)}{J'_1(R_1 b_n) - J'_1(R_2 b_n)} \left[ t + \lambda_1 + \lambda_1 e^{-t/\lambda_1} + \frac{1}{\lambda_1^2} \right]
\]
\[
\times \sum_{h=0}^{\infty} \sum_{l=0}^{h} \frac{h! \lambda_2^h}{l! (h-l)!} \left[ \frac{-\nu b_n^2}{\lambda_1} \right]^h \left\{ (\lambda_2 \nu b_n^2 - 1) G_{1,l-h-1,h+1}(-\lambda_1^{-1}, t)
\right\}
\]
\[
+ (\lambda_2 \nu b_n^2 - \lambda_1) G_{1,l-h-1,h+2}(-\lambda_1^{-1}, t) \right\}
\]

4. Limiting cases
Solutions for Maxwell and classical Newtonian fluids, executing the same flow, can be obtained as limiting cases of our general solutions.

4.1. Maxwell fluid
By setting \( \lambda_2 \to 0 \) in equations (3.17) and (3.25), the expressions for velocity and shear stress associated to Maxwell fluid can be recovered.

4.2. Classical Newtonian fluid
By taking \( \lambda_1, \lambda_2 \to 0 \) in equations (3.17) and (3.25), the velocity field and related shear stress for classical Newtonian fluid can be obtained.

5. Graphical representation and discussion
In this section, we illustrate the obtained results graphically and discuss the effects of various substantial parameters on velocity field and shear stress.

Figure 2. Effect of kinematic viscosity on velocity.
Figures 2 and 3 demonstrate the changes in velocity and shear stress related to kinematic viscosity $\nu$. From these figures, it can be observed that the larger value of $\nu$ decreases both velocity and shear stress (in absolute value).

![Figure 3. Effect of kinematic viscosity on shear stress.](image1)

Figures 4 and 5 elaborate the effects of relaxation parameter $\lambda_1$ on velocity and shear stress. It can be seen that with the increase of $\lambda_1$, the velocity increases while shear stress (in absolute value) varies inversely with this parameter.

![Figure 4. Effect of relaxation parameter on velocity.](image2)
Figure 5. Effect of relaxation parameter on shear stress.

Figure 6. Effect of retardation parameter on velocity.

Figures 6 and 7 depict variations in velocity and shear stress due to retardation parameter $\lambda_2$. From here, it can be clearly observed that the larger value of $\lambda_2$ increase both velocity and shear stress (in absolute value).

6. Conclusions

The present study is focused on the derivation of velocity and shear stress for unsteady flow of incompressible Oldroyd-B fluid between two infinite concentric rotating circular cylinders. The motion of the fluid is produced by two cylinders which at time $t = 0^+$ begin to rotate around their common axis. Series solutions of governing differential equations have been derived by using Laplace and Hankel transforms which is most effective method for the proposed problem. For $\lambda_2 \to 0$ or $\lambda_1 \to 0$ and $\lambda_2 \to 0$, similar solutions for Maxwell fluids, respectively, classical Newtonian fluids can be recovered as limiting cases of our general results. Moreover, the acquired results are sketched graphically, and the effects of pertinent parameters on velocity and shear stress are discussed thoroughly.
The obtained results have many engineering applications, e.g., lubrication oil between two rotating cylinders/shafts which appears in many engineering designs especially in Mechanical Machinery. The derived results categorically indicate the following findings:

- The fluid velocity decreases as we increase the value of kinematic viscosity $\nu$, while velocity of fluid increases with increasing values of both relaxation $\lambda_1$ and retardation $\lambda_2$ parameters.
- The shear stress (in absolute value) decreases with increasing values of kinematic viscosity $\nu$ and relaxation parameter $\lambda_1$, while the influence of retardation parameter $\lambda_2$ on shear stress is contrary to that of $\nu$ and $\lambda_1$.

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References


On global universality for zeros of random polynomials

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Abstract

In this work, we study asymptotic zero distribution of random multi-variable polynomials which are random linear combinations $\sum_j a_j P_j(z)$ with i.i.d coefficients relative to a basis of orthonormal polynomials $\{P_j\}$ induced by a multi-circular weight function $Q$ defined on $\mathbb{C}^m$ satisfying suitable smoothness and growth conditions. In complex dimension $m \geq 3$, we prove that $\mathbb{E}[(\log(1+|a_j|))^m] < \infty$ is a necessary and sufficient condition for normalized zero currents of random polynomials to be almost surely asymptotic to the (deterministic) extremal current $\frac{i}{\pi} \frac{\partial}{\partial V} Q$. In addition, in complex dimension one, we consider random linear combinations of orthonormal polynomials with respect to a regular measure in the sense of Stahl & Totik and we prove analogous results in this setting.

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Keywords. Random polynomial, distribution of zeros, global universality

1. Introduction

A random Kac polynomial is of the form

$$f_n(z) = \sum_{j=0}^{n} a_j z^j$$

where coefficients $a_j$ are independent complex Gaussian random variables of mean zero and variance one. A classical result due to Kac and Hammersley [16,19] asserts that normalized zeros of Kac random polynomials of large degree tend to accumulate on the unit circle $S^1 = \{|z| = 1\}$. This ensemble of random polynomials has been extensively studied (see eg. [17,18,22,30] and references therein). Recently, Ibragimov and Zaporozhets [18] proved that for independent and identically distributed (i.i.d.) real or complex random variables $a_j$

$$\mathbb{E}[\log(1+|a_j|)] < \infty \quad (1.1)$$

is a necessary and sufficient condition for zeros of random Kac polynomials to accumulate near the unit circle. In particular, under the condition (1.1) asymptotic zero distribution of Kac polynomials is independent of the choice of the probability law of random coefficients. We refer to this phenomenon as global universality for zeros of Kac polynomials.

In [32], Shiffman and Zelditch remarked that it was an implicit choice of an inner product that produced the concentration of zeros of Kac polynomials around the unit circle $S^1$. 

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More generally, for a simply connected domain $\Omega \Subset \mathbb{C}$ with real analytic boundary $\partial \Omega$ and a fixed orthonormal basis (ONB) $\{P_j\}_{j=1}^{n+1}$ induced by a measure $\rho(z)dz$ where $\rho \in \mathcal{C}^{1}(\partial \Omega)$ and $|dz|$ denote arc-length, Shiffman and Zelditch proved that zeros of random polynomials

$$f_n(z) = \sum_{j=1}^{n+1} a_j P_j(z)$$

where $a_j$ i.i.d standard complex Gaussians concentrate near the boundary $\partial \Omega$ as $n \to \infty$. Furthermore, the empirical measures of zeros

$$\frac{1}{n} \sum_{\{z: f_n(z) = 0\}} \delta_z$$

converge weakly to the equilibrium measure $\mu_\Omega$. Recall that for a non-polar compact set $K \subset \mathbb{C}$ the equilibrium measure $\mu_K$ is the unique minimizer of the logarithmic energy functional

$$\nu \to \int \int \log \left| \frac{1}{z - w} \right| d\nu(z) d\nu(w)$$

over all probability measures supported on $K$. Later, Bloom [10] observed that $\Omega$ can be replaced by a regular compact set $K \subset \mathbb{C}$, the inner product can be defined in terms of any Bernstein Markov measure (see also [11] for a generalization of this result to $\mathbb{C}^m$ for Gaussian random pluricomplex polynomials). More recently, Pritsker and Ramachandran [23] observed that (1.1) is a necessary and sufficient condition for zeros of random linear combinations of Szegö, Bergman, or Faber polynomials (associated with Jordan domains bounded with analytic curves) to accumulate near the support of the corresponding equilibrium measure.

The purpose of this work is to study global universality for normalized zero currents of random multi-variable complex polynomials. Asymptotic zero distribution of multivariate random polynomials has been studied by several authors (see eg. [1–3, 5, 8, 11, 13, 31]). We remark that randomization of the space of polynomials in these papers is different than that of [18, 21, 23]. Namely, in the former ones each $\mathcal{P}_n$ is endowed with a $d_n := \text{dim}(\mathcal{P}_n)$ fold product probability measure which leads to a sequence of polynomials (with $n^{th}$ coordinate has total degree at most $n$) chosen independently at random according to the $n^{th}$ product measure. On the other hand, the papers [18, 21, 23] fix a random sequence of scalars for which one considers random linear combinations of a fixed basis for $\mathcal{P}_n$. We adopt the approach of [18, 21, 23] in the present note.

The setting is as follows: let $Q : \mathbb{C}^m \to \mathbb{R}$ be a weight function satisfying

$$Q(z) \geq (1 + \epsilon) \log \|z\| \text{ for } \|z\| \gg 1 \quad (1.2)$$

for some fixed $\epsilon > 0$. Throughout this note (unless otherwise stated), we assume that the function $Q : \mathbb{C}^m \to [0, \infty)$ is of class $\mathcal{C}^2$ and it is invariant under the action of the real torus $\mathbb{S}^m$, the latter means that

$$Q(z_1, \ldots, z_m) = Q(|z_1|, \ldots, |z_m|) \text{ for all } (z_1, \ldots, z_m) \in \mathbb{C}^m. \quad (1.3)$$

One can define an associated weighted extremal function

$$V_Q(z) := \sup \{u(z) : u \in \mathcal{L}(\mathbb{C}^m), u \leq Q \text{ on } \mathbb{C}^m\}$$

where $\mathcal{L}(\mathbb{C}^m)$ denotes the Lelong class of pluri-subharmonic (psh) functions $u$ that satisfies $u(z) - \log^+ \|z\| = O(1)$. We also denote by

$$\mathcal{L}^+(\mathbb{C}^m) := \{u \in \mathcal{L}(\mathbb{C}^m) : u(z) \geq \log^+ \|z\| + C_u \text{ for some } C_u \in \mathbb{R}\}.$$ 

Seminal results of Siciak and Zakharyuta (see [29] and references therein) imply that $V_Q \in \mathcal{L}^+(\mathbb{C}^m)$ and that $V_Q$ verifies

$$V_Q(z) = \max_{p \in \mathbb{C}^m} \left\{ \frac{1}{\deg p} \log |p(z)| : p \text{ is a polynomial and } \max_{z \in \mathbb{C}^m} |p(z)| e^{-\deg(p)Q(z)} \leq 1 \right\}. \quad (1.4)$$
Moreover, a result of Berman [7, Proposition 2.1] implies that $V_Q$ is of class $C^{1,1}$.

Next, we define an inner product on the space $\mathcal{P}_n$ of multi-variable polynomials of degree at most $n$ by setting

$$\langle f_n, g_n \rangle_n := \int_{C_m} f_n(z) \overline{g_n(z)} e^{-2nQ(z)} dV_m(z)$$

(1.5)where $dV_m$ denotes the Lebesgue measure on $\mathbb{C}^m$. We also let $\{P_j^n\}_{j=1}^{d_n}$ be the orthonormal basis (ONB) for $\mathcal{P}_n$ obtained by applying Gram-Schmidt algorithm in the Hilbert space $(\mathcal{P}_n, \langle \cdot, \cdot \rangle_n)$ to the monomials $\{z^J\}_{|J| \leq n}$ where $J = (j_1, \ldots, j_m)$ is a multi-index and we assume that the monomials $\{z^J\}_{|J| \leq n}$ are ordered with respect to lexicographical ordering. Note that since $Q$ is $m$–circular we have $P^n_j(z) = c^j_n z^j$ for some deterministic constant $c^j_n$ for $J \in \mathbb{N}^m$.

Let $a_1, a_2, \ldots$ be a sequence of i.i.d. real or complex random variables whose probability law denoted by $\mathbf{P}$. Throughout this note, we assume that $a_j$ are non-degenerate, roughly speaking this means that $\mathbf{P}[a_j = z] < 1$ for every $z \in \mathbb{C}$ (see §2.1.) A random polynomial is of the form

$$f_n(z) = \sum_{j=1}^{d_n} a_j P^n_j(z)$$

where $d_n := \dim(\mathcal{P}_n) = \binom{n+m}{n}$. We also let $\mathcal{H} := \bigcup_{n=1}^{\infty} \mathcal{P}_n$ and denote the corresponding probability space of polynomials by $(\mathcal{H}, \mathbf{P})$.

**Theorem 1.1.** Let $a_j$ be i.i.d. non-degenerate real or complex random variables satisfying

$$\mathbb{E}[(\log(1 + |a_j|))^{m}] < \infty.$$  

(1.6)

If the dimension of complex Euclidean space $m \geq 3$ then almost surely in $\mathcal{H}$

$$\frac{1}{n} \log |f_n(z)| \xrightarrow{n \to \infty} V_Q(z)$$

in $L^1_{\text{loc}}(\mathbb{C}^m)$. In particular, almost surely in $\mathcal{H}$

$$\frac{1}{n} \partial \overline{\partial} \left( \frac{1}{n} \log |f_n(z)| \right) \xrightarrow{n \to \infty} \frac{i}{\pi} \partial \overline{\partial} V_Q(z)$$

in the sense of currents as $n \to \infty$.

Furthermore, for all dimensions $m \geq 1$, we have convergence in probability

$$\frac{1}{n} \partial \overline{\partial} \left( \frac{1}{n} \log |f_n(z)| \right) \xrightarrow{n \to \infty} \frac{i}{\pi} \partial \overline{\partial} V_Q(z)$$

in the sense of currents as $n \to \infty$.

Note that Theorem 1.1 provides an optimal condition on random coefficients for a random version of Siciak-Zakharyuta theorem in this context (cf. [1, 3, 8, 9]). In the univariate case we have $\frac{1}{\pi} \partial \overline{\partial} = \frac{1}{2\pi i} \Delta$ where $\Delta$ denotes the Laplacian and we denote the corresponding equilibrium measure by $\mu_Q := \frac{1}{\pi} \partial \overline{\partial} V_Q$. An important example is $Q(z) = \frac{|z|^2}{2}$ and $\mu_Q = \frac{1}{2\pi} \partial \overline{\partial} dz$ where $\partial \overline{\partial}$ denotes closed the unit disc in the complex plane [29, pp 245]. Then a routine calculation shows that

$$P^n_j(z) = \sqrt{\frac{n^j}{2\pi j^!}} z^j \text{ for } j = 0, 1, \ldots, n$$

form an ONB for $\mathcal{P}_n$. A random Weyl polynomial is of the form

$$W_n(z) = \sum_{j=0}^{n} a_j \sqrt{\frac{n^j}{j^!}} z^j.$$ 

In particular, Theorem 1.1 generalizes a special case of [21, Theorem 2.5] to the several complex variables.
Let us denote the Euclidean volume in $\mathbb{C}^m$ by $\text{Vol}_{2m}$ and for an open set $U \subset \mathbb{C}^m$, we define

$$\mathcal{V}_U := \frac{1}{(m-1)!} \int_U \frac{i}{2\pi} \partial \bar{\partial} V_Q \wedge (\frac{i}{\pi} \partial \bar{\partial} \|z\|^2)^{m-1}.$$ 

Next result indicates that in higher dimensions the condition $(1.6)$ is also necessary for zero divisors of random polynomials to be almost surely equidistributed with the extremal current $\frac{i}{\pi} \partial \bar{\partial} V_Q$.

**Theorem 1.2.** Let $a_j$ be i.i.d. non-degenerate real or complex valued random variables and assume that the dimension of complex Euclidean space $m \geq 3$. The logarithmic moment

$$\mathbb{E}[(\log(1 + |a_j|))^m] < \infty$$

if and only if

$$\mathbb{P}\left\{ \{f_n\}_{n \geq 0} : \lim_{n \to \infty} \frac{1}{n} \text{Vol}_{2m-2}(Z_{f_n} \cap U) = \mathcal{V}_U \right\} = 1 \quad (1.7)$$

for every open set $U \subset (\mathbb{C}^*)^m$ such that $\partial U$ has zero Lebesgue measure.

Note that when $m = 1$ the volume $\text{Vol}_{2m-2}(Z_{f_n} \cap U)$ becomes the number of zeros of $f_n$ in $U$ which we denote by

$$N_n(U, f_n) := \#\{z \in U : f_n(z) = 0\}.$$ 

The following result is an immediate consequence of Theorem 1.1 together with Theorem 1.2 and provides a weak universality result for zeros of univariate random polynomials:

**Corollary 1.3.** Let $a_j$ be i.i.d. non-degenerate real or complex valued random variables. If the logarithmic moment

$$\mathbb{E}[\log(1 + |a_j|)] < \infty$$

then for every $\epsilon > 0$

$$\lim_{n \to \infty} \text{Prob}_n\left\{ f_n : |\frac{1}{n} N_n(U, f_n) - \mu_Q(U) | \geq \epsilon \right\} = 0 \quad (1.8)$$

for every open set $U \subset \mathbb{C}^*$ such that $\partial U$ has zero Lebesgue measure.

We remark that the condition $(1.8)$ is called convergence in probability in the context of probability theory. Moreover, $(1.8)$ is equivalent to the following statement: for every subsequence $n_k$ of positive integers there exists a further subsequence $n_{kj}$ such that

$$\frac{1}{n_{kj}} N_{n_{kj}}(U, f_{n_{kj}}) \to \mu_Q(U) \text{ with probability one in } \mathfrak{M}.$$ 

Next, we consider random elliptic polynomials which are of the form

$$G_n(z) = \sum_{|j| = n} a_j \binom{n}{j} \frac{1}{m^{|j|}} z^j$$

where $\binom{n}{j} = \frac{n!}{(n-j)!j! \cdots m!}$ and $a_j$ are non-degenerate i.i.d. random variables.

Let us denote by

$$\mathcal{M}_U := \frac{1}{(m-1)!} \int_U \frac{i}{2\pi} \partial \bar{\partial} \log(1 + \|z\|^2) \wedge (\frac{i}{\pi} \partial \bar{\partial} \|z\|^2)^{m-1}.$$ 

The following result is an analogue of Theorem 1.2 in the present setting (see §4.1 for details):

**Theorem 1.4.** Let $a_j$ be i.i.d. non-degenerate real or complex valued random variables and assume that the dimension of complex Euclidean space $m \geq 3$. The logarithmic moment

$$\mathbb{E}[(\log(1 + |a_j|))^m] < \infty$$

if and only if the zero loci of elliptic polynomials satisfy

$$\mathbb{P}\left\{ \{G_n\}_{n \geq 0} : \lim_{n \to \infty} \frac{1}{n} \text{Vol}_{2m-2}(Z_{G_n} \cap U) = \mathcal{M}_U \right\} = 1 \quad (1.9)$$
for every open set $U \in (\mathbb{C}^*)^m$ such that $\partial U$ has zero Lebesgue measure.

Finally, we consider random linear combinations of univariate orthonormal polynomials of regular asymptotic behavior (cf. [28, §3]). Orthogonal polynomials of regular $n^{th}$ root asymptotic behavior are natural generalizations of classical orthogonal polynomials on the real line. More precisely, let $\mu$ be a measure Borel measure with compact support $S_\mu \subset \mathbb{C}$. We assume that the support $S_\mu$ contains infinitely many points and its logarithmic capacity $\text{Cap}(S_\mu) > 0$. We let $\Omega := \mathbb{C} \setminus S_\mu$ and $g_\Omega(z, \infty)$ denotes the Green function with logarithmic pole at infinity. Then the equilibrium measure of the support $S_\mu$ is given by $\nu_{S_\mu} := \Delta g_\Omega(z, \infty)$. We say that $\Omega$ is regular if $g(z, \infty) \equiv 0$ on $S_\mu$. It is well known that if $\Omega$ is regular then $g(z, \infty)$ is continuous on $\mathbb{C}$. Next, we define the inner product induced by $\mu$:

$$\langle f, g \rangle := \int_{\mathbb{C}} f(z) \overline{g(z)} d\mu$$

on the space of polynomials $P_n$. Then one can find uniquely defined orthonormal polynomials

$$P_n^\mu(z) = \gamma_n(\mu) z^n + \cdots, \text{ where } \gamma_n(\mu) > 0 \text{ and } n \in \mathbb{N}.$$ 

We say that $\mu$ is regular, denoted by $\mu \in \text{Reg}$, if

$$\lim_{n \to \infty} \gamma_n(\mu)^{1/n} = \frac{1}{\text{Cap}(S_\mu)}.$$  \hspace{1cm} (1.10)

For a fixed $\mu \in \text{Reg}$, we consider random linear combinations of orthonormal polynomials

$$f_n(z) = \sum_{j=0}^n a_j P_j^\mu(z)$$

and we obtain the following generalization:

**Theorem 1.5.** Let $\mu \in \text{Reg}$ such that $\Omega := \mathbb{C} \setminus S_\mu$ is connected and regular. Assume that the convex hull $\text{Co}(S_\mu)$ has Lebesgue measure zero (hence, $\text{Co}(S_\mu)$ is a line segment). If the logarithmic moment

$$\mathbb{E}[\log(1 + |a_j|)] < \infty$$

then for every $\epsilon > 0$

$$\lim_{n \to \infty} \text{Prob}_n \left\{ f_n : \frac{1}{n} \left| N_n(U, f_n) - \nu_{S_\mu}(U) \right| \geq \epsilon \right\} = 0$$

for every open set $U \in \mathbb{C}^*$ such that $\partial U$ has zero Lebesgue measure.

We remark that if $\mu$ is a Bernstein-Markov measure with compact support in $\mathbb{C}$ then $\mu \in \text{Reg}$ ([9, Proposition 3.4]). In particular, any Bernstein-Markov measure $\mu$ supported on an interval of the real line falls in the framework of Theorem 1.5. The latter class contains classical orthogonal polynomials such as Chebyshev or Jacobi polynomials.

2. Background

2.1. Probabilistic preliminaries

For a complex (respectively real) random variable $\eta$ we let $P$ denote its probability law and denote its concentration function by

$$Q(\eta, r) := \sup_{z \in \mathbb{C}} P[\eta \in B(z, r)]$$

where $B(z, r)$ denotes the Euclidean ball (respectively interval) centered at $z$ and of radius $r > 0$. We say that $\eta$ is non-degenerate if $Q(\eta, r) < 1$ for some $r > 0$. If $\eta$ and $\xi$ are independent complex random variables and $r, c > 0$ then we have

$$Q(\eta + \xi, r) \leq \min\{Q(\eta, r), Q(\xi, r)\} \text{ and } Q(c\xi, r) = Q(\zeta, \frac{r}{c}).$$  \hspace{1cm} (2.1)
Let $a_1, a_2, \ldots$ be independent and identically distributed (real or complex valued) random variables. The following lemma is standard in the literature and it will be useful in the sequel.

**Lemma 2.1.** Let $a_j$ be a sequence of i.i.d. real or complex valued random variables for $j = 1, 2, \ldots$

(i) If $E[(\log(1 + |a_j|))^m] < \infty$ then for each $\epsilon > 0$ almost surely

$$|a_j| < e^{\sqrt{m\epsilon}}$$

for sufficiently large $j$.

(ii) If $E[(\log(1 + |a_j|))^m] = \infty$ then almost surely

$$\limsup_{j \to \infty} |a_j|^\frac{1}{m} = \infty.$$

**Proof.** For a non-negative random variable $X$ we have

$$\sum_{j=1}^{\infty} P[X \geq j] \leq E[X] \leq 1 + \sum_{j=1}^{\infty} P[X \geq j].$$

Letting $X = \frac{1}{m}(\log(1 + |a_1|))^m$ and using the assumption that $a_j$ are identically distributed, we obtain

$$\sum_{j=1}^{\infty} P[a_j \in \mathbb{C} : |a_j| \geq e^{\sqrt{mj}}] < \infty.$$

Hence, by independence of $a_j$’s and Borel-Cantelli lemma we have almost surely

$$|a_j| < e^{\sqrt{mj}}$$

for sufficiently large $j$.

For (ii), we define the event $A_j^M := \{a_j \in \mathbb{C} : |a_j|^m \geq M\}$ where $M > 1$ is fixed. Then by (2.3)

$$\sum_{j=1}^{\infty} P_n[A_j^M] = \infty$$

and second Borel-Cantelli lemma implies that almost surely $|a_j|^m \geq M$ for infinitely many values of $j$. Now, we let $M_n > 0$ be a sequence such that $M_n \uparrow \infty$. Then by previous argument the event

$$F_n := \{|a_j|^m \geq M_n \text{ for infinitely many } j\}$$

has probability one. Thus letting $F = \cap_{n=1}^{\infty} F_n$ has also probability one and (ii) follows. \(\square\)

### 2.2. Pluripotential theory

#### 2.2.1. Global extremal function.

Let $\Sigma \subset \mathbb{C}^m$ be a closed set. Recall that an *admissible weight function* $Q : \mathbb{C}^m \to \mathbb{R}$ is a lower semi-continuous function that satisfies

1. \(\{z \in \Sigma : Q(z) < \infty\}\) is not pluripolar
2. \(\lim_{\|z\| \to \infty} (Q(z) - \log \|z\|) = \infty\) if $\Sigma$ is unbounded.

The *weighted extremal function* associated to the pair $(\Sigma, Q)$ is defined by

$$V_{\Sigma, Q} = \sup\{u(z) : u \in \mathcal{L}(\mathbb{C}^m), u \leq Q \text{ on } \Sigma\}. \quad (2.4)$$

If $\Sigma = \mathbb{C}^m$ and $Q$ is an admissible weight function we write $V_Q$ for short. We also let $V_{\Sigma, Q}^*$ denote the upper semi-continuous regularization of $V_{\Sigma, Q}$ that is $V_{\Sigma, Q}^*(z) := \limsup_{\zeta \to z} V_{\Sigma, Q}(\zeta)$. 
It is well known that \( V_{\Sigma,Q}^* \in L^+(\mathbb{C}^m) \) (see [29, Appendix B]). Moreover, for an admissible weight function \( Q \) the set
\[
\{ z \in \mathbb{C}^m : V_{\Sigma,Q}(z) < V_{\Sigma,Q}^*(z) \}
\]
is pluripolar. We also remark that when \( Q \equiv 0 \) and \( \Sigma \) is a non-pluripolar compact set the function \( V_{\Sigma}^* \) is nothing but the pluricomplex Green function of \( \Sigma \) (see [20, §5]). We let \( B(r) \) denote the ball in \( \mathbb{C}^m \) centered at the origin and with radius \( r > 0 \). Then it is well known [29, Appendix B] that for sufficiently large \( r \)
\[
V_Q = V_{B(r),Q} \text{ on } \mathbb{C}^m
\]
for every admissible weight function \( Q \). It also follows from a result of Siciak [27, Proposition 2.16] that if \( Q \) is a continuous admissible weight function then \( V_Q = V_Q^* \) on \( \mathbb{C}^m \). We refer the reader to the manuscript [29, Appendix B] for further properties of the weighted global extremal function.

2.2.2. Bergman kernel asymptotics. In the sequel we will assume that \( Q : \mathbb{C}^m \to \mathbb{R} \) is a \( C^2 \) weight function satisfying (1.2) and (1.3). The Bergman kernel for the Hilbert space of weighted polynomials \( \mathcal{P}_n \) may be defined as
\[
S_n(z,w) := d_n^{n/2} \sum_{j=1}^{d_n} P_n^j(z) \overline{P_n^j(w)}
\]
where \( \{P_n^j\}_{j=1}^{d_n} \) is an ONB for \( \mathcal{P}_n \) as in the introduction. The restriction of the Bergman kernel over the diagonal is given by
\[
S_n(z,z) = d_n \sum_{j=1}^{d_n} |P_n^j(z)|^2.
\]
It is well known [8, §6] (cf. [2, 7]) that
\[
\frac{1}{2n} \log S_n(z,z) \to V_Q(z) \text{ locally uniformly on } \mathbb{C}^m.
\]

3. Proofs

Proof of Theorem 1.1. By [8, Proposition 4.4] it is enough to prove that almost surely in \( \mathcal{H} \), for any subsequence \( I \) of positive integers
\[
(\limsup_{n \in I} \frac{1}{n} \log |f_n(z)|)^* = V_Q(z)
\]
for all \( z \in \mathbb{C}^m \). To this end we fix a subsequence \( I \) of positive integers.

Step 1: Proof of upper bound. Note that by Lemma 2.1 for each \( \epsilon > 0 \) there exists \( j_0 \in \mathbb{N} \) such that almost surely
\[
\sum_{j=j_0}^{d_n} |a_j|^2 \leq d_n e^{2 \sqrt{\epsilon d_n}}.
\]
Then using \( d_n = O(n^m) \) and by Cauchy-Schwarz inequality almost surely in \( \mathcal{H} \)
\[
\limsup_{n \in I} \frac{1}{n} \log |f_n(z)| = \limsup_{n \in I} \left( \frac{1}{n} \log \frac{|f_n(z)|}{\sqrt{S_n(z,z)}} + \frac{1}{2n} \log S_n(z,z) \right)
\leq \limsup_{n \to \infty} \left( \frac{1}{2n} \log(\sum_{j=1}^{d_n} |a_j|^2) + \frac{1}{2n} \log S_n(z,z) \right)
\leq \epsilon + V_Q(z)
\]
on $\mathbb{C}^m$. Thus, it follows from [9, Lemma 2.1] that $(\limsup_{n \to \infty} \frac{1}{n} \log |f_n(z)|)^* \in L(\mathbb{C}^m)$ and

$$F(z) := (\limsup_{n \to \infty} \frac{1}{n} \log |f_n(z)|)^* \leq V_Q(z)$$

holds on $\mathbb{C}^m$ almost surely in $\mathcal{H}$.

**Step 2: Proof of lower bound.** In order to get the lower bound first we prove the following lemma which is a generalization of [4, Proposition 2.1]:

**Lemma 3.1.** For every $\epsilon > 0$ and $z \in (\mathbb{C}^*)^m$ there exists $\delta > 0$ such that for sufficiently large $n \in \mathbb{N}$

$$\# \{ j \in \{ 1, \ldots, d_n \} : P_j^n(z) > e^{n(V_Q(z)-3\epsilon)} \} \geq \delta d_n.$$ 

**Proof.** We denote the probability measures $\mu_n := \frac{1}{b_n} e^{-2nQ(z)} dV_m$ where the normalizing constants $b_n := \int_{\mathbb{C}^m} e^{-2nQ(z)} dV_m$. It follows that the sequence of measures $\{ \mu_n \}_{n=1}^\infty$ satisfies large deviation principle (LDP) on $\mathbb{C}^m$ with the rate function $\mathcal{I}(z) = 2[Q(z) - \inf_{w \in \mathbb{C}^m} Q(w)]$ (see e.g. [12, 1.1.5]). More precisely, for $A \subset \mathbb{C}^m$ letting

$$\mathcal{I}(A) := \inf_{z \in A} \mathcal{I}(z)$$

we have

$$\limsup_{n \to \infty} \frac{1}{n} \log \mu_n(K) \leq -\mathcal{I}(K) \text{ and } \liminf_{n \to \infty} \frac{1}{n} \log \mu_n(U) \geq -\mathcal{I}(U)$$

for every closed set $K \subset \mathbb{C}^m$ and every open set $U \subset \mathbb{C}^m$.

Next, we define

$$c_{nT}^n := \left( \int_{\mathbb{C}^m} |z|^2 e^{-2nQ(z)} dV_m \right)^{-\frac{1}{2}}$$

where $T \in [0, 1]^m$ is a multi-index and $z^T = z_1^{t_1} \cdots z_m^{t_m}$. Then by Varadhan’s lemma [12, Theorem 2.1.10] and (1.2), for every such $T = (t_1, \ldots, t_m)$

$$- \lim_{n \to \infty} \frac{1}{n} \log c_{nT}^n = \sup_{r \in \mathbb{R}^m_+} \left( \sum_{j=1}^m t_j \log r_j - Q(r_1, \ldots, r_m) \right) = \sup_{S \in \mathbb{R}^m} \left( \langle S, T \rangle - Q(e^{s_1}, \ldots, e^{s_m}) \right) =: u(T).$$

Let us denote by

$$\Phi(S) := Q(e^{s_1}, \ldots, e^{s_m})$$

where $S = (s_1, \ldots, s_m) \in \mathbb{R}^m$ and Legendre-Fenchel transform of $\Phi$ is by definition given by

$$\Phi^*(T) := \sup_{S \in \mathbb{R}^m} \left( \langle S, T \rangle - \Phi(S) \right) = \sup_{S \in \mathbb{R}^m_{\geq 0}} \left( \langle S, T \rangle - \Phi(S) \right),$$

where the second equality follows from $Q \geq 0$. Since $u(T) = \Phi^*(T)$ for $T \in [0, 1]^m$ the function $u(T)$ is lower-semicontinuous and convex on $[0, 1]^m$.

On the other hand, denoting by $\Psi(S) := V_Q(e^{s_1}, \ldots, e^{s_m})$ since $\Psi$ is a $C^{1,1}$ convex function we have

$$\Psi(S) = \Psi^{**}(S).$$

Thus, for every $\epsilon > 0$ and $S \in \mathbb{R}^m$ there exists $T_0 \in R_{\geq 0}^m$ such that

$$\Psi(S) - \epsilon < \langle S, T_0 \rangle - \Phi^*(T_0) \leq \langle S, T_0 \rangle - \Phi^*(T_0)$$
where the latter inequality follows from the inequality $V_Q \leq Q$ on $\mathbb{C}^m$. Moreover, it follows from [24, Theorem 23.5] and $V_Q \in \mathcal{E}^{1,1}(\mathbb{C}^m)$ that $T_0 = \nabla \Psi(S)$ and hence by using $V_Q \in \mathcal{L}$ we conclude that $T_0 \in [0, 1]^m$. Thus, for every $\epsilon > 0$ and $S \in \mathbb{R}^m$ there exists $T_0 \in [0, 1]^m$ such that
\[
\langle S, T_0 \rangle - u(T_0) > V_Q(e^{x_1}, \ldots, e^{x_m}) - \epsilon
\]
and by lower-semicontinuity of $u$ there exists a product of intervals $J \subset [0, 1]^m$ containing $T_0$ such that the Lebesgue measure $|J| > 0$ and
\[
\langle S, T \rangle - u(T) > V_Q(e^{x_1}, \ldots, e^{x_m}) - 2\epsilon \text{ for every } T \in J.
\]

Now, for fixed $z \in (\mathbb{C}^*)^m$ letting $S = (\log |z_1|, \ldots, \log |z_m|)$ then for sufficiently large $n$ we have
\[
\frac{1}{n} \log (e^{n|z|^T} n) > V_Q(z) - 3\epsilon
\]
for every $T \in J$. Finally, letting $J_n := \{J \in \mathbb{N}^m : |J| \leq n \text{ and } \frac{1}{n} J \in J\}$ where $\frac{1}{n} J := (\frac{j}{n}, \ldots, \frac{j}{n})$ we see that for sufficiently large $n$ we have
\[
\#J_n \geq \frac{d_n}{2} |J|
\]
where $|J|$ denotes Lebesgue measure of $J \subset \mathbb{R}^m$.

Now, we turn back to proof of the lower bound. For fixed $z \in (\mathbb{C}^*)^m$ and for every $\epsilon > 0$ by Lemma 3.1 there exists a product interval $J \subset [0, 1]^m$ such that
\[
P^m_J(z) > e^{n(V_Q(z) - \epsilon)}
\]
where $P^m_J(z) = C_J^m z^J$ and $J \in J_n := \{\frac{1}{n} J \in J\}$. Next, we define the random variables
\[
X_n := \sum_{J \in J_n} a_J \alpha_J \text{ and } Y_n := \sum_{J \not\in J_n} a_J \alpha_J
\]
where
\[
\alpha_J := e^{-n(V_Q(z) - \epsilon)} P^m_J(z).
\]

Then by (2.1) and sufficiently large $n$ we have
\[
\text{Prob}_n[f_n : |f_n(z)| < e^{n(V_Q(z) - 2\epsilon)}] \leq \Omega(X_n + Y_n, e^{-\epsilon n}) \leq \Omega(X_n, e^{-\epsilon n}). \tag{3.2}
\]

Now, it follows from Kolmogorov-Rogozin inequality [14] and $\alpha_J > 1$ that
\[
\Omega(X_n, e^{-\epsilon n}) \leq C_1 \left( \sum_{J \in J_n} (1 - \Omega(a_J, e^{-\epsilon n}))^{-\frac{1}{2}} \right) \leq C_2 |J_n|^{-\frac{1}{2}} \leq C_3 (d_n)^{-\frac{1}{2}}. \tag{3.3}
\]

Hence combining (3.2) and (3.3) we obtain: for every $z \in (\mathbb{C}^*)^m$ there exists $C_\epsilon > 0$ such that
\[
\text{Prob}_n[f_n : \frac{1}{n} \log |f_n(z)| < V_Q(z) - \epsilon] \leq \frac{C_\epsilon}{\sqrt{n^m}}. \tag{3.4}
\]

Since $m \geq 3$, it follows from Borel-Cantelli lemma and (3.4) that with probability one in $\mathcal{H}$
\[
\liminf_{n \to \infty} \frac{1}{n} \log |f_n(z)| \geq V_Q(z). \tag{3.5}
\]

Thus, we conclude that for each $z \in (\mathbb{C}^*)^m$ there exists a subset $\mathcal{C}_z \subset \mathcal{H}$ of probability one such that for every sequence $\{f_n\}_{n \in \mathbb{N}} \in \mathcal{C}_z$
\[
F(z) = (\limsup_{n \in I} \frac{1}{n} \log |f_n(z)|)^* = V_Q(z) \tag{3.6}
\]
Next, we fix a countable dense subset $D := \{z_j\}_{j \in \mathbb{N}} \subset \mathbb{C}^m$ such that $z_j \in (\mathbb{C}^*)^m$ and (3.6) holds. Then, we define
\[
\mathcal{E} := \cap_{j=1}^\infty \mathcal{C}_{z_j}.
\]
Note that $C \subset H$ is also of probability one. Since $V_Q(z)$ is continuous on $C^m$ we have
\[
V_Q(z) = \lim_{z_j \in D,z_j \to z} V_Q(z_j) \leq \limsup_{z_j \in D,z_j \to z} F(z_j) \leq F(z)
\]
where the second inequality follows from (3.5) and the last one follows from upper-semicontinuity of $F(z)$. We deduce that for every $\{f_n\}_{n \in \mathbb{N}} \in C$
\[
F(z) = V_Q(z)
\]
for every $z \in (\mathbb{C}^*)^m$. Since $\{z \in \mathbb{C}^m : z_1 \cdots z_m = 0\}$ has Lebesgue measure zero, by a well-known property of psh functions we conclude that
\[
F(z) = V_Q(z)
\]
for every $z \in \mathbb{C}^m$. This completes the proof for dimensions $m \geq 3$.

On the other hand, it follows from [8, Proposition 4.4], Step 1, (3.4) and the preceding argument that for every $\epsilon > 0$, open set $U \subset \mathbb{C}^m$ and sufficiently large $n$
\[
\text{Prob}_n[f_n \in P_n : \|1/n \log |f_n| - V_Q\|_{L^1(U)} \geq \epsilon] \leq C \epsilon \sqrt{n/m}
\]
which gives the second assertion.

**Proof of Theorem 1.2.** First, we prove that (1.6) is a sufficient condition for (1.7). We fix an open set $U \subset (\mathbb{C}^*)^m$ such that $\partial U$ has zero Lebesgue measure. Let us denote by
\[
\Theta := \frac{1}{(m-1)!} \frac{i}{\pi} \partial \bar{\partial} V_Q \land (\frac{i}{2} \partial \bar{\partial} \|z\|^2)^{m-1}.
\]
For $\delta > 0$ arbitrary, we fix real valued smooth functions $\varphi_1, \varphi_2$ such that $0 \leq \varphi_1 \leq \chi_U \leq \varphi_2 \leq 1$ and
\[
\int_U \Theta - \delta \leq \int_{\mathbb{C}^m} \varphi_1 \Theta \leq \int_{\mathbb{C}^m} \varphi_2 \Theta \leq \int_U \Theta + \delta.
\]
Now, letting
\[
\psi_j := \frac{\varphi_j}{(m-1)!} (\frac{i}{2} \partial \bar{\partial} \|z\|^2)^{m-1}
\]
for $j = 1, 2$ by Wirtinger’s theorem we have
\[
Vol_{2m-2}(Z_{f_n} \cap U) \leq \int_{Z_{f_n}} \psi_2.
\]
Then by Theorem 1.1
\[
\limsup_{n \to \infty} \frac{1}{n} Vol_{2m-2}(Z_{f_n} \cap U) \leq \int_{\mathbb{C}^m} \varphi_2 \Theta \leq \int_U \Theta + \delta.
\]
Similarly one can obtain
\[
\liminf_{n \to \infty} \frac{1}{n} Vol_{2m-2}(Z_{f_n} \cap U) \geq \int_U \Theta - \delta.
\]
Since $\delta > 0$ is arbitrary the assertion follows.

Next, we prove that (1.6) is a necessary condition for (1.7). We will prove the assertion by contradiction. Assume that
\[
\mathbb{E}[(\log(1 + |a_j|))^m] = \infty.
\]
By assumption $U \subset (\mathbb{C}^*)^m$ so we have $0 \leq b_n := \min_{j=1,...,d_n} \inf_{z \in U} |P_j^n(z)|$. For $\epsilon > 0$ small we let
\[
t_n := \left(\frac{\epsilon^{n(M_Q+\epsilon)}}{b_n}\right)^m
\]
where \( M_Q := \sup_{U} V_Q \). Then by the argument in the proof of Lemma 2.1 (ii) for each \( n \in \mathbb{N}_+ \) the set
\[
F_n := \{ |a_j|^{m/j} \geq t_n \text{ for infinitely many } j \}
\]
has probability one. This implies that
\[
F := \cap_{n=1}^{\infty} F_n
\]
has also probability one. Thus, we may assume that for infinitely many values of \( n \) there exists \( j_n \in \{1, \ldots, d_n\} \) such that
\[
\max_{j=1, \ldots, d_n} |a_j|^{1/j} = |a_{j_n}|^{1/j_n} \quad \text{and} \quad |a_{j_n}| \geq t_n^{j_n/m}.
\]
(3.7)
For simplicity of notation let us assume \( j_n = d_n \). Now, we will show that the random polynomial \( f_n(z) = \sum_{|J| \leq n} a_J P^n_J(z) \) has no zeros in \( U \) for infinitely many values of \( n \). Denoting \( a' := (a_j)^{d_n-1} \), by Cauchy-Schwarz inequality, uniform convergence of the Bergman kernel on \( U \) and (3.7) we have
\[
\left| \sum_{j=1}^{d_n-1} a_J P^n_J(z) \right| \leq ||a'||S_n(z, z)^{1/2} \\
\leq \sqrt{d_n} |a_{d_n}|^{d_n-1} \exp(n(V_Q(z) + \frac{c}{2})) \\
\leq |a_{d_n}|^{d_n-1} \exp(n(M_Q + c)) \\
= \frac{\exp(n(M_Q + c)) |a_{d_n}|}{|a_{d_n}|} \\
\leq b_n |a_{d_n}|
\]
for infinitely many values of \( n \). Hence,
\[
\sup_{z \in U} \left| \sum_{j=1}^{d_n-1} a_J P^n_J(z) \right| < \inf_{z \in U} |a_{d_n} P^n_{d_n}(z)|.
\]

4. Generalizations and concluding remarks

4.1. Elliptic polynomials

Recall that a random elliptic polynomial in \( \mathbb{C}^m \) is of the form
\[
G_n(z) = \sum_{|J| \leq n} a_J \binom{n}{J}^{1/2} z^J
\]
where \( \binom{n}{J} = \frac{n!}{(n-|J|)!j_1! \ldots j_m!} \) and \( a_J \) are non-degenerate i.i.d. random variables. These polynomials induced by taking \( Q(z) = \frac{1}{2} \log(1 + \|z\|^2) \) i.e. the potential of the standard Fubini-Study Kähler metric on the complex projective space \( \mathbb{P}^m \). In this case, the scaled monomials \( \binom{n}{J}^{1/2} z^J \) form an ONB with respect to the inner product
\[
\langle F_n, G_n \rangle_n := \int_{\mathbb{C}^m} F_n(z) G_n(z) \frac{dV_m(z)}{(1 + \|z\|^2)^{n+m+1}}
\]
Moreover, since \( Q(z) \) is itself a Lelong class of psh function the weighted extremal function in this setting is given by
\[
V_Q(z) = Q(z) = \frac{1}{2} \log(1 + \|z\|^2).
\]
Specializing further, if the coefficients $a_J$ are standard i.i.d. complex Gaussians this ensemble is known as $SU(m + 1)$ polynomials and their zero distribution was studied extensively among others by [6, 31].

**Proof of Theorem 1.4.** Since the proof is very similar to that of Theorems 1.1 and 1.2 we explain the modifications in the present setting.

By [8, Proposition 4.4] it is enough to prove that almost surely in $\mathcal{H}$, for any subsequence $I$ of positive integers

$$F(z) := (\limsup_{n \in I} \frac{1}{n} \log |f_n(z)|)^* = V_Q(z) \text{ for all } z \in \mathbb{C}^m$$

In order to prove the upper bound $F(z) \leq V_Q(z)$, we use the same argument as in Theorem 1.1 together with the Bergman kernel asymptotics. Namely, letting $S_n(z, z) := \sum_{|J| \leq n} \binom{n}{J} z^{2J}$ a routine calculation gives

$$\frac{1}{2n} \log S_n(z, z) \to \frac{1}{2} \log(1 + \|z\|^2)$$

locally uniformly on $\mathbb{C}^m$ (see eg. [31]). On the other hand, for the lower bound (3.5), we need an analogue of Lemma 3.1. Note that $Q(z) = \frac{1}{2} \log(1 + \|z\|^2)$ is a multi-circular weight function whose infimum is 0 attained at $z = 0$. Then proceeding as in the proof Lemma 3.1, one can show that the sequence of measures $\mu_n := \frac{1}{a_n} e^{-2nQ(z)} dV_m$ verifies a LDP with rate function $J(z) = 2Q(z)$. This result and Kolmogorov-Rogozin inequality allow us to prove an analogue of (3.4) in the present setting. This together with the argument in the first part of the proof of Theorem 1.2 finish the proof of sufficiency of (1.6). In order to prove necessity, we use the Bergman kernel asymptotics and we apply the same argument as in the second part of the proof of Theorem 1.2. $\square$

### 4.2. Regular orthonormal polynomials

**Proof of Theorem 1.5.** We proceed as in the proof of Theorems 1.1 and 1.2. To this end we fix a subsequence $n_k$ of positive integers. It follows from [28, Theorem 3.1(ii) ] that

$$\lim_{n \to \infty} \frac{1}{n} \log |P_n^\mu(z)| = g_\Omega(z, \infty) \tag{4.1}$$

holds locally uniformly on $\mathbb{C} \setminus Co(S_\mu)$. Denoting the Bergman kernel by

$$S_n(z, z) := \sum_{j=0}^n |P_n^\mu(z)|^2$$

we infer that

$$\frac{1}{2n} \log S_n(z, z) \to g_\Omega(z, \infty)$$

locally uniformly on $\mathbb{C} \setminus Co(S_\mu)$. Thus, by Lemma 2.1 and Cauchy-Schwarz inequality almost surely in $\mathcal{H}$ we have

$$\limsup_{n_k \to \infty} \frac{1}{n_k} \log |f_{n_k}(z)| \leq g_\Omega(z, \infty)$$

for every $z \in \mathbb{C} \setminus Co(S_\mu)$.

In order to prove the lower bound, we use the local uniform convergence (4.1) which replaces Lemma 3.1. This in turn together with Kolmogorov-Rogozin inequality give

$$\text{Prob}_n[f_n : \frac{1}{n} \log |f_n(z)| < g_\Omega(z, \infty) - \epsilon] \leq \frac{C_{\epsilon}}{\sqrt{n}}$$

for every $z \in \mathbb{C}^* \setminus Co(S_\mu)$. Then applying the argument in Theorem 1.2 using the assumption $Co(S_\mu)$ has Lebesgue measure zero we obtain the assertion. $\square$
In order to get almost sure convergence in Theorems 1.1 and 1.2 for complex dimensions $m \leq 2$ we need a stronger form of Kolmogorov-Rogozin inequality. More precisely, for a fixed unit vector $u^{(n)} \in \mathbb{C}^n$, i.i.d. real or complex random variables $a_j$ for $j = 1, \ldots, n$ and $\epsilon \geq 0$ we consider the small ball probability

$$p_\epsilon(u^{(n)}) := P_n[\{|a^{(n)} : ||(a^{(n)}, u^{(n)})|| \leq \epsilon\}]$$

where $P_n$ is the product probability measure induced by the law of $a_j$'s and $(a^{(n)}, u^{(n)}) := \sum_{j=1}^n a_j u_j^{(n)}$. In order to obtain the lower bound in Theorem 1.1 we need for every $\epsilon > 0$

$$\sum_{n \geq 1} p_{\epsilon \rightarrow \infty}(u^{(dn)}) < \infty \quad (4.2)$$

for every unit vector $u^{(dn)} \in \mathbb{C}^{dn}$.

We remark that if the random variables $a_j$ are standard (real or complex) Gaussians then the probability $p_\epsilon(u^{(n)}) \sim \epsilon$. In particular, $p_\epsilon(u^{(n)})$ does not depend on the direction of the vector $u^{(n)}$. However, for most other distributions, $p_\epsilon(u^{(n)})$ does depend on the direction of $u^{(n)}$. For instance if $a_j$ are Bernoulli random variables (i.e. taking values $\pm 1$ with probability $\frac{1}{2}$) then $p_0((1, 1, 0, \ldots, 0)) = \frac{1}{2}$ on the other hand, $p_0((1, 1, 1, \ldots, 1)) \sim n^{-\frac{1}{2}}$. Determining small ball probabilities is a classical theme in probability theory. We refer the reader to the manuscripts [15, 25, 26, 33] and references therein for more details.

Another interesting problem is to find a necessary and sufficient condition for almost sure convergence of normalized zero currents when the space of polynomials $\mathbb{P}_n$ is endowed with $d_n$-fold product probability measure. A sufficient condition was obtained in [1]. Namely, let $a_j^n$ be iid random variables whose probability $P$ has a bounded density and logarithmically decaying tails i.e.

$$P\{a_j \in \mathbb{C} : \log |a_j| > R\} = O(R^{-\rho}) \text{ as } R \to \infty \text{ for some } \rho > m + 1. \quad (4.3)$$

We consider random polynomials of the form $f_n(z) = \sum_{j=1}^{d_n} a_j^n P_j^n(z)$. If (4.3) holds then almost surely normalized zero currents $\frac{1}{n} \{Z_{f_n}\}$ converges weakly to the extremal current $\frac{i}{\pi} \partial \bar{\partial} V_Q$.

4.3.1. Higher codimensions. In [1, Theorem 1.2] (see also [3]) it is proved that if the coefficients of random polynomials $f_n(z) = \sum_{j=1}^{d_n} a_j^n P_j^n(z)$ are i.i.d random variables whose distribution law verifies (4.3) then almost surely normalized empirical measure of zeros

$$\frac{1}{n^m} \sum_{\{z \in \mathbb{C}^m : f_1(z) = \ldots = f_m(z) = 0\}} \delta_z$$

of $m$ i.i.d. random polynomials $f_1^n, \ldots, f_m^n$ converges weakly to the weighted equilibrium measure $(\frac{i}{\pi} \partial \bar{\partial} V^*_{Z,Q})^m$. In the present paper, we have observed that for codimension one we no longer need $a_j$ to have a density with respect to Lebesgue measure. For instance, $a_j$ can be discrete such as Bernoulli random variables. It would be interesting to know if [1, Theorem 1.2] or a weaker form of it (e.g. convergence with high probability) also generalizes to the setting of discrete random variables.

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On global universality for zeros of random polynomials

References


Divisor function and bounds in domains with enough primes

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Abstract

In this note, first we show that there is no uniform divisor bound for the Bézout identity using Dirichlet’s theorem on arithmetic progressions. Then, we discuss for which rings the absolute value bound for the Bézout identity is not trivial and the answer depends on the number of small primes in the ring.

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1. Introduction and definitions

Let \( d(n) \) be the number of positive divisors of a given positive integer \( n \). More precisely,
\[
d(n) = |\{m \geq 1 : m \text{ divides } n\}| = \sum_{m \mid n} 1.
\]

For instance for a prime number \( p \), we have \( d(p) = 2 \). If \( p_1^{\alpha_1} \cdots p_k^{\alpha_k} \) is the prime factorization of \( n \), then
\[
d(n) = d(p_1^{\alpha_1} \cdots p_k^{\alpha_k}) = (\alpha_1 + 1) \cdots (\alpha_k + 1).
\]

The divisor function and its extensions have been studied extensively in terms of both analytic and arithmetic properties. It is well-known that (see [2, Chapter 3])
\[
\sum_{n \leq x} d(n) \sim x \log x + (2\gamma - 1)x,
\]
where \( \gamma \) is Euler’s constant. Using equation (1.1), one can prove that for a given \( \varepsilon > 0 \) there exist \( n_0 = n_0(\varepsilon) \geq 1 \) and \( C_\varepsilon > 0 \) such that if \( n \geq n_0 \) then \( d(n) \leq C_\varepsilon n^{\varepsilon} \). The remainder term in the asymptotic expansion (1.2) is an important problem in number theory and it is called the Dirichlet divisor problem. Details can be found in [3, 6].

Now let \( R \) be a domain. The arithmetic version of Hilbert’s Nullstellensatz states that if the polynomials \( f_1, \ldots, f_s \) belong to the ring \( R[X_1, \ldots, X_n] \) without a common zero in an algebraically closed field containing \( R \), then there exist \( a \) in \( R \setminus \{0\} \) and \( h_1, \ldots, h_s \) in \( R[X_1, \ldots, X_n] \) such that
\[
a = f_1 h_1 + \cdots + f_s h_s.
\]
One can see that Nullstellensatz implies its arithmetic version. Finding degree bounds for \( h_1, \ldots, h_s \) in equation (1.3) has received continuous attention, see [7–10]. By \( \text{deg } f \), we
mean the total degree of the polynomial $f$ in several variables. More generally given a field $K$, if $f_0, f_1, \ldots, f_s$ in $K[X_1, \ldots, X_n]$ all have degree less than $D$ and $f_0$ is in the ideal $\langle f_1, \ldots, f_s \rangle$, then

$$f_0 = \sum_{i=1}^{s} f_i h_i$$

for certain $h_i$ whose degree is bounded by a constant $c_1(n, D)$ depending only on $n$ and $D$, not to $K$, the number of generators $s$ or the polynomials $f_1, \ldots, f_s$. This result was first validated in a paper of G. Hermann [8], where her pattern was based on linear algebra and computational methods.

Throughout this note $R$ stands for an integral domain and $K$ for its field of fractions.

Recall that an absolute value on $R$ is a map $| \cdot | : R \to [0, \infty)$ such that

- $|x| = 0$ if and only if $x = 0$,
- $|xy| = |x||y|$,
- $|x + y| \leq |x| + |y|$.

For a polynomial $f \in R[X_1, \ldots, X_n]$, we put

$$|f| = \max_i \{|a_i|\}$$

where $a_i$ occurs as a coefficient in the monomial expression of $f$. If there is an absolute value on $R$ then it extends to $K$.

The following theorem follows immediately from [8].

**Theorem 1.1.** Let $R$ be a ring with an absolute value $| \cdot |$. For all $n \geq 1$, $D \geq 1$, $H \geq 1$ there are two constants $c_1(n, D)$ and $c_2(n, D, H)$ such that if $f_1, \ldots, f_s$ in $R[X_1, \ldots, X_n]$ have no common zero in the algebraic closure of $K$ with $\deg(f_i) \leq D$ and $|f_i| \leq H$, then there exist nonzero $a$ in $R$ and $h_1, \ldots, h_s$ in $R[X_1, \ldots, X_n]$ such that

(i) $a = f_1 h_1 + \cdots + f_s h_s$
(ii) $\deg(h_i) \leq c_1$
(iii) $|a|, |h_i| \leq c_2$

**Proof.** The degree bound $c_1$ reduces the Bézout identity $a = f_1 h_1 + \cdots + f_s h_s$ to a system of $K$-linear equations. Applying Gauss-Jordan method to this linear system one obtains an estimate for the absolute value of $a$ and the polynomials $h_i$. In other words, the existence of the constant $c_1$ yields the existence of $c_2$. \hfill $\square$

**Remark 1.2.** The constants $c_1$ and $c_2$ do not depend on $s$ because the vector space

$$V(n, D) = \{ f \in K[X_1, \ldots, X_n] : \deg(f) \leq D \}$$

is finite dimensional over $K$. In fact the dimension is $q = q(n, D) = \binom{n+D}{n}$. Given $1 = f_1 h_1 + \cdots + f_s h_s$ with $f_i \in V(n, D)$, we may always assume $s \leq q$.

Generally, the Gauss-Jordan method gives a very large value for $c_2$. In order to obtain more effective and sharp results for $c_2$, there is no technique for an arbitrary domain. In this note, our aim is to discuss for which domains the bound $c_2$ in Theorem 1.1 is not trivial and it is worth to try sharper estimates than the one given by the Gauss-Jordan method. We also show that there is no unique divisor bound for the Bézout identity (1.3) when $R = \mathbb{Z}$. This means that Theorem 1.1 fails when we replace the absolute value with the divisor function with a method based on Dirichlet’s theorem on arithmetical progressions.

Now we explain when the bound $c_2$ is trivial. If $R$ is a ring equipped with an absolute value which has a non-zero element $\varepsilon$ of absolute value $< 1$, then we can multiply both sides of the Bézout identity

$$a = f_1 h_1 + \cdots + f_s h_s,$$  \hfill (1.4)

by some power of $\varepsilon \in R$. Thus, we get that

$$a \varepsilon^k = f_1 \cdot (\varepsilon^k h_1) + \cdots + f_s \cdot (\varepsilon^k h_s).$$
Divisor function and bounds in domains with enough primes

Therefore, the bound $c_2$ in Theorem 1.1 can be taken 1 and Theorem 1.1 becomes trivial by [8]. In this note, we also answer when the bound $c_2$ exists non-trivially.

From now on, our ring $R$ is a unique factorization domain, UFD for short, endowed with an absolute value $|\cdot|$. By a small element in $R$, we mean an element of absolute value less than 1. An element is called big if it is of absolute value larger than 1. If we multiply both sides of equation (1.4) by an element $\varepsilon$ of absolute value $< 1$, we see that the elements $a, h_1, \ldots, h_s$ have a common divisor $\varepsilon$. However, if there is a unit $u$ with $|u| < 1$, then multiplying both sides of the equation with powers of $u$, the absolute value bound $c_2$ can be made small again as before, as units do not affect the prime factorization. So the interesting case is when there are no small units, which is equivalent to all the units have absolute value 1. So imposing the condition

$$\gcd(a, a_1, \ldots, a_m) = 1$$

where $a_1, \ldots, a_m$ are all elements that occur as some coefficient of some $h_i$ in equation (1.4) with all the units have absolute value 1 will make perfect sense for effective and sharper results as it prevents us from multiplying both sides of equation (1.4) by elements of small absolute value.

We give a criterion when we can choose $a$ such that $\gcd(a, a_1, \ldots, a_m) = 1$ where $a_1, \ldots, a_m$ are all elements that occur as some coefficient of some $h_i$ in equation (1.4) and also ensure a uniform bound for $a, a_1, \ldots, a_m$ depending on $n, D$ and the absolute values of $f_i$ as in Theorem 1.1. Interestingly, the answer depends on the number of primes in $R$ of absolute value less than 1, and this suggests us the following two definitions.

**Definition 1.3.** We say that $R$ is a UFD with the $p$-property if $R$ is a unique factorization domain endowed with an absolute value such that every unit has absolute value 1 and if there are primes $p$ and $q$ satisfying

$$|p| < 1 < |q|,$$

then there exists another prime $r$ non-associated to $p$ with $|r| < 1$.

**Definition 1.4.** We say that $R$ is a UFD with the 1-property if $R$ is a unique factorization domain equipped with an absolute value such that every unit has absolute value 1, and there is only one prime $p$ of absolute value less than 1, and there exists a prime $q$ of absolute value greater than 1.

Next, we extend the divisor function to $\mathbb{Z}$. We let $d(0) = 0$ and if $a < 0$ then we define $d(a) = d(-a)$. For a polynomial $f \in \mathbb{Z}[X_1, \ldots, X_n]$, we put

$$d(f) = \max_i \{d(a_i)\}$$

where $a_i$ occurs as a coefficient in the monomial expression of $f$. Now we can state our results.

**Theorem 1.5.** There is no uniform divisor bound for the Bézout identity. Precisely, there exist polynomials $f_{n1}, f_{n2}, g_{n1}, g_{n2}$ in $\mathbb{Z}[X]$ with

$$d(f_{n1}), d(f_{n2}), d(g_{n1}), d(g_{n2}) \leq 3$$

such that $f_{n1}, f_{n2}$ do not have a common zero in $\mathbb{C}$, the polynomials $g_{n1}, g_{n2}$ also do not have a common zero in $\mathbb{C}$, and for every $u_{n1}, u_{n2}, v_{n1}, v_{n2}$ in $\mathbb{Z}[X]$ and $a_{n1}, a_{n2} \in \mathbb{Z} \setminus \{0\}$ with

$$f_{n1}u_{n1} + f_{n2}u_{n2} = a_{n1}$$

and

$$g_{n1}v_{n1} + g_{n2}v_{n2} = a_{n2}$$

we have

$$\lim_{n \to \infty} d(u_{n1}) = \lim_{n \to \infty} d(a_{n2}) = \infty.$$
The condition (iv) in the following theorem makes the computation of the absolute value constant $c_2$ non-trivial.

**Theorem 1.6.** Let $R$ be a domain with an absolute value $|\cdot|$. For all $n \geq 1$, $D \geq 1$, $H \geq 1$ there are two constants $c_1(n, D)$ and $c_2(n, D, H)$ such that if $f_1, \ldots, f_s$ in $R[X_1, \ldots, X_n]$ have no common zero in the algebraic closure of $K$ with $\deg(f_i) \leq D$ and $|f_i| \leq H$, then there exist nonzero $a$ in $R$ and $h_1, \ldots, h_s$ in $R[X_1, \ldots, X_n]$ such that

(i) $a = f_1 h_1 + \cdots + f_s h_s$  
(ii) $\deg(h_i) \leq c_1 + D + 1$  
(iii) $|a|, |h_i| \leq c_2$  
(iv) If $R$ is a UFD with the $p$-property, then we can choose $a$ and $a_1, \ldots, a_m$ such that $\gcd(a, a_1, \ldots, a_m) = 1$ where $a_1, \ldots, a_m$ are all elements that occur as some coefficient of some $h_i$.  

Moreover, if $R$ is a UFD with the 1-property, then we cannot ensure the existence of $c_2$ and $\gcd(a, a_1, \ldots, a_m) = 1$ simultaneously.

Note also that if $|ab| < 1$ then $|a|$ can be very large and $|b|$ can be very small. So cancellation can make the absolute values larger if there are sufficiently small and big elements in the ring. Thus for the equation  

$$a = f_1 h_1 + \cdots + f_s h_s,$$

simply dividing by $\gcd(a, a_1, \ldots, a_m)$ may not work in order to obtain (iv) in the previous theorem.

2. Preliminaries

In this section, we prove several lemmas and we also give some examples of UFD with the $p$-property and 1-property. First, we recall Dirichlet’s theorem on arithmetic progressions.

**Fact 2.1.** (Dirichlet) For any two positive coprime integers $a$ and $q$, there are infinitely many primes of the form

$$a + nq,$$

where $n$ is a non-negative integer. In other words, there are infinitely many primes $p$ which are congruent to a modulo $q$.

The proof of Dirichlet’s theorem is based on the non-vanishing of Dirichlet $L$-functions at 1. For more on Dirichlet’s theorem, its generalizations and some special values of $L$-functions, we refer the reader to [1, 5]. The next lemma will play a key role in the proof of Theorem 1.5.

**Lemma 2.2.** There exist two sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in $\mathbb{N}$ such that $d(\alpha_n) \leq 2$ and $d(\beta_n) \leq 2$ but

$$\lim_{n \to \infty} d(\alpha_n + \beta_n) = \infty.$$

**Proof.** Let $n \geq 1$ be a positive integer and $p$ be a given prime number. Note that $p^n$ and $p^n - 1$ are coprime positive integers. By Dirichlet’s theorem on arithmetic progressions, the arithmetic progression

$$p^n - 1, p^n - 1 + p^n, \ldots, p^n - 1 + kp^n, \ldots$$

contains infinitely many prime numbers. Let $p_n$ be a prime number in this arithmetic progression. Let $\alpha_n = 1$ and $\beta_n = p_n$. Clearly $d(\alpha_n) = 1$ and $d(\beta_n) = 2$. However, the sum $\alpha_n + \beta_n$ is divisible by $p^n$ and so $d(\alpha_n + \beta_n) \geq n + 1$. Hence we are done. □

Now we give two examples of rings which have the $p$-property. At a first glance, the existence of a ring with the 1-property is not clear.
**Example 2.3.**

- \( \mathbb{Z} \) is a UFD with the p-property whose all primes have absolute value greater than 1.
- \( \mathbb{Z}_p \) (p-adic integers) is a UFD with the p-property whose only prime \( p \) has absolute value \( 1/p \).

Next, we recall the Gauss lemma. Let \( f = a_0 + a_1 X + \cdots + a_d X^d \) be in \( \mathbb{Q}[X] \). For any prime number \( p \) in \( \mathbb{N} \) we define
\[
|f|_p = \max_i \{|a_i|_p\},
\]
where \( |\cdot|_p \) is the p-adic absolute value on \( \mathbb{Q} \) with \( |p|_p = 1/p \).

**Lemma 2.4.** (Gauss lemma [4, 1.6.3]) Suppose that \( f \) and \( g \) are in \( \mathbb{Q}[X] \). For any prime number \( p \), we have \( |fg|_p = |f|_p |g|_p \).

Now we give an example of a ring with the p-property which has infinitely many small and big primes. We also give an example of a ring with the 1-property.

**Lemma 2.5.** There exist rings \( S_1 \) and \( S_2 \) such that \( S_1 \) is a UFD with the p-property which has infinitely many small and big primes and \( S_2 \) is a UFD with the 1-property.

**Proof.** Let \( \gamma \in (0, 1) \) be a transcendental number. Then the ring \( S_1 = \mathbb{Z}[\gamma] \) can be seen as a unique factorization domain since it is isomorphic to \( \mathbb{Z}[X] \) and its units are only 1 and -1. We put the usual absolute value on \( S_1 \) as it is a subset of \( \mathbb{R} \). Then \( S_1 \) has infinitely many primes \( p \) with \( |p| < 1 \) and infinitely many primes \( q \) with \( |q| > 1 \). In particular \( S_1 \) is a UFD with the p-property.

Now let \( p \) be a prime number in \( \mathbb{N} \). On \( \mathbb{Z}[X] \), we define
\[
|a_0 + a_1 X + \cdots + a_k X^k| := \max_p |a_i|_p = \left| a_0 + \frac{a_1}{p} X + \cdots + \frac{a_k}{p^k} X^k \right|_p.
\]
Then \( S_2 = \mathbb{Z}[X] \) becomes a UFD with the 1-property by the Gauss lemma with the absolute value above, and the only small prime is \( p \) in \( S_2 \) which is of absolute value \( 1/p \).

**Lemma 2.6.** Suppose \( R \) is a UFD with the p-property. If there are primes \( p \) and \( q \) with \( |p| < 1 < |q| \), then there are infinitely many non-associated primes with absolute value strictly less than 1 and infinitely many non-associated primes with absolute value strictly larger than 1.

**Proof.** By definition, we know there are at least two non-associated primes with absolute value less than 1. Let \( p_1, \ldots, p_k \) (for \( k \geq 2 \)) be non-associated primes with absolute value less than 1. Put \( A = p_1 \cdots p_k \). Now choose \( m \) large enough such that
\[
\left| \sum_{i=1}^k (A/p_i)^m \right| < 1.
\]
Since this element is not a unit as all the units have absolute value 1, it must be divisible by a prime whose absolute value is strictly less than 1. This yields us a new prime. For the second part, given \( q_1, \ldots, q_n \) primes of absolute value larger than 1, for large \( n \) the element \( q_1^n q_2 \cdots q_k + 1 \) provides a new prime that has absolute value greater than 1.

### 3. Proof of Theorem 1.5

Set \( f_n = \alpha_n + X + \beta_n^2 X^2 \) and \( f_n = X^3 \) where \( \alpha_n \) and \( \beta_n \) are as in Lemma 2.2. Recall that \( \alpha_n = 1 \) for all \( n \geq 1 \). Then \( d(f_n) \) and \( d(f_n) \) are bounded by 3 and they have no common zero in \( \mathbb{C} \). However, whenever we write
\[
a_{n1} = f_{n1} u_{n1} + f_{n2} u_{n2}
\]
where \(a_{n1}\) is non-zero, then \(u_{n1}\) must have degree bigger than 2 and the first three coefficients of \(u_{n1}\) are uniquely determined: if
\[
u_{n1}(X) = e_0 + e_1X + e_2X^2 + \cdots + e_kX^k
\]
then automatically we have \(e_0 = a_{n1}, e_1 = -a_{n1}\) and \(e_2 = a_{n1}(\alpha_n - \beta_n)(\alpha_n + \beta_n)\). Hence
\[
d(u_{n1}) \geq d(e_2) \geq d(\alpha_n + \beta_n) \geq n + 1.
\]
Moreover if we put \(g_{n1} = \alpha_n + X\) and \(g_{n2} = \beta_n - X\) then they have no common zero. Similarly, whenever we write
\[
a_{n2} = g_{n1}v_{n1} + g_{n2}v_{n2},
\]
then we see that \(d(a_{n2}) \geq d(\alpha_n + \beta_n) \geq n + 1\). Thus \(a_{n2}\) has many divisors although \(d(g_{n1})\) and \(d(g_{n2})\) are bounded by 2.

### 4. Proof of Theorem 1.6

We already know the existence of \(c_1\) and \(c_2\) by Theorem 1.1. Now we prove (iv) and we still keep (i), (ii) and (iii). Clearly we may assume that \(s \geq 2\) and \(a\) is not invertible. Assume \(R\) is a UFD with the p-property. We need to choose \(a\) and \(a_1, \ldots, a_m\) such that
\[
gcd(a, a_1, \ldots, a_m) = 1
\]
where \(a_1, \ldots, a_m\) are all elements that occur as some coefficient of some \(h_i\). If all the primes in \(R\) have absolute value larger than 1 or smaller than 1 (like \(R\) is \(\mathbb{Z}\) or \(\mathbb{Z}_p\) respectively), then we can divide both sides of the equation
\[
a = f_1h_1 + f_2h_2 + \cdots + f_sh_s
\]
by \(\gcd(a, a_1, \ldots, a_m)\) and get the result because if all the primes in \(R\) have absolute value greater than 1, then cancellation makes the absolute value smaller and if all the primes in \(R\) have absolute value less than 1 then we can take \(c_2\) to be 1. The remaining case is when there are primes of absolute value larger than 1 and primes of absolute value smaller than 1. Let \(d\) be the greatest common divisor of all the coefficients of \(f_1\) and \(f_2\). Then, the coefficients of \(f_1/d\) and \(f_2/d\) have no common divisor. On the other hand, since there are both small and large elements in the ring, the element \(d\) can be very small and so \(f_1/d\) and \(f_2/d\) may have very large absolute values. Let \(p_1, \ldots, p_k\) be the all prime divisors of \(a\). By Lemma 2.6, there are infinitely many primes with absolute value strictly less than 1. Now choose a prime \(p\) such that \(|p| < 1\) and \(p\) does not divide \(a\), in other words \(p\) is not in the finite set \(\{p_1, \ldots, p_k\}\). Choose a natural number \(k\) such that \(p^{k1}/d\) and \(p^{k2}/d\) have absolute values less than \(c_2\). Put \(v = c_1(n, D) + 1\). Then, we have
\[
0 = f_1 \cdot \frac{p^kX^i_1f_2}{d} - f_2 \cdot \frac{p^kX^i_1f_1}{d}.
\]
Therefore, by adding the previous equation to \(a = f_1h_1 + f_2h_2 + \cdots + f_s h_s\), we obtain that
\[
a = f_1\left(h_1 + \frac{p^kX^i_1f_2}{d}\right) + f_2\left(h_2 - \frac{p^kX^i_1f_1}{d}\right) + \cdots + f_s h_s
\]
\[
= f_1 t_1 + f_2 t_2 + \cdots + f_s t_s
\]
where \(\deg t_i \leq c_1 + D + 1\) and \(|t_i| \leq c_2\). Observe that
\[
gcd(a, a_1, \ldots, a_m) = 1
\]
where \(a_1, \ldots, a_m\) are all elements that occur as some coefficient of some \(t_i\).

Finally, we prove the remaining part of the Theorem. Let \(p\) be the unique small prime in \(R\) of absolute value less than 1. The reason behind the last part of the Theorem is the fact that an element has small absolute value if and only if its \(p\)-adic valuation is very large. Let \(B\) be an element in \(R\) of absolute value very big and coprime to \(p\). Choose \(m\) minimal such that \(|p^m B| \leq 1\). Similarly choose \(k\) minimal such that \(|p^k B| \leq c_2\). Note
that as $B$ is very large then so are $m$ and $k$. Let $n = D = H = 1$. Set $f_1 = p^{2m+1} + p^{2m}X$ and $f_2 = p^mB - p^mBX$. Clearly $f_1$ and $f_2$ have no common zero since

$$p^{2m}B(p + 1) = Bf_1 + p^m f_2$$

and $p$ is not -1. Whenever we write $a = f_1h_1 + f_2h_2$, we get that $p^m$ divides $h_2$ and $B$ divides $h_1$. Also we have that $p^{2m}B$ divides $a$. Now suppose $|h_i| \leq c_2$ for $i = 1, 2$. Since $B$ divides $h_1$, we see that $p^k$ divides $h_1$ since $p$ is the unique small prime in $R$. Thus $p^k$ divides $a$, $h_1$ and $h_2$. Furthermore, we may assume that the only prime divisor of $a$, $h_1$ and $h_2$ is $p$, because if there is $q$ dividing all of them which is coprime to $p$, then there is $\ell \geq k$ such that $p^\ell$ divides $h_1$ in order to make the absolute value of $h_1$ less than $c_2$. Similar observation shows that $p^\ell$ also divides $h_2$ and $a$. Therefore, in order to satisfy the coprimality in the theorem, we need to divide $a$, $h_1$ and $h_2$ by $p^k$. So the absolute value of $h_1/p^k$ becomes larger than $B$.

We end our note by posing the following question:

**Question 4.1.** What is the condition on $f_1, ..., f_s$ to obtain a uniform divisor bound for $a$ and $h_1, ..., h_s$ in (1.3)? For which rings that are a UFD with the $p$-property which have infinitely many small and big primes, we can obtain sharper estimates for $c_2$ which is better than the constant given by the Gauss-Jordan method?

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**References**

Projective coordinate spaces over modules

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Abstract

In this paper, we investigate some properties of the (right) modules constructed over the local ring and also construct a projective coordinate space over the (right) module. Finally, in a 3-dimensional projective coordinate space, the incidence matrix for a line that combines the certain two points and also all points of a line given with the incidence matrix are found by the help of Maple programme.

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1. Introduction

Jukl, in [5], introduced the real plural algebra of order m and so investigated the linear forms on a free finite dimensional module M, especially their kernel. Jukl continued to study on free finite dimensional modules in [6]. In [3], Erdoğan et. al. investigated some properties of the (left) modules constructed over the real plural algebra and later, in [2], Çiftçi and Erdoğan obtained an n-dimensional projective coordinate space over (n + 1)-dimensional (left) module constructed by the help of this real plural algebra. For more detailed information on modules, see [8]. For the algebraic and linear algebraic notions that will be used throughout this paper, we refer to [4] and [9].

In this paper we will study by the algebra \( A := F\eta_0 + F\eta_1 + F\eta_2 + ... + F\eta_{m-1} \) with a basis \( \{1, \eta_1, \eta_2, \eta_3, ..., \eta_{m-1}\} \) such that \( \eta_i\eta_j = 0 \) for \( \eta_i \notin F \) (where \( F \) is a field). We immediately state that this algebra is not isomorphic to the real plural algebra of order \( m \). For this reason, by taking this algebra instead of the real plural algebra of order \( m \), we will reconsider almost all of the results that are obtained in [2, 3]. So, we will be able to investigate some properties of the (right) modules constructed over the algebra and also to construct an \( (m - 1) \)-dimensional projective coordinate space over the \( m \)-dimensional (right) module.

The remainder of the paper is organized as follows. In Section 2, there are some basic definitions and results from the literature. In Section 3, we investigate some properties of the (right) modules constructed over \( A \). In Section 4, we construct an projective coordinate space over the (right) module. Finally, in a 3-dimensional projective coordinate space, the incidence matrix for a line that combines the certain two points and also all points of a line given with the incidence matrix are found by the help of Maple programme.

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2. Preliminaries

In this section, first of all, we will start by recalling some definitions and results from [5].

**Definition 2.1** ([5, Def. 1.1]). The real plural algebra of order \( n \) is every linear algebra \( \mathcal{A} \) on \( \mathbb{R} \) having as a vector space over \( \mathbb{R} \) a basis \( \{1, \eta, \eta^2, \ldots, \eta^{n-1}\} \) where \( \eta^n = 0 \) for \( \eta \notin \mathbb{R} \).

By Definition 2.1, we see that an element \( x \) of \( \mathcal{A} \) is of the form \( x = a_0 + a_1 \eta + a_2 \eta^2 + \cdots + a_{n-1} \eta^{n-1} \) where \( a_i \in \mathbb{R} \) for \( 0 \leq i \leq n - 1 \).

A ring with identity element is called local if the set of its non-units form an ideal.

Now we can state the following two results without proof.

**Proposition 2.2** ([5, Prop. 1.3]). An element \( x = a_0 + a_1 \eta + a_2 \eta^2 + \cdots + a_{n-1} \eta^{n-1} \in \mathcal{A} \) is a unit if and only if \( a_0 \neq 0 \).

**Proposition 2.3** ([5, Prop. 1.5]). \( \mathcal{A} \) is a local ring with the maximal ideal \( \eta \mathcal{A} \). The ideals \( \eta^j \mathcal{A}, 1 \leq j \leq n \), are all ideals in \( \mathcal{A} \).

In [5, Prop. 1.7], it is stated that \( \mathcal{A} \) is isomorphic to the linear algebra of matrix \( M_{nn}(\mathbb{R}) \) of the form

\[
\begin{pmatrix}
 b_0 & b_1 & b_2 & \cdots & b_{n-2} & b_{n-1} \\
 0 & b_0 & b_1 & \ddots & \vdots & b_{n-2} \\
 \vdots & 0 & b_0 & \ddots & b_1 & b_2 \\
 \vdots & \vdots & \ddots & b_0 & b_1 \\
 0 & 0 & 0 & \cdots & 0 & b_0
\end{pmatrix},
\]

where \( b_i \in \mathbb{R} \) for \( 0 \leq i \leq n - 1 \) (for the detailed proof of this fact, see [3]).

A module that is constructed over a local ring \( \mathcal{A} \) is called an \( \mathcal{A} \)-module. So, we can give the following definition.

**Definition 2.4.** Let \( \mathcal{A} \) be a local ring. Let \( M \) be a finitely generated \( \mathcal{A} \)-module. Then \( M \) is an \( \mathcal{A} \)-space of finite dimension if there exists \( E_1, E_2, \ldots, E_n \) in \( M \) with

i) \( M = E_1 \mathcal{A} \oplus E_2 \mathcal{A} \oplus \ldots \oplus E_n \mathcal{A} \)

ii) the map \( \mathcal{A} \rightarrow E_i \mathcal{A} \) defined by \( x \rightarrow E_ix \) is an isomorphism for \( 1 \leq i \leq n \).

Let \( F \) be a field. Consider \( \mathcal{A} := F \eta_0 + F \eta_1 + F \eta_2 + \ldots + F \eta_{n-1} \) with componentwise addition and multiplication as follows:

\[
a \cdot b = (a_0 + a_1 \eta_1 + a_2 \eta_2 + \cdots + a_{n-1} \eta_{n-1}) \cdot (b_0 + b_1 \eta_1 + b_2 \eta_2 + \cdots + b_{n-1} \eta_{n-1})
\]
\[
= a_0 b_0 + (a_0 b_1 + a_1 b_0) \eta_1 + (a_0 b_2 + a_2 b_0) \eta_2 + \cdots + (a_0 b_{n-1} + a_{n-1} b_0) \eta_{n-1},
\]

where \( a_i \eta_j = 0 \) for \( 1 \leq i, j \leq n - 1 \) and the set \( \{1, \eta_1, \eta_2, \ldots, \eta_{n-1}\} \) is a basis of \( \mathcal{A} \). Then, \( \mathcal{A} \) is a unital, commutative and associative local ring with the maximal ideal \( I = \mathcal{A} = \{a_1 \eta_1 + a_2 \eta_2 + \cdots + a_{n-1} \eta_{n-1} \mid a_i \in F, 1 \leq i \leq n - 1\} \). So, we can reach the result that an element \( \alpha = a_0 + a_1 \eta_1 + a_2 \eta_2 + \cdots + a_{n-1} \eta_{n-1} \in \mathcal{A} \) is a unit if and only if \( a_0 \neq 0 \).

In that case, note that \( \alpha^{-1} = a_0^{-1} - a_1 \eta_1 a_0^{-1} - a_2 \eta_2 a_0^{-1} - \cdots - a_{n-1} \eta_{n-1} a_0^{-1} \).

Moreover, the local ring we will study on is considered as the vector space \( F(n) := F \times F^{n-1} = \{(a_0, v) \mid v = (a_1, a_2, \ldots, a_{n-1}) \in F^{n-1}\} \) with componentwise addition and multiplication as follows:

\[
a \cdot b = (a_0, v) \cdot (b_0, w)
\]
\[
= (a_0 b_0, a_0 w + v b_0)
\]
\[
= (a_0 b_0, (a_0 b_1 + a_1 b_0, a_0 b_2 + a_2 b_0, \ldots, a_0 b_{n-1} + a_{n-1} b_0)),
\]
where \( v = (a_1, a_2, \ldots, a_{n-1}) \), \( w = (b_1, b_2, \ldots, b_{n-1}) \in F^{n-1} \). In this case, \( F(n) \) is local with \( I = \{0\} \times F^{n-1} \) as ideal of non-units. For more detailed information on \( F(n) \), see [1].

Hence, it is clear that the local ring \( A \) is not isomorphic to the real plural algebra of order \( n \). But, it is isomorphic to \( F(n) \). Throughout this paper we restrict ourselves to the local ring \( A \).

3. \( A \)-Modules

In this section, we investigate some properties of the (right) modules constructed over \( A \). First, we give the following result, the analogue of Theorem 6 in [3].

**Proposition 3.1.** None of the units of \( A \) are zero divisors, namely for every \( \alpha, \beta \in A \):

\[
\alpha = a_0 + a_1 \eta_1 + a_2 \eta_2 + \cdots + a_{n-1} \eta_{n-1}, \quad a_0 \neq 0 \quad \text{and} \quad \beta = b_0 + b_1 \eta_1 + b_2 \eta_2 + \cdots + b_{n-1} \eta_{n-1} \quad \text{if} \quad \alpha \cdot \beta = 0 \\
\text{or} \quad \beta \cdot \alpha = 0, \quad \text{then} \quad \beta = 0.
\]

Also for \( 1 \leq k \leq n-1 \) and \( \alpha = a_k \eta_k + a_{k+1} \eta_{k+1} + \cdots + a_{n-1} \eta_{n-1} \), \( a_k \neq 0 \) if \( \alpha \cdot \beta = 0 \) or \( \beta \cdot \alpha = 0 \) then \( \beta = b_1 \eta_1 + b_2 \eta_2 + \cdots + b_{n-1} \eta_{n-1} \).

**Proof.** If \( \alpha \) is a unit, then there is an inverse element \( \alpha^{-1} \) and since \( A \) is associative, \( \alpha \cdot \beta = 0 \Rightarrow \alpha^{-1}(\alpha \cdot \beta) = \alpha^{-1} \cdot 0 \Rightarrow \beta = 0 \). For \( \beta \cdot \alpha = 0 \), it is easily seen that \( \beta = 0 \) by similar calculations.

Now let \( \alpha = a_k \eta_k + a_{k+1} \eta_{k+1} + \cdots + a_{n-1} \eta_{n-1}, \quad a_k \neq 0 \) for \( 1 \leq k \leq n-1 \), we have

\[
\alpha \cdot \beta = (a_k \eta_k + a_{k+1} \eta_{k+1} + \cdots + a_{n-1} \eta_{n-1})(b_0 + b_1 \eta_1 + b_2 \eta_2 + \cdots + b_{n-1} \eta_{n-1}) = 0.
\]

Thus we have \( \beta = b_1 \eta_1 + b_2 \eta_2 + \cdots + b_{n-1} \eta_{n-1} \) if \( a_k \neq 0 \), and since \( a_k \neq 0 \), we find \( b_0 = 0 \). Thus we have \( \beta = b_1 \eta_1 + b_2 \eta_2 + \cdots + b_{n-1} \eta_{n-1}, \quad b_i \in F, \quad 1 \leq i \leq n-1 \).

Now, we can state the following result without proof, the analogue of Proposition 7 in [3].

**Proposition 3.2.** Let \( K = M_{nn}(F) \) be the (linear) algebra of matrix of the form

\[
\begin{pmatrix}
a_0 & a_1 & a_2 & a_3 & \cdots & a_{n-1} \\
0 & a_0 & 0 & \cdots & \cdots & 0 \\
0 & \cdots & a_0 & 0 & \cdots & 0 \\
\vdots & \cdots & \cdots & \cdots & \cdots & \vdots \\
\vdots & \cdots & \cdots & \cdots & \cdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & a_0
\end{pmatrix}_{n \times n}
\]

where \( a_i \in F \) for \( 0 \leq i \leq n-1 \). Then the map \( f : A \to K = M_{nn}(F) \) which is given as

\[
f(\alpha) = (a_{ij}) = \begin{cases} 
a_{ii} = a_0, & 1 \leq i \leq n \\
a_{ij} = a_{j-1}, & 2 \leq j \leq n \\
a_{ij} = 0, & \text{otherwise}
\end{cases}
\]

for every \( \alpha = a_0 + a_1 \eta_1 + a_2 \eta_2 + \cdots + a_{n-1} \eta_{n-1} \in A \) is an isomorphism.

Now we would like to find a basis of \( K = M_{nn}(F) \). Let us take any element of \( K \) such that

\[
a = \begin{pmatrix}
a_0 & a_1 & a_2 & \cdots & a_{n-1} \\
0 & a_0 & 0 & \cdots & \cdots & 0 \\
0 & \cdots & a_0 & 0 & \cdots & 0 \\
\vdots & \cdots & \cdots & \cdots & \cdots & \vdots \\
\vdots & \cdots & \cdots & \cdots & \cdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & a_0
\end{pmatrix} \in K.
\]

Then, the element can be written in the following form:
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Thus we have \( a = a_0 I_n + a_1 \eta_1 + a_2 \eta_2 + \ldots + a_{n-1} \eta_{n-1} \). Moreover, the set \( \{ \eta_0 = I_n, \eta_1, \eta_2, \ldots, \eta_{n-1} \} \) is a basis of \( K \). We can express any element of this set in general as follows: for \( 1 \leq k \leq n-1 \), \( \eta_k = (a_{ij})_{n \times n} \), where

\[
(a_{ij}) = \begin{cases}
  a_{ij} = 1 & j = k + 1, \\
  a_{ij} = 0 & \text{otherwise}.
\end{cases}
\]

Now, we will construct a (right) module \( M \) over the algebra \( A \), by the following proposition, although a (left) module is obtained in Proposition 8 of [3]. Thanks to this, we will obtain a basis of \( M \).

**Proposition 3.3.** \( M = F_n^m \) is a right module over the linear algebra of matrix \( K = M_{nn}(F) \). Then the following set as a basis of \( K \)-(right)module \( M \).

\[
E_1 = \begin{pmatrix}
  1 & 0 & 0 & \ldots & 0 \\
  0 & 0 & 0 & \ldots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & 0 & \ldots & 0
\end{pmatrix} ,
E_2 = \begin{pmatrix}
  0 & 0 & 0 & \ldots & 0 \\
  1 & 0 & 0 & \ldots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & 0 & \ldots & 0
\end{pmatrix}, \ldots ,
E_m = \begin{pmatrix}
  0 & 0 & 0 & \ldots & 0 \\
  0 & 0 & 0 & \ldots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  1 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

**Proof.** Linear independence of this set is obvious. Moreover for every \( X \in M \), \( X \) can be written as follows:

\[
X = \begin{pmatrix}
  x_{11} & x_{12} & x_{13} & \ldots & x_{1n} \\
  x_{21} & x_{22} & x_{23} & \ldots & x_{2n} \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  x_{m1} & x_{m2} & x_{m3} & \ldots & x_{mn}
\end{pmatrix}_{m \times n}
\]

\[
= \begin{pmatrix}
  1 & 0 & 0 & \ldots & 0 \\
  0 & 0 & 0 & \ldots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & 0 & \ldots & 0
\end{pmatrix}_{m \times n} \begin{pmatrix}
  x_{11} & x_{12} & x_{13} & \ldots & x_{1n} \\
  0 & x_{11} & 0 & \ldots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  0 & \ldots & \ldots & 0 & x_{11}
\end{pmatrix}_{n \times n}
\]
Let $A$ holds, then a submodule of $M$ module with unity over $R$ is called a free finitely dimensional module. Proposition 3.6. Proposition 3.5. Definition 3.4. Thus $[E_1, E_2, \cdots, E_m] = M$. Consequently, the set $\{E_1, E_2, \cdots, E_m\}$ is a basis of $K$-module $M$. Now, from [7], we give a definiton, will be used in the next section.

**Definition 3.4.** Let $R$ be a local ring, $R_0$ be the maximal ideal of $R$ and $M$ be a free module with unity over $R$. Let $S$ be a non-empty subset of the module $M$. Let $M_0$ be a submodule of $M$ constructed over $R_0$. For $x_1, x_2, \cdots, x_k \in S$ and $\alpha_1, \alpha_2, \cdots, \alpha_k \in R$, if

$$\sum_{i=1}^{k} \alpha_i x_i \in M_0 \Rightarrow \alpha_i \in R_0 \text{ for every } i$$

holds, then $S$ is called $R$-independent. Otherwise, $S$ is called an $R$-dependent subset.

Finally, we would like to complete this section by giving two results, without proof, on $A$-spaces. They are the analogues of Theorem 9 and Proposition 10 in [3], respectively.

**Proposition 3.5.** Let $M = A^n$. Then, for $u_1, u_2, \ldots, u_k \in A \setminus I$ and $x_{ij} \in I$, there are linearly independent vectors such that $\alpha_1 = (u_1, x_{21}, x_{31}, \ldots, x_{n1})$, $\alpha_2 = (x_{12}, u_2, x_{32}, \ldots, x_{n2})$, $\alpha_3 = (x_{13}, x_{23}, u_3, \ldots, x_{n3})$, $\ldots$, $\alpha_k = (x_{1k}, x_{2k}, x_{3k}, \ldots, u_k)$. For $k = n$, the set $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is a basis for $M$.

**Proposition 3.6.** An $A$-module $M$ over a local ring $A$ is an $A$-space if and only if it is a free finitely dimensional module.

4. **Construction of a projective coordinate space**

In this section, an $(m - 1)$-dimensional projective coordinate space over the right module obtained in the previous section will be constructed with the help of equivalence classes, by following the similar method given in [2]. So, the points and lines of this space are determined and the points are classified.

We know from the previous section that, the set $M = F_m^n$ is an $m$-dimensional right-module over the local ring $K = M_{nn}(F)$ and the set $\{E_1, E_2, \ldots, E_m\}$ is a basis of $M$. Each element of a $K$-module $M$ can be expressed uniquely as a linear combination of $E_1, E_2, \ldots, E_m$. Furthermore a maximal ideal of $K$ is denoted by

$$I = \left\{ \begin{pmatrix} 0 & a_1 & \cdots & \cdots & a_{n-1} \\ 0 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} : a_i \in F, 1 \leq i \leq n - 1 \right\}.$$
Now let us define the set
\[ M_0 = \left\{ \sum_{i=1}^{m} E_iA_i | A_i \in I, \ 1 \leq i \leq m \right\}. \]

Then, we get
\[ M_0 = \left\{ \begin{pmatrix} 0 & x_{12} & \cdots & x_{1n} \\ 0 & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & x_{m2} & \cdots & x_{mn} \end{pmatrix} \mid x_{ij} \in F \right\}. \]

Now, we consider equivalence relation on the elements of
\[ M^* = M \setminus M_0 = \left\{ \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{pmatrix} \mid 1 \leq i \leq m, \ \exists x_{i1} \neq 0 \right\}, \]

whose equivalence classes are the one-dimensional right submodules of \( M \) with the set \( M_0 \) deleted. Thus, if \( X, Y \in M^* \), then \( X \) is equivalent to \( Y \) if \( Y = X\lambda \) for \( \lambda \in K^* = K \setminus I \). The set of equivalence classes is denoted by \( P(M) \). Then \( P(M) \) is called an \((m-1)\)-dimensional projective coordinate space and the elements of \( P(M) \) are called points; the equivalence class of vector \( X \) is the point \( \overline{X} \). Consequently, \( X \) is called a coordinate vector for \( \overline{X} \) or that \( X \) is a vector representing of \( \overline{X} \). In this case, \( X\lambda \) with \( \lambda \in K^* \) also represents \( \overline{X} \); that is, by \( X\lambda = \overline{X} \). Thus, \( \overline{X} \) can be expressed as follows:

\[
\overline{X} = \begin{pmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{1n} \\ x_{21} & x_{22} & x_{23} & \cdots & x_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & x_{m3} & \cdots & x_{mn} \end{pmatrix}
\begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ 0 & a_0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & \cdots & 0 & a_0 \end{pmatrix}_{n \times n}
\]

\[
= \begin{pmatrix} x_{11}a_0 & x_{11}a_1 + x_{12}a_0 & \cdots & x_{11}a_{n-1} + x_{1n}a_0 \\ x_{21}a_0 & x_{21}a_1 + x_{22}a_0 & \cdots & x_{21}a_{n-1} + x_{2n}a_0 \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1}a_0 & x_{m1}a_1 + x_{m2}a_0 & \cdots & x_{m1}a_{n-1} + x_{mn}a_0 \end{pmatrix}_{m \times n},
\]

where \( a_0 \neq 0 \ \land \ 1 \leq i \leq m, \ \exists x_{i1} \neq 0. \)

Let \( \overline{X}, \overline{Y}, \cdots \) be \( p+1 \) points such that any two of them are \( K \)-independent. Then the set \( \Pi_p = \text{Sp} \{ \overline{X}, \overline{Y}, \cdots \} \setminus M_0 \) is called a subspace of dimension \( p \) or \( p \)-space.

In \( P(M) \), a point is a subspace of dimension \( 0 \) and a line is a subspace of dimension \( 1 \).

For \( X \in M^* \), the set \( \overline{X} = \{ X\lambda | \lambda \in K^* \} \) is a 0-dimensional subspace of \( P(M) \). So, \( \overline{X} \) is a point of \( P(M) \).

Now, we investigate the condition of being \( K \)-independent for two different points \( \overline{X} \) and \( \overline{Y} \) of \( P(M) \).

Firstly, let us denote the coordinate vectors for the points \( \overline{X} \) and \( \overline{Y} \) by \( X \) and \( Y \), respectively. We form a linear combination as
Let us denote the coefficient matrix of \((4.1)\) by

\[
A = \begin{pmatrix}
    x_{11} & y_{11} \\
    x_{21} & y_{21} \\
    \vdots & \vdots \\
    x_{m1} & y_{m1}
\end{pmatrix}.
\]

If \(\text{rank} A = 2\), then we get \(a_0 = b_0 = 0\). So this shows that

\[
\begin{pmatrix}
    a_0 & a_1 & \cdots & a_{n-1} \\
    0 & a_0 & 0 & \cdots & 0 \\
    0 & \ddots & \ddots & \ddots & \ddots \\
    \vdots & \ddots & \ddots & \ddots & \ddots \\
    0 & \cdots & 0 & 0 & \ddots \\
\end{pmatrix}
+ \begin{pmatrix}
    b_0 & b_1 & \cdots & b_{n-1} \\
    0 & b_0 & 0 & \cdots & 0 \\
    0 & \ddots & \ddots & \ddots & \ddots \\
    \vdots & \ddots & \ddots & \ddots & \ddots \\
    0 & \cdots & 0 & 0 & b_0 \\
\end{pmatrix}
= \begin{pmatrix}
    (x_{11}a_0 + y_{11}b_0) & (x_{11}a_1 + y_{11}b_0) & \cdots & (x_{11}a_{n-1} + y_{11}b_0) \\
    (x_{21}a_0 + y_{21}b_0) & (x_{21}a_1 + y_{21}b_0) & \cdots & (x_{21}a_{n-1} + y_{21}b_0) \\
    \vdots & \vdots & \ddots & \vdots \\
    (x_{m1}a_0 + y_{m1}b_0) & (x_{m1}a_1 + y_{m1}b_0) & \cdots & (x_{m1}a_{n-1} + y_{m1}b_0)
\end{pmatrix}.
\]

\[
x_{11}a_0 + y_{11}b_0 = 0, \\
x_{21}a_0 + y_{21}b_0 = 0, \\
\vdots \\
x_{m1}a_0 + y_{m1}b_0 = 0.
\]

Let us denote the coefficient matrix of \((4.1)\) by

\[
A = \begin{pmatrix}
    x_{11} & y_{11} \\
    x_{21} & y_{21} \\
    \vdots & \vdots \\
    x_{m1} & y_{m1}
\end{pmatrix}.
\]

If \(\text{rank} A = 2\), then we get \(a_0 = b_0 = 0\). So this shows that

\[
\begin{pmatrix}
    0 & a_1 & \cdots & a_{n-1} \\
    0 & 0 & \cdots & 0 \\
    \vdots & \ddots & \ddots & \ddots \\
    0 & \cdots & 0 & 0 \\
\end{pmatrix}
+ \begin{pmatrix}
    0 & b_1 & \cdots & b_{n-1} \\
    0 & 0 & \cdots & 0 \\
    \vdots & \ddots & \ddots & \ddots \\
    0 & \cdots & 0 & 0 \\
\end{pmatrix}
\in I.
\]

In that case, the coordinate vectors \(X\) and \(Y\) for the points \(\overline{X}\) and \(\overline{Y}\), respectively, are \(K\)-independent if and only if the rank of the coefficient matrix is equal to 2. That is, first columns of the coordinate vectors \(X\) and \(Y\) are linearly independent vectors.

Let the set \(Sp\{\overline{X}, \overline{Y}\} = \{X\lambda + Y\gamma | \lambda, \gamma \in K^*\}\) be a 1-dimensional subspace of \(P(M)\) such that \(\overline{X}\) and \(\overline{Y}\) are \(K\)-independent elements. Then \(Sp\{\overline{X}, \overline{Y}\}\) is a line of \(P(M)\). It is
denoted by

\[
\begin{pmatrix}
    x_{11} & x_{12} & x_{13} & \cdots & x_{1n} \\
    x_{21} & x_{22} & x_{23} & \cdots & x_{2n} \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    x_{m1} & x_{m2} & x_{m3} & \cdots & x_{mn}
\end{pmatrix}
\begin{pmatrix}
    a_0 & a_1 & a_2 & \cdots & a_{n-1} \\
    0 & a_0 & 0 & \cdots & 0 \\
    0 & \ddots & a_0 & \cdots & 0 \\
    \vdots & \ddots & \ddots & \ddots & \vdots \\
    \vdots & \ddots & \ddots & \ddots & 0 \\
    0 & 0 & \cdots & 0 & a_0
\end{pmatrix}
\]

\[
\begin{pmatrix}
    y_{11} & y_{12} & y_{13} & \cdots & y_{1n} \\
    y_{21} & y_{22} & y_{23} & \cdots & y_{2n} \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    y_{m1} & y_{m2} & y_{m3} & \cdots & y_{mn}
\end{pmatrix}
\begin{pmatrix}
    b_0 & b_1 & b_2 & \cdots & b_{n-1} \\
    0 & b_0 & 0 & \cdots & 0 \\
    0 & \ddots & b_0 & \cdots & 0 \\
    \vdots & \ddots & \ddots & \ddots & \vdots \\
    \vdots & \ddots & \ddots & \ddots & 0 \\
    0 & 0 & \cdots & 0 & b_0
\end{pmatrix}
\]

where \(a_0 \neq 0 \land 1 \leq i \leq m, \exists x_{i1} \neq 0\) or \(b_0 \neq 0 \land 1 \leq i \leq m, \exists y_{i1} \neq 0\).

We know that for every coordinate vector \(X \in M^*\) of the point \(\overline{X} \in P(M)\), \(X\) can be written uniquely as a linear combination of the vectors \(E_1, E_2, \cdots, E_m\). So the matrix \(X\) is expressed as

\[X = (X_1, X_2, \cdots, X_m) \in \mathbb{K}^m,\]

where

\[
X_1 = \begin{pmatrix}
    x_{11} & x_{12} & x_{13} & \cdots & x_{1n} \\
    0 & x_{11} & 0 & \cdots & 0 \\
    \vdots & 0 & \ddots & \ddots & \vdots \\
    \vdots & \vdots & \ddots & \ddots & 0 \\
    0 & 0 & \cdots & 0 & x_{11}
\end{pmatrix},
\]

\[
X_2 = \begin{pmatrix}
    x_{21} & x_{22} & x_{23} & \cdots & x_{2n} \\
    0 & x_{21} & 0 & \cdots & 0 \\
    \vdots & 0 & \ddots & \ddots & \vdots \\
    \vdots & \vdots & \ddots & \ddots & 0 \\
    0 & 0 & \cdots & 0 & x_{21}
\end{pmatrix},
\]

\[
\cdots, X_m = \begin{pmatrix}
    x_{m1} & x_{m2} & x_{m3} & \cdots & x_{mn} \\
    0 & x_{m1} & 0 & \cdots & 0 \\
    \vdots & 0 & \ddots & \ddots & \vdots \\
    \vdots & \vdots & \ddots & \ddots & 0 \\
    0 & 0 & \cdots & 0 & x_{m1}
\end{pmatrix}.
\]

There are two cases:

**Case 1:** For the first component of the coordinate vector \(X\) of the point \(\overline{X}\), if \(x_{11} \neq 0\), then \(X_1 \notin I\) and

\[
X_1 = \begin{pmatrix}
    x_{11} & x_{12} & x_{13} & \cdots & x_{1n} \\
    0 & x_{11} & 0 & \cdots & 0 \\
    \vdots & 0 & \ddots & \ddots & \vdots \\
    \vdots & \vdots & \ddots & \ddots & 0 \\
    0 & 0 & \cdots & 0 & x_{11}
\end{pmatrix}
\]
is a unit element so there is an inverse of \(X_1\). If we multiply both sides of the equation with the inverse matrix \(X_1^{-1}\), we get

\[
X = (I_n, X_2, \cdots, X_m) = \begin{pmatrix}
1 & 0 & \cdots & 0 \\
x_{21} & x_{22} & \cdots & x_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
x_{m1} & x_{m2} & \cdots & x_{mn}
\end{pmatrix}.
\]

Thus, this type of points are called proper points.

**Case 2:** For the first component of the coordinate vector \(X\) of the point \(X\), if \(x_{11} = 0\), then \(X_1 \in I\). So, the inverse of the matrix \(X_1\) does not exist. Thus we call the points of \(P(M)\) whose coordinate vectors are in the form

\[
\begin{pmatrix}
0 & x_{12} & \cdots & x_{1n} \\
x_{21} & x_{22} & \cdots & x_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
x_{m1} & x_{m2} & \cdots & x_{mn}
\end{pmatrix}
\]

as ideal points.

Now, by giving a definition we will handle a special example related to the definition.

**Definition 4.1.** An \(s\)-space is the set of points whose representing vectors

\[
\begin{pmatrix}
x_{11} & x_{12} & \cdots & x_{1n} \\
x_{21} & x_{22} & \cdots & x_{n} \\
\vdots & \vdots & \ddots & \vdots \\
x_{m1} & x_{m2} & \cdots & x_{mn}
\end{pmatrix}
\]

of the points \(X\) satisfy the equations \(XA = 0\), where \(A\) is an \(m \times ((m - 1) - s)\) matrix of rank \((m - 1) - s\) with coefficients in \(K\).

Now let us take \(m = 4\) and \(n = 2\), so we study an example of a 3-dimensional projective coordinate space \(P(M)\). For the 3-dimensional projective coordinate space, first we will determine all points of a line whose incidence matrix is given and then we will determine the incidence matrix of a line that goes through the given points.

**Example 4.2.** In the 3-dimensional projective coordinate space \(P(M)\), any line, 1-dimensional subspace \(\Pi_1\) is the set of points whose representing vectors

\[
\begin{pmatrix}
x_{11} & x_{12} \\
x_{21} & x_{22} \\
\vdots & \vdots \\
x_{m1} & x_{m2}
\end{pmatrix}
\]

of the points \(X\) satisfy the equations \(XA = 0\), where \(A\) is a \(4 \times 2\) matrix of rank 2 with coefficients in \(K\). Thus \(\Pi_1 = \{X \mid AX = 0, A \in K_2^4 \setminus I_2^4\}\) is obtained. First, we identify all points of a line whose incidence matrix is

\[
\begin{bmatrix}
a & e \\
b & f \\
c & g \\
d & h
\end{bmatrix} = \begin{bmatrix}
a_0 & a_1 \\
b_0 & b_1 \\
c_0 & c_1 \\
d_0 & d_1
\end{bmatrix} \in K_2^4 \setminus I_2^4.
\]

As a consequence of the incidence matrix, it is trivial to see that \(\exists a_0, b_0, c_0, d_0, e_0, f_0, g_0, h_0 \neq 0\).

For \(XA = 0\), we have the following cases:
**Case 1:** For the coordinate vector $X$ of the point $\bar{X}$, if $x_{11} \neq 0$, then $X = (l_2, X_2, X_3, X_4) \in K^4$. Thus we obtain the following equations from $XA = 0$:

\begin{align*}
a_0 + x_{21}b_0 + x_{31}c_0 + x_{41}d_0 &= 0, \\
a_1 + x_{21}b_1 + x_{22}b_0 + x_{31}c_1 + x_{32}c_0 + x_{41}d_1 + x_{42}d_0 &= 0, \\
e_0 + x_{21}f_0 + x_{31}g_0 + x_{41}h_0 &= 0, \\
e_1 + x_{21}f_1 + x_{22}f_0 + x_{31}g_1 + x_{32}g_0 + x_{41}h_1 + x_{42}h_0 &= 0.
\end{align*}

If we solve (4.2) by using the Maple programme, we get the following solutions:

\begin{align*}
x_{21} &= \frac{-(a_0g_0 - c_0e_0) + (d_0g_0 - c_0h_0)x_{41}}{b_0g_0 - f_0c_0}, \\
x_{22} &= \frac{a'}{c_0^2f_0^2 + b_0^2g_0^2 - 2b_0g_0f_0c_0} \\
x_{31} &= \frac{(-b_0e_0 + f_0a_0) + (f_0d_0 - b_0h_0)x_{41}}{b_0g_0 - f_0c_0} \\
x_{32} &= \frac{b'}{c_0^2f_0^2 + b_0^2g_0^2 - 2b_0g_0f_0c_0}
\end{align*}

where

\[ a' = \begin{pmatrix}
  2b_1a_0 - e_1f_0c_0^2 + c_0g_1f_0a_0 + g_0a_1f_0c_0 - g_0c_0f_1a_0 \\
  -g_0c_1f_0a_0 - g_0b_1c_0c_0 + b_0g_0c_0c_1 + b_0g_0c_0c_0 - \\
  b_0c_0g_1c_0 + f_1c_0^2c_0 - b_0g_0^2a_1 \\
  -h_1f_0c_0^2 + f_1c_0^2h_0 - b_0g_0^2d_1 + c_0g_1f_0d_0 + g_0^2b_1d_0 \\
  +g_0d_1f_0c_0 - g_0c_1f_0d_0 - g_0f_1c_0d_0 - g_0b_1c_0h_0 + \\
  b_0g_0c_0h_1 + b_0g_0c_0h_0 - b_0c_0g_1h_0 \\
  (-h_0f_0c_0^2 - b_0g_0^2d_0 + g_0d_0f_0c_0 + b_0g_0c_0h_0) x_{42}
\end{pmatrix} + x_{41} \]

and

\[ b' = \begin{pmatrix}
  b_0g_1e_0 - b_0g_1e_0 - c_1f_0^2a_0 + a_1f_0^2c_0 - b_0c_1f_0c_0 - b_0e_1f_0c_0 \\
  -f_1a_0b_0g_0 + b_0g_1f_0a_0 + f_1c_0b_0c_0 - f_0b_0a_1g_0 + f_0b_1c_0c_0 + \\
  f_0b_1a_0g_0 - f_0b_1c_0e_0 \\
  b_0^2h_1g_0 + b_0^2h_1g_0 - b_0h_1f_0c_0 - f_1d_0b_0g_0 + b_0g_1f_0d_0 + f_1c_0b_0h_0 \\
  -c_1f_0^2d_0 + d_1f_0^2c_0 + f_0b_0c_1h_0 - f_0b_0d_1g_0 - f_0b_1c_0h_0 + f_0b_1d_0g_0 + \\
  (b_0^2h_0g_0 - b_0h_0f_0c_0 + d_0f_0^2c_0 - f_0b_0d_0a_0) x_{42}
\end{pmatrix} \]

**Case 2:** For the coordinate vector $X$ of the point $\bar{X}$, if $x_{11} = 0$, then $\bar{X}$ is an ideal point of the form

\[ X_1 = \begin{pmatrix} 0 & x_{12} \\ 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} x_{21} & x_{22} \\ 0 & x_{21} \end{pmatrix}, \quad X_3 = \begin{pmatrix} x_{31} & x_{32} \\ 0 & x_{31} \end{pmatrix}, \quad X_4 = \begin{pmatrix} x_{41} & x_{42} \\ 0 & x_{41} \end{pmatrix}. \]

Here, we know that $\exists x_{22}, x_{23}, x_{24} \neq 0$. Thus we obtain the following equations from $XA = 0$ :

\begin{align*}
x_{12}a_0 + x_{21}b_1 + x_{22}b_0 + x_{31}c_1 + x_{32}c_0 + x_{41}d_1 + x_{42}d_0 &= 0, \\
x_{12}a_0 + x_{21}b_1 + x_{22}b_0 + x_{31}c_1 + x_{32}c_0 + x_{41}d_1 + x_{42}d_0 &= 0,
\end{align*}

and

\begin{align*}
x_{12}a_0 + x_{21}b_1 + x_{22}b_0 + x_{31}c_1 + x_{32}c_0 + x_{41}d_1 + x_{42}d_0 &= 0, \\
x_{12}a_0 + x_{21}b_1 + x_{22}b_0 + x_{31}c_1 + x_{32}c_0 + x_{41}d_1 + x_{42}d_0 &= 0, \\
x_{12}a_0 + x_{21}b_1 + x_{22}b_0 + x_{31}c_1 + x_{32}c_0 + x_{41}d_1 + x_{42}d_0 &= 0.
\end{align*}
If we solve (4.3) by using the Maple programme, we get the following solutions:

\[
\begin{align*}
    x_{21} &= \frac{(-d_0g_0 + h_0c_0)x_{41}}{-f_0c_0 + b_0g_0}, \\
    x_{22} &= \frac{a''}{g_0^2b_0^2 + f_0^2c_0^2 - 2f_0c_0b_0g_0}, \\
    x_{31} &= \frac{(-f_0d_0 + b_0h_0)x_{41}}{-f_0c_0 + b_0g_0}, \\
    x_{32} &= \frac{b''}{g_0^2b_0^2 + f_0^2c_0^2 - 2f_0c_0b_0g_0}, \\
    x_{41} &= x_{41}, \quad x_{42} = x_{42}, \quad x_{12} = x_{12}
\end{align*}
\]

where

\[
\begin{align*}
    a'' &= \begin{pmatrix}
        (-b_0g_0^2a_0 + b_0g_0c_0c_0 + g_0f_0a_0c_0 - f_0c_0^2c_0) x_{12} + \\
        (-b_0g_0^2a_1 + b_0g_0c_1c_0 + b_0g_0c_0h_0 - b_0c_0g_1h_0 + b_1d_0g_0^2) - c_0f_0g_1d_0 - f_0c_0^2h_1
    \end{pmatrix},
\end{align*}
\]

\[
\begin{align*}
    b'' &= \begin{pmatrix}
        (e_0b_0^2g_0 - b_0f_0c_0c_0 + f_0^2a_0c_0 - f_0a_0b_0g_0) x_{12} \\
        (-b_0f_0h_1c_0 + b_0f_0g_1d_0 - b_0f_1g_0d_0 + b_0f_1h_0c_0 + h_1b_0^2g_0 - g_1b_0^2h_0) + f_0^2d_0c_0 - b_0h_0f_0c_0 - f_0c_1h_0h_0 - f_0d_1b_0g_0
    \end{pmatrix}.
\end{align*}
\]

Now conversely, we have a new situation. We determine the incidence matrix of a line whose points are given. This also has two cases:

**Case 1:** Let us take the coordinate vectors

\[
X = \begin{pmatrix}
    1 & 0 \\
    x_{21} & x_{22} \\
    x_{31} & x_{32} \\
    x_{41} & x_{42}
\end{pmatrix}
\quad \text{and} \quad
Y = \begin{pmatrix}
    1 & 0 \\
    y_{21} & y_{22} \\
    y_{31} & y_{32} \\
    y_{41} & y_{42}
\end{pmatrix},
\]

of proper points \(X\) and \(Y\), respectively. Then we search the incidence matrix of the form

\[
A = \begin{pmatrix}
    a & e & c & d \\
    b & f & g & h
\end{pmatrix} = \begin{pmatrix}
    (a_0 & a_1) & (e_0 & e_1) \\
    (b_0 & b_1) & (f_0 & f_1) \\
    (c_0 & c_1) & (g_0 & g_1) \\
    (d_0 & d_1) & (h_0 & h_1)
\end{pmatrix} \in K^4_2 \setminus I^4_2.
\]

we know that the coordinate vectors of these points are as follows

\[
X = \begin{pmatrix}
    1 & 0 \\
    0 & 1 \\
    x_{21} & x_{22} \\
    x_{31} & x_{32} \\
    x_{41} & x_{42}
\end{pmatrix}.
\]

and

\[
Y = \begin{pmatrix}
    1 & 0 \\
    0 & 1 \\
    y_{21} & y_{22} \\
    y_{31} & y_{32} \\
    y_{41} & y_{42}
\end{pmatrix}.
\]
Thus we obtain the following equations from $XA = 0$ and $YA = 0$:

$$
\begin{align*}
    a_0 + x_{21}b_0 + x_{31}c_0 + x_{41}d_0 &= 0, \\
    a_1 + x_{21}b_1 + x_{22}b_0 + x_{31}c_1 + x_{32}c_0 + x_{41}d_1 + x_{42}d_0 &= 0, \\
    c_0 + x_{21}f_0 + x_{31}g_0 + x_{41}h_0 &= 0, \\
    e_1 + x_{21}f_1 + x_{22}f_0 + x_{31}g_1 + x_{32}g_0 + x_{41}h_1 + x_{42}h_0 &= 0.
\end{align*}
$$

(4.4)

If we solve (4.4) by using the Maple programme, then we get the following solutions:

$$
\begin{align*}
a_0 &= \frac{(x_{21}y_{31} - y_{21}x_{31})c_0 + (x_{21}y_{41} - y_{21}x_{41})d_0}{-y_{21} + x_{21}}, & a_1 &= \frac{-y_{21} + x_{21}}{-y_{21} + x_{21}}, \\
b_0 &= \frac{(x_{31} - y_{31})c_0 + (x_{41} - y_{41})d_0}{-y_{21} + x_{21}}, & b_1 &= \frac{-y_{21} + x_{21}}{-y_{21} + x_{21}}, \\
c_0 &= \frac{-y_{21} + x_{21}}{-y_{21} + x_{21}}, & c_1 &= \frac{-y_{21} + x_{21}}{-y_{21} + x_{21}}.
\end{align*}
$$

where

$$
\begin{align*}
    a_1' &= \begin{pmatrix}
        y_{41}x_{21}^2 - x_{21}y_{22}x_{41} - y_{21}x_{22}y_{41} - y_{21}x_{22}x_{41} - x_{22}y_{41}^2 \\
        y_{21}x_{22}y_{41} + y_{21}x_{22}x_{41} + x_{22}y_{41}^2 \\
        y_{21}x_{22}x_{41} - y_{21}y_{22}x_{41}^2 \\
        y_{21}x_{22}x_{41}^2 + y_{21}x_{22}y_{41}^2 \\
        (y_{31}x_{21} + y_{32}x_{21} + y_{32}x_{21})^2 + (y_{31}x_{21} + y_{32}x_{21} + y_{32}x_{21})^2 \\
    \end{pmatrix} d_0, \\
    b_1' &= \begin{pmatrix}
        -y_{21}x_{21} + y_{21}x_{21} \\
        y_{31}x_{21} - y_{31}x_{21} \\
        x_{21}y_{22}x_{31} + x_{21}y_{22}x_{31} + x_{21}y_{22}x_{31} + x_{21}y_{22}x_{31} + x_{21}y_{22}x_{31} \\
        y_{21}y_{22}x_{31} - y_{21}x_{22}x_{31} \\
        y_{41}x_{21}^2 - y_{21}y_{22}x_{31} + y_{21}x_{22}x_{31} \\
    \end{pmatrix} c_1, \\
    e_1' &= \begin{pmatrix}
        y_{42}x_{21}^2 - x_{21}y_{22}x_{41} - y_{21}x_{22}y_{41} - y_{21}x_{22}x_{41} - x_{22}y_{41}^2 \\
        y_{21}x_{22}y_{41} + y_{21}x_{22}x_{41} + x_{22}y_{41}^2 \\
        y_{21}x_{22}x_{41} - y_{21}y_{22}x_{41}^2 \\
        y_{21}x_{22}x_{41}^2 + y_{21}x_{22}y_{41}^2 \\
        (y_{31}x_{21} + y_{32}x_{21} + y_{32}x_{21})^2 + (y_{31}x_{21} + y_{32}x_{21} + y_{32}x_{21})^2 \\
        y_{21}y_{22}x_{31} - y_{21}x_{22}x_{31} \\
        x_{21}y_{22}x_{31} + x_{21}y_{22}x_{31} + x_{21}y_{22}x_{31} + x_{21}y_{22}x_{31} + x_{21}y_{22}x_{31} \\
        y_{41}x_{21}^2 - y_{21}y_{22}x_{31} + y_{21}x_{22}x_{31} \\
        y_{42}x_{21}^2 - x_{21}y_{22}x_{31} - y_{21}x_{22}x_{31} - y_{21}y_{22}x_{31} - y_{21}x_{22}x_{31} + y_{41}x_{21}^2 \\
        + x_{22}y_{41}^2 - y_{21}y_{22}x_{31} + y_{21}x_{22}x_{31} \\
        + y_{41}x_{21}^2 - y_{21}y_{22}x_{31} + y_{21}x_{22}x_{31} + x_{22}y_{41}^2 \\
        + y_{31}x_{21}^2 - y_{31}x_{21}^2 \\
    \end{pmatrix} h_0, \\
    f_1' &= \begin{pmatrix}
        y_{42}x_{21} + y_{42}x_{21} + y_{32}x_{21} - y_{32}x_{21} - y_{32}x_{21} - y_{32}x_{21} - y_{32}x_{21} \\
        y_{42}x_{21} + y_{42}x_{21} + y_{32}x_{21} - y_{32}x_{21} - y_{32}x_{21} - y_{32}x_{21} - y_{32}x_{21} \\
        y_{42}x_{21} + y_{42}x_{21} + y_{32}x_{21} - y_{32}x_{21} - y_{32}x_{21} - y_{32}x_{21} - y_{32}x_{21} \\
        y_{42}x_{21} + y_{42}x_{21} + y_{32}x_{21} - y_{32}x_{21} - y_{32}x_{21} - y_{32}x_{21} - y_{32}x_{21} \\
        y_{42}x_{21} + y_{42}x_{21} + y_{32}x_{21} - y_{32}x_{21} - y_{32}x_{21} - y_{32}x_{21} - y_{32}x_{21} \\
    \end{pmatrix} g_0, \\
    g_1' &= \begin{pmatrix}
        -y_{41}x_{21} - y_{41}x_{21} + y_{41}x_{21} + y_{41}x_{21} + y_{41}x_{21} \\
        y_{31}x_{21} - y_{31}x_{21} \\
    \end{pmatrix} h_1.
\end{align*}
$$

Case 2: Let us take the coordinate vectors

$$
X = \begin{pmatrix}
    1 & 0 & x_{21} & x_{22} \\
    0 & 1 & x_{31} & x_{32} \\
    0 & 0 & x_{41} & x_{42}
\end{pmatrix}
$$

and

$$
Y = \begin{pmatrix}
    0 & y_{12} & y_{21} & y_{22} \\
    0 & y_{31} & y_{32} & y_{33} \\
    y_{41} & y_{42} & y_{43} & y_{44}
\end{pmatrix}
$$
of proper and ideal points \( X \) and \( Y \), respectively. Here for the point \( Y \), we know that \( \exists \) \( y_{21}, y_{31}, y_{41} \neq 0 \). The coordinate vectors of these points as follows:

\[
X = \left( \begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array} \right)
, \left( \begin{array}{ccc}
x_{21} & x_{2} & x_{31} \\
x_{31} & x_{32} & x_{41} \\
0 & x_{41} & x_{42}
\end{array} \right)
, \left( \begin{array}{ccc}
x_{21} & x_{2} & x_{31} \\
x_{31} & x_{32} & x_{41} \\
0 & x_{41} & x_{42}
\end{array} \right)
, \left( \begin{array}{ccc}
x_{21} & x_{2} & x_{31} \\
x_{31} & x_{32} & x_{41} \\
0 & x_{41} & x_{42}
\end{array} \right)
\]

and

\[
Y = \left( \begin{array}{ccc}
0 & y_{12} & 0 \\
0 & 0 & y_{21} \\
0 & 0 & y_{31}
\end{array} \right)
, \left( \begin{array}{ccc}
y_{21} & y_{22} & y_{31} \\
y_{31} & y_{32} & y_{41} \\
y_{41} & y_{42} & y_{41}
\end{array} \right)
, \left( \begin{array}{ccc}
y_{21} & y_{22} & y_{31} \\
y_{31} & y_{32} & y_{41} \\
y_{41} & y_{42} & y_{41}
\end{array} \right)
\]

Then we obtain the following equations from \( XA = 0 \) and \( YA = 0 \):

\[
a_0 + x_{21}b_0 + x_{31}c_0 + x_{41}d_0 = 0,
\]

\[
a_1 + x_{21}b_1 + x_{22}b_0 + x_{31}c_1 + x_{32}c_0 + x_{41}d_1 + x_{42}d_0 = 0,
\]

\[
e_0 + x_{21}f_0 + x_{31}g_0 + x_{41}h_0 = 0,
\]

\[
e_1 + x_{21}f_1 + x_{22}f_0 + x_{31}g_1 + x_{32}g_0 + x_{41}h_1 + x_{42}h_0 = 0,
\]

\[
y_{12}a_0 + y_{21}b_1 + y_{22}b_0 + y_{31}c_1 + y_{32}c_0 + y_{41}d_1 + y_{42}d_0 = 0,
\]

\[
y_{12}e_0 + y_{21}f_1 + y_{22}f_0 + y_{31}g_1 + y_{32}g_0 + y_{41}h_1 + y_{42}h_0 = 0.
\]

If we solve (4.5) by using the Maple programme, then we get the following solutions:

\[
a_0 = \frac{(x_{41}y_{21} - y_{41}x_{21}) b_0 + (-y_{41}x_{31} + x_{41}y_{31}) c_0}{y_{41}}, \quad a_1 = \frac{d''}{y_{41}},
\]

\[
b_0 = b_0, \quad b_1 = b_1, \quad c_0 = c_0, \quad c_1 = c_1,
\]

\[
d_0 = -\frac{y_{21}b_0 + y_{31}c_0}{y_{41}}, \quad d_1 = -\frac{d''}{y_{41}},
\]

\[
e_0 = \frac{(-y_{41}x_{31} + x_{41}y_{31}) g_0 + (-y_{41}x_{21} + x_{41}y_{21}) f_0}{y_{41}}, \quad e_1 = \frac{e''}{y_{41}},
\]

\[
h_0 = -\frac{y_{21}f_0 + y_{31}g_0}{y_{41}}, \quad h_1 = -\frac{h''}{y_{41}},
\]

\[
f_0 = f_0, \quad f_1 = f_1, \quad g_0 = g_0, \quad g_1 = g_1,
\]

where

\[
a''_1 = \left( \begin{array}{c}
(x_{41}y_{41}y_{22} - x_{41}y_{12}y_{41}x_{21} - x_{41}y_{12}y_{41}x_{21} + y_{12}x_{31}y_{21} + y_{41}x_{12}y_{41}x_{21} - y_{41}^2 x_{22}) b_0 \\
+ (x_{41}y_{41}y_{22} - y_{41}^2 x_{21}) b_1
\end{array} \right),
\]

\[
d''_1 = (y_{41}y_{21} - y_{41}y_{21} - y_{12}y_{41}x_{21} + y_{12}x_{41}y_{21}) b_0 + (y_{41}y_{21}) b_1 + (y_{41}y_{32} + y_{12}y_{41}y_{31} - y_{42}y_{31} - y_{12}y_{41}x_{31}) c_0 + (x_{41}y_{41}y_{31}) c_1
\]

\[
e''_1 = \left( \begin{array}{c}
(y_{41}x_{42}y_{21} - y_{41}^2 x_{22} + y_{12}^2 x_{31}y_{21} + x_{41}x_{12}y_{41}x_{21} - x_{12}y_{41}x_{21} - y_{41}y_{12}y_{41}x_{21}) f_0 \\
+ (y_{41}x_{12}y_{41}x_{21} - y_{41}^2 x_{21}) f_1
\end{array} \right),
\]

\[
h''_1 = (y_{12}x_{41}y_{21} + y_{12}y_{22} - y_{12}y_{41}x_{21} - y_{42}y_{21}) f_0 + (y_{41}y_{21}) f_1 + (y_{41}y_{31} + y_{12}x_{41}y_{31} - y_{12}y_{41}x_{31} - y_{41}y_{41}y_{31} + (x_{41}y_{41}y_{31}) g_0 + (x_{41}y_{41}y_{31} - y_{41}^2 x_{31} x_1) g_1.
\]
References

A few remarks on boundedness in topological modules and topological groups

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Abstract

Let $X$, $Y$, and $Z$ be topological modules over a topological ring $R$. In the first part of the paper, we introduce three different classes of bounded bigroup homomorphisms from $X \times Y$ into $Z$ with respect to the three different uniform convergence topologies. We show that these spaces form again topological modules over $R$. In the second part, we characterize bounded sets in the arbitrary product of topological groups with respect to the both product and box topologies.

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Keywords. Bigroup homomorphism, topological module, boundedness, topological group

1. Introduction and preliminaries

In [4], some notions for bounded group homomorphisms on a topological ring have been introduced. Also, it has been proved that each class of bounded group homomorphisms on a topological ring, with respect to an appropriate topology, forms a topological ring. Also, an analogous statement for topological groups and bounded homomorphisms between them has been investigated in [3]. Since every topological ring can be viewed as a topological module over itself, it is not a hard job to see that we can consider the concepts of topological modules of bounded group homomorphisms on a topological module. In fact, the results in [4], can be generalized to topological modules in a natural way. Recall that a topological module $X$ is a module with a topology over a topological ring $R$ such that the addition (as a map from $X \times X$ into $X$), and the multiplication (as a map from $R \times X$ into $X$) are continuous. There are many examples of topological modules, for instance, every topological vector space is a topological module over a topological field, every abelian topological group is a topological module over $\mathbb{Z}$, where $\mathbb{Z}$ denotes the ring of integers with the discrete topology, and also every topological ring is a topological module over each of its subrings. So, it is of independent interest if we consider possible relations between algebraic structures of a module and its topological properties. In the first part of the present paper, we are going to consider bounded bigroup homomorphisms between topological modules. We endow each class of bounded bigroup homomorphisms to a uniform convergence topology and we show that under the assumed topology, each class of
them, forms a topological module. In addition, we see that if each class of bounded bigroup homomorphisms is uniformly complete. In the following, by a bigroup homomorphism on Cartesian product \( X \times Y \), we mean a map which is group homomorphism on \( X \) and \( Y \), respectively. Also, note that if \( X \) is a topological module over topological ring \( R \), then, \( B \subseteq X \) is said to be bounded if for each zero neighborhood \( W \subseteq X \), there exists zero neighborhood \( V \subseteq R \) such that \( VB \subseteq W \). Finally, as a special case, we consider bounded sets in arbitrary Cartesian products of abelian topological groups. For more information about topological modules, topological rings, topological groups, and the related notions, see [1–8].

2. Bounded bigroup homomorphisms

**Definition 2.1.** Let \( X, Y, \) and \( Z \) be topological modules over a topological ring \( R \). A bigroup homomorphism \( \sigma : X \times Y \to Z \) is said to be:

i. \( n \)-bounded if there exist some zero neighborhoods \( U \subseteq X \) and \( V \subseteq Y \) such that \( \sigma(U, V) \) is bounded in \( Z \).

ii. \( \frac{2}{n} \)-bounded if there exists a zero neighborhood \( U \subseteq X \) such that for each bounded set \( B \subseteq Y \), \( \sigma(U, B) \) is bounded in \( Z \).

iii. \( b \)-bounded if for every bounded subsets \( B_1 \subseteq X \) and \( B_2 \subseteq Y \), \( \sigma(B_1, B_2) \) is bounded in \( Z \).

The first point is that these concepts of bounded bigroup homomorphisms are far from being equivalent. In prior to anything, we show this.

**Example 2.2.** Let \( X = \mathbb{R}^N \), the space of all real sequences, with the coordinate-wise topology and the pointwise product. It is easy to see that \( X \) is a topological module over itself. Consider the bigroup homomorphism \( \sigma : X \times X \to Y \) defined by \( \sigma(x, y) = xy \), in which the product is given by pointwise. It is not difficult to see that \( \sigma \) is \( b \)-bounded but since \( X \) is not locally bounded, it can not be \( n \)-bounded.

Also, the above example may apply to determine a \( b \)-bounded bigroup homomorphism which is not \( \frac{2}{n} \)-bounded.

**Example 2.3.** Let \( X \) be \( \ell_\infty \), the space of all bounded real sequences, with the topology induced by the uniform norm and pointwise product. Suppose \( Y \) is \( \ell_\infty \), with the coordinate-wise topology and pointwise product. Consider the bigroup homomorphism \( \sigma \) from \( X \times X \) to \( Y \) as in Example 2.2. It is easy to see that \( \sigma \) is \( \frac{2}{n} \)-bounded but it is not \( n \)-bounded. For, suppose \( \epsilon > 0 \) is arbitrary. Assume that \( N^{(0)}_\epsilon \) is the ball with centre zero and radius \( \epsilon \) in \( X \). If \( U \) is an arbitrary zero neighborhood in \( Y \), without loss of generality, we may assume that \( U \) is of the form

\[
(-\epsilon_1, \epsilon_1) \times \ldots \times (-\epsilon_r, \epsilon_r) \times \mathbb{R} \times \mathbb{R} \times \ldots,
\]

in which, \( \epsilon_i > 0 \). Fix \( 0 < \delta < \min\{\epsilon_i\} \). Consider the sequence \( (a_n) \subseteq U \) defined by \( a_n = (\delta, \ldots, \delta, n, \ldots, n, 0, \ldots) \), in which \( \delta \) is appeared \( r \) times and \( n \) equips \( n - r \) components. Now, it is not difficult to see that \( \sigma(N^{(0)}_\epsilon, (a_n)) \) can not be a bounded subset of \( Y \).

**Example 2.4.** Let \( X \) be \( \ell_\infty \), with pointwise product and the uniform norm topology, and \( Y \) be \( \ell_\infty \), with the zero multiplication and the topology induced by norm. Consider \( \sigma \) from \( X \times Y \) to \( X \) as in Example 2.2. Then, \( \sigma \) is \( n \)-bounded but it is not \( \frac{2}{n} \)-bounded. For, suppose \( \epsilon > 0 \) is arbitrary. Consider the sequence \( (a_n) \) in \( Y \) defined by \( a_n = (\frac{1}{\epsilon}, \ldots, \frac{1}{\epsilon}, 0, \ldots) \). \( (a_n) \) is bounded in \( Y \) but \( \sigma(N^{(0)}_\epsilon, (a_n)) \) contains the sequence \( (1, \ldots, n, 0, \ldots) \) which is not bounded in \( X \).
Since topological modules are topological spaces, we can consider the concept of jointly continuity for a bigroup homomorphism between topological modules. The interesting result in this case, in spite of the case related to topological vector spaces and topological groups notions, is that there is no relation between jointly continuous bigroup homomorphisms and bounded ones; see [3,7] for more details on these concepts. To see this, consider the following example.

**Example 2.5.** Let $X$ be $\ell_\infty$, with the pointwise product and coordinate-wise topology, and $Y$ be $\ell_\infty$, with the zero multiplication and the uniform norm topology. Consider the bigroup homomorphism $\sigma$ from $X \times X$ into $Y$ as in Example 2.2. Indeed, $\sigma$ is $b$-bounded and $n$-bounded but it is easy to see that $\sigma$ can not be jointly continuous.

The class of all $n$-bounded bigroup homomorphisms on a topological module $X$ is denoted by $B_n(X \times X)$ and is equipped with the topology of uniform convergence on some zero neighborhoods, namely, a net $(\sigma_\alpha)$ of $n$-bounded bigroup homomorphisms converges uniformly to zero on some zero neighborhoods $U, V \subseteq X$ if for each zero neighborhood $W \subseteq X$ there is an $\alpha_0$ with $\sigma_\alpha(U, V) \subseteq W$ for each $\alpha \geq \alpha_0$. The set of all $\frac{n}{2}$-bounded bigroup homomorphisms on a topological module $X$ is denoted by $B_{\frac{n}{2}}(X \times X)$ and it is assigned with the topology of $\sigma$-uniformly convergence on some zero neighborhood. We say that a net $(\sigma_\alpha)$ of $\frac{2}{n}$-bounded bigroup homomorphisms converges $\sigma$-uniformly to zero on some zero neighborhood if there exists a zero neighborhood $U \subseteq X$ such that for each bounded set $B \subseteq X$ and for each zero neighborhood $W \subseteq X$ there is an $\alpha_0$ with $\sigma_\alpha(U, B) \subseteq W$ for each $\alpha \geq \alpha_0$. Finally, the class of all $b$-bounded bigroup homomorphisms on a topological module $X$ is denoted by $B_b(X \times X)$ and is endowed with the topology of uniform convergence on bounded sets which means a net $(\sigma_\alpha)$ of $b$-bounded bigroup homomorphisms converges uniformly to zero on bounded sets $B_1, B_2 \subseteq X$ if for each zero neighborhood $W \subseteq X$ there is an $\alpha_0$ with $\sigma_\alpha(B_1, B_2) \subseteq W$ for each $\alpha \geq \alpha_0$. In this part of the paper, we show that the operations of addition and module multiplication are continuous in each of the topological modules $B_n(X \times X), B_{\frac{n}{2}}(X \times X)$, and $B_b(X \times X)$ with respect to the assumed topology, respectively. So, each of them forms a topological $R$-module.

**Theorem 2.6.** The operations of addition and module multiplication in $B_n(X \times X)$ are continuous with respect to the topology of uniform convergence on some zero neighborhoods.

**Proof.** Suppose two nets $(\sigma_\alpha)$ and $(\gamma_\alpha)$ of $n$-bounded bigroup homomorphisms converge to zero uniformly on some zero neighborhoods $U, V \subseteq X$. Let $W$ be an arbitrary zero neighborhood in $X$. So, there is a zero neighborhood $W_1$ with $W_1 + W_1 \subseteq W$. There are some $\alpha_0$ and $\alpha_1$ such that $\sigma_\alpha(U, V) \subseteq W_1$ for each $\alpha \geq \alpha_0$ and $\gamma_\alpha(U, V) \subseteq W_1$ for each $\alpha \geq \alpha_1$. Choose an $\alpha_2$ with $\alpha_2 \geq \alpha_0$ and $\alpha_2 \geq \alpha_1$. If $\alpha \geq \alpha_2$ then $(\sigma_\alpha + \gamma_\alpha)(U, V) \subseteq \sigma_\alpha(U, V) + \gamma_\alpha(U, V) \subseteq W_1 + W_1 \subseteq W$. Thus, the addition is continuous. Now, we show the continuity of the module multiplication. Suppose $(r_\alpha)$ is a net in $R$ which is convergent to zero. There are some neighborhoods $V_1 \subseteq R$ and $W_2 \subseteq X$ such that $V_1 W_2 \subseteq W$. Find an $\alpha_3$ with $\sigma_\alpha(U, V) \subseteq W_2$ for each $\alpha \geq \alpha_3$. Take an $\alpha_4$ such that $(r_\alpha) \subseteq V_1$ for each $\alpha \geq \alpha_4$. Choose an $\alpha_5$ with $\alpha_5 \geq \alpha_0$ and $\alpha_5 \geq \alpha_4$. If $\alpha \geq \alpha_5$ then $r_\alpha \sigma_\alpha(U, V) \subseteq V_1 W_2 \subseteq W$, as asserted. \hfill $\Box$

**Theorem 2.7.** The operations of addition and module multiplication in $B_{\frac{n}{2}}(X \times X)$ are continuous with respect to the topology of $\sigma$-uniform convergence on some zero neighborhood.

**Proof.** Suppose two nets $(\sigma_\alpha)$ and $(\gamma_\alpha)$ of $\frac{2}{n}$-bounded bigroup homomorphisms converge to zero $\sigma$-uniformly on some zero neighborhood $U \subseteq X$. Fix a bounded set $B \subseteq X$. Let $W$ be an arbitrary zero neighborhood in $X$. So, there is a zero neighborhood $W_1$ with $W_1 + W_1 \subseteq W$. There are some $\alpha_0$ and $\alpha_1$ such that $\sigma_\alpha(U, B) \subseteq W_1$ for each $\alpha \geq \alpha_0$ and
\[ \gamma_{\alpha}(U, B) \subseteq W_1 \] for each \( \alpha \geq \alpha_1 \). Choose an \( \alpha_2 \) with \( \alpha_2 \geq \alpha_0 \) and \( \alpha_2 \geq \alpha_1 \). If \( \alpha \geq \alpha_2 \) then \( (\sigma_\alpha + \gamma_{\alpha})(U, B) \subseteq \sigma_\alpha(U, B) + \gamma_{\alpha}(U, B) \subseteq W_1 + W_1 \subseteq W \). Thus, the addition is continuous. Now, we show the continuity of the module multiplication. Suppose \( (r_\alpha) \) is a net in \( R \) which is convergent to zero. There are some neighborhoods \( V_1 \subseteq R \) and \( W_2 \subseteq X \) such that \( V_1 W_2 \subseteq W \). Find an \( \alpha_3 \) with \( \gamma_{\alpha}(U, B) \subseteq W_2 \) for each \( \alpha \geq \alpha_3 \). Take an \( \alpha_4 \) such that \( (r_\alpha) \subseteq V_1 \) for each \( \alpha \geq \alpha_4 \). Choose an \( \alpha_5 \) with \( \alpha_5 \geq \alpha_3 \) and \( \alpha_5 \geq \alpha_4 \). If \( \alpha \geq \alpha_5 \) then \( r_\alpha \sigma_\alpha(U, B) \subseteq V_1 W_2 \subseteq W \), as we wanted. \( \square \)

**Theorem 2.8.** The operations of addition and module multiplication in \( B_b(X \times X) \) are continuous with respect to the topology of uniform convergence on bounded sets.

**Proof.** Suppose two nets \( (\sigma_\alpha) \) and \( (\gamma_{\alpha}) \) of \( b \)-bounded bigroup homomorphisms converge to zero uniformly on bounded sets. Fix two bounded sets \( B_1, B_2 \subseteq X \). Let \( W \) be an arbitrary zero neighborhood in \( X \). So, there is a zero neighborhood \( W_1 \) with \( W_1 + W_1 \subseteq W \). There are some \( \alpha_0 \) and \( \alpha_1 \) such that \( \sigma_\alpha(B_1, B_2) \subseteq W_1 \) for each \( \alpha \geq \alpha_0 \) and \( \gamma_{\alpha}(B_1, B_2) \subseteq W_1 \) for each \( \alpha \geq \alpha_1 \). Choose an \( \alpha_2 \) with \( \alpha_2 \geq \alpha_0 \) and \( \alpha_2 \geq \alpha_1 \). If \( \alpha \geq \alpha_2 \) then \( (\sigma_\alpha + \gamma_{\alpha})(B_1, B_2) \subseteq \sigma_\alpha(B_1, B_2) + \gamma_{\alpha}(B_1, B_2) \subseteq W_1 + W_1 \subseteq W \). Thus, the addition is continuous. Now, we show the continuity of the module multiplication. Suppose \( (r_\alpha) \) is a net in \( R \) which is convergent to zero. There are some neighborhoods \( V_1 \subseteq R \) and \( W_2 \subseteq X \) such that \( V_1 W_2 \subseteq W \). Find an \( \alpha_3 \) with \( \gamma_{\alpha}(B_1, B_2) \subseteq W_2 \) for each \( \alpha \geq \alpha_3 \). Take an \( \alpha_4 \) such that \( (r_\alpha) \subseteq V_1 \) for each \( \alpha \geq \alpha_4 \). Choose an \( \alpha_5 \) with \( \alpha_5 \geq \alpha_3 \) and \( \alpha_5 \geq \alpha_4 \). If \( \alpha \geq \alpha_5 \) then \( r_\alpha \sigma_\alpha(B_1, B_2) \subseteq V_1 W_2 \subseteq W \), as asserted. \( \square \)

In this step, we investigate whether each class of bounded bigroup homomorphisms is uniformly complete. The answer for \( B_b(X \times X) \) is affirmative but for other cases there exist counterexamples.

**Remark 2.9.** The class \( B_n(X \times X) \) can contain a Cauchy sequence whose limit is not an \( n \)-bounded bigroup homomorphism. Let \( X = \mathbb{R}^N \), the space of all real sequences, with the coordinate-wise topology and the pointwise product. Define the bigroup homomorphisms \( \sigma_n \) on \( X \) as follows:

\[ \sigma_n(x, y) = (x_1 y_1, \ldots, x_n y_n, 0, \ldots), \]

in which \( x = (x_i)_{i=1}^\infty \) and \( y = (y_i)_{i=1}^\infty \). Each \( \sigma_n \) is \( n \)-bounded. For, if \( U_n = \{ x \in X \mid |x_j| < 1, j = 0, 1, \ldots, n \} \), then, \( \sigma_n(U_n, U_n) \) is bounded in \( X \). Also, \( (\sigma_n) \) is a Cauchy sequence in \( B_n(X \times X) \). Because if \( W \) is an arbitrary zero neighborhood in \( X \), without loss of generality, we may assume that it is of the form

\[ W = (-\varepsilon_1, \varepsilon_1) \times \cdots \times (-\varepsilon_r, \varepsilon_r) \times \mathbb{R} \times \mathbb{R} \times \cdots, \]

in which \( \varepsilon_i > 0 \). So, for \( m, n > r \); we have \( (\sigma_n - \sigma_m)(X, X) \subseteq W \). Also, \( (\sigma_n) \) converges uniformly on \( (X, X) \) to the bigroup homomorphism \( \sigma \) defined by

\[ \sigma(x, y) = (x_1 y_1, x_2 y_2, \ldots). \]

But we have seen in Example 2.2 that \( \sigma \) is not \( n \)-bounded.

**Remark 2.10.** The class \( B_{2}(X \times X) \) can contain a Cauchy sequence whose limit is not an \( \frac{\ell_\infty}{2} \)-bounded bigroup homomorphism. Let \( X \) be \( \ell_\infty \), with the pointwise product and the uniform norm topology, and \( Y \) be \( \ell_\infty \), with the zero multiplication and the topology induced by norm. Consider bigroup homomorphisms \( \sigma_n \) from \( X \times Y \) to \( X \) as in Remark 2.9. It is not difficult to see that each \( \sigma_n \) is \( \frac{n}{2} \)-bounded. Also, \( (\sigma_n) \) is a Cauchy sequence in \( B_{2}(X \times X) \) which is convergent \( \sigma \)-uniformly on \( X \) to the bigroup homomorphism \( \sigma \) described in Example 2.4, so that it is not an \( \frac{\ell_\infty}{2} \)-bounded bigroup homomorphism.

**Proposition 2.11.** Suppose a net \( (\sigma_\alpha) \) of \( b \)-bounded bigroup homomorphisms converges to a bigroup homomorphism \( \sigma \) uniformly on bounded sets. Then \( \sigma \) is also \( b \)-bounded.
Fix bounded sets $B_1, B_2 \subseteq X$. Let $W$ be an arbitrary zero neighborhood in $X$. There is a zero neighborhood $W_1$ such that $W_1 + W_1 \subseteq W$. Choose a zero neighborhood $V_1 \subseteq R$ and a zero neighborhood $V_2 \subseteq X$ with $V_1 V_2 \subseteq W$. There is an $\alpha_0$ such that $(\sigma_a - \sigma)(B_1, B_2) \subseteq W_2$ for each $\alpha \geq \alpha_0$. Fix an $\alpha \geq \alpha_0$. So, there is a zero neighborhood $V_2 \subseteq V_1$ with $V_2 \alpha(B_1, B_2) \subseteq W_2$. Therefore,

$$V_2 \alpha(B_1, B_2) \subseteq V_2 \alpha(B_1, B_2) + V_2 W_2 \subseteq W_2 + V_1 W_2 \subseteq W_1 + W_1 \subseteq W.$$ 

\[ \square \]

3. Bounded sets in topological groups

Let us start with some remarks on boundedness which clarify the context. Suppose $X$ is a topological vector space. When one wants to define a bounded set in $X$, there are two absolutely fruitful tools; scalar multiplication and absorbing neighborhoods at zero. These objects help us to match our intrinsic of boundedness in topological vector spaces; namely, a subset is bounded if it lies in a big enough ball. Now, consider the case when $G$ is a topological group. These two handy material are not available. Of course, it is possible to define bounded sets in a topological group by replacing scalar multiplication with group multiplication in the definition of a bounded set in a topological vector space but this does not meet our intuition of a bounded set since for example the multiplicative group $S^1$ is not bounded in this manner. In addition, it is also possible to consider boundedness in a topological group like totally boundedness in a topological vector space but this one also does not match our intrinsic since it is similar to compactness in the additive group $\mathbb{R}$. Following [3], a subset $B$ in an abelian topological group $G$ is called bounded if for each neighborhood $U$ of the identity, there is an $n \in \mathbb{N}$ such that $B \subseteq nU$. Let $(G_\alpha)_{\alpha \in \Lambda}$ be a family of abelian topological groups and $G = \prod_{\alpha \in \Lambda} G_\alpha$. It is an easy job to see that $G$ is again an abelian topological group with respect to the both product and box topologies. In this step, we characterise bounded sets of $G$ in terms of bounded sets of $(G_\alpha)'s$. All topological groups are assumed to be abelian and Hausdorff.

First, we improve [3, Theorem 1].

**Theorem 3.1.** Let $(G_\alpha)_{\alpha \in \Lambda}$ be a family of abelian topological groups and $G = \prod_{\alpha \in \Lambda} G_\alpha$ with the product topology. Then $B \subseteq G$ is bounded if and only if there exists a family of subsets $(B_\alpha)_{\alpha \in \Lambda}$ such that each $B_\alpha \subseteq G_\alpha$ is bounded and $B \subseteq \prod_{\alpha \in \Lambda} B_\alpha$.

**Proof.** Suppose $B \subseteq G$ is bounded. Put

$$B_\alpha = \{ x \in G_\alpha : \exists y = (y_\beta) \in B \text{ and } x \text{ is } \alpha \text{-th coordinate of } y \}.$$ 

Each $B_\alpha$ is bounded. For, if $U_\alpha$ is a neighborhood of identity in $G_\alpha$, put

$$U = U_\alpha \times \prod_{\beta \neq \alpha} G_\beta.$$ 

Indeed, $U$ is a neighborhood of identity in $G$. Therefore there is a positive integer $n$ with $B \subseteq nU$ so that $B_\alpha \subseteq nU_\alpha$. Now, it is not difficult to see that $B \subseteq \prod_{\alpha \in \Lambda} B_\alpha$. The converse is a consequence of [3, Theorem 1]. \[ \square \]

Note that in a general abelian topological group, every singleton is not necessarily bounded; in other words, not every neighborhood at identity is absorbing. For example, let $G$ be an abelian topological group. Put $H = G \times \mathbb{Z}_2$ with the product topology. Then $G \times \{0\}$ is a zero neighborhood which is not absorbing. On the other hand, when $G$ is a connected abelian group, by [2, Chapter III, Theorem 6], $G$ is absorbed by positive powers of any neighborhood at identity so that singletons will be bounded. Nevertheless, connectedness is a sufficient condition; consider the additive group $\mathbb{Q}$. In the case when in a topological group $G$, singletons are bounded, compact sets are bounded and therefore we can consider the notion "Heine-Borel" property. Recall that $G$ has the Heine-Borel
property if every closed bounded subset of $G$ is compact. Now, we have the following result.

**Corollary 3.2.** Suppose $(G_\alpha)_{\alpha \in \Lambda}$ are a family of topological groups in which singletons are bounded and $G = \prod_{\alpha \in \Lambda} G_\alpha$ with the product topology. Then singletons are also bounded in $G$.

**Corollary 3.3.** Let $(G_\alpha)_{\alpha \in \Lambda}$ be a family of topological groups in which singletons are bounded and $G = \prod_{\alpha \in \Lambda} G_\alpha$ with the product topology. Then $G$ has the Heine-Borel property if and only if each $G_\alpha$ has.

**Theorem 3.4.** Let $(G_\alpha)_{\alpha \in \Lambda}$ be a family of abelian topological groups and $G = \prod_{\alpha \in \Lambda} G_\alpha$ with the box topology. Then $B \subseteq G$ is bounded if and only if there exists a finite set $\{\alpha_1, \ldots, \alpha_k\}$ of indices such that $B \subseteq B_{\alpha_1} \times \ldots \times B_{\alpha_k} \times \prod_{\beta \in \Lambda - \{\alpha_1, \ldots, \alpha_k\}} \{e_\beta\}$, where $e_\beta$ denotes the identity element of $G_\beta$.

**Proof.** Suppose $B \subseteq G$ is bounded and there is a net $(c_\alpha)$ of non-identity elements of $(G_\alpha)$ such that each $c_\alpha$ belongs to a component of $B$. There is a neighborhood $U_\alpha$ at identity element $e_\alpha$ in $G_\alpha$ such that $c_\alpha \notin U_\alpha$. Partition the index set to a countable collection $(A_n)$. For each $\alpha \in A_n$, take a neighborhood $V_\alpha$ at identity with $n V_\alpha \subseteq U_\alpha$. Put $B_1 = \prod_{\alpha \in A_n} \{c_\alpha\}$.

Obviously, $B_1$ should be bounded in $G$. Now, suppose $V$ is a neighborhood at identity of the form $V = \prod_{\alpha \in \Lambda} V_\alpha$.

If it is not a difficult job to see that there is no $M > 0$ with $B_1 \subseteq M V$: for, in this case, $c_\alpha \in M V_\alpha \subseteq n V_\alpha \subseteq U_\alpha$, a contradiction. Therefore, for all but finitely many components, $B$ should have identity elements. Also, by a similar argument that we had in the first direction of the proof of Theorem 3.1, we conclude that for some $\{\alpha_1, \ldots, \alpha_k\}$, $B \subseteq B_{\alpha_1} \times \ldots \times B_{\alpha_k} \times \prod_{\beta \in \Lambda - \{\alpha_1, \ldots, \alpha_k\}} \{e_\beta\}$, in which, each $B_{\alpha_i}$ is bounded in $G_{\alpha_i}$. The other direction is trivial. \qed

**Remark 3.5.** Considering the proof of Theorem 3.4, we conclude that in the box topology, singletons are never bounded so that in such spaces compact sets are not bounded in general.

**Corollary 3.6.** Let $(G_\alpha)$ be a family of abelian topological groups and $G = \prod_{\alpha \in \Lambda} G_\alpha$ with the box topology. Then singletons are not bounded, in general. In particular, $G$ is never connected by [2, Chapter III, Theorem 6] even when all of $G_\alpha$’s are connected. Nevertheless, consider this point that by [2, Chapter III, Exercise 8], product topology preserves connectedness.

**References**


On generalized weakly symmetric 
\((LCS)_n\)-manifolds

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Abstract

The object of the present paper is to study generalized weakly symmetric and weakly Ricci symmetric \((LCS)_n\)-manifolds. Our aim is to bring out different type of curvature restrictions for which \((LCS)_n\)-manifolds are sometimes Einstein and some other time remain \(\eta\)-Einstein. Finally, the existence of such manifold is ensured by a non-trivial example.

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1. Introduction

The notion of Lorentzian concircular structure manifolds (briefly \((LCS)_n\)-manifolds) has been initiated by Shaikh [25]. Thereafter, a lot of study has been carried out. For details we refer [5, 12, 19, 27–30, 33] and the references therein.

The notion of weakly symmetric Riemannian manifold have been introduced by Tamássy and Binh [34]. Thereafter, a lot research has been carried out in this topic. For details, we refer to see [1, 2, 10, 13, 14, 21–24, 26, 31, 32] and the references there in.

In the spirit of Tamássy and Binh [34], a Riemannian manifold \((M^n, g)(n > 2)\), is said to be a weakly symmetric manifold, if its curvature tensor \(\bar{R}\) of type \((0, 4)\) is not identically zero and admits the identity

\[
\]

(1.1)

where \(A_1, B_1\) & \(D_1\) are non-zero 1-forms defined by \(A_1(X) = g(X, \sigma_1)\), \(B_1(X) = g(X, g_1)\) and \(D_1(X) = g(X, \pi_1)\), for all \(X\) and \(\bar{R}(Y, U, V, W) = g(\bar{R}(Y, U)V, W)\), \(\nabla\) being the operator of the covariant differentiation with respect to the metric tensor \(g\). An \(n\)-dimensional Riemannian manifold of this kind is denoted by \((WS)_n\)-manifold.
Keeping in tune with Dubey [11], the author have introduced the notion of a generalized weakly symmetric Riemannian manifold (which is abbreviated hereafter as \((GWS)_n\)-manifold). An \(n\)-dimensional Riemannian manifold is said to be generalized weakly symmetric if it admits the equation

\[
(\nabla_X \bar{R})(Y, U, V, W) = A_1(X)\bar{R}(Y, U, V, W) + B_1(Y)\bar{R}(X, U, V, W) + B_1(W)\bar{R}(Y, U, V, X) + D_1(Y)\bar{R}(X, U, V, W)
\]

\[
+ D_1(W)\bar{R}(Y, U, V, X) + A_2(X)\bar{G}(Y, U, V, W) + B_1(Y)\bar{G}(X, U, V, W) + B_1(U)\bar{G}(Y, X, V, W)
\]

\[
\]

(1.2)

where

\[
\bar{G}(Y, U, V, W) = [g(U, V)g(Y, W) - g(Y, V)g(U, W)]
\]

and \(A_i, B_i \& D_i\) are non-zero \(1\)-forms defined by \(A_i(X) = g(X, \sigma_i)\), \(B_i(X) = g(X, \tilde{\sigma}_i)\), and \(D_i(X) = g(X, \pi_i)\), for \(i = 1, 2\). The beauty of such \((GWS)_n\)-manifold is that it has the flavour of

(i) locally symmetric space [7] (for \(A_1 = B_1 = D_1 = 0\)),
(ii) locally recurrent space [36] (for \(A_1 \neq 0\), \(A_2 = B_1 = D_1 = 0\)),
(iii) generalized recurrent space [11] (for \(A_1 \neq 0\), \(B_1 = D_1 = 0\)),
(iv) pseudo symmetric space [8] (\(A_1 = B_1 = D_1 = H_1 \neq 0\), \(A_2 = B_2 = D_2 = 0\)),
(v) generalized pseudo symmetric space [3] (for \(A_1 = B_1 = D_1 = H_1 \neq 0\)),
(vi) semi-pseudo symmetric space [35] (\(A_1 = B_1 = D_1 \neq 0\)),
(vii) generalized semi-pseudo symmetric space [4] (\(A_1 = 0, B_1 = D_1 \neq 0\)),
(viii) almost pseudo symmetric space [9] (for \(A_1 = H_1 + K_1, B_1 = D_1 = H_1 \neq 0\) and \(A_2 = 0\)),
(ix) almost generalized pseudo symmetric space [6] (\(A_1 = H_1 + K_1, B_1 = D_1 = H_1 \neq 0\)),
(x) weakly symmetric space [34] (for \(A_1, B_1, D_1 \neq 0\), \(A_2 = B_2 = D_2 = 0\)).

Analogously, we have introduced generalized weakly Ricci symmetric \((LCS)_n\)-manifold which is defined as follows:

An \(n\)-dimensional Riemannian manifold is said to be generalized weakly Ricci symmetric if it admits the equation

\[
(\nabla_X S)(Y, Z) = A_1(X)S(Y, Z) + B_1(Y)S(X, Z) + D_1(Z)S(Y, X) + A_2(Y)g(Y, Z) + B_2(Y)g(X, Z) + D_2(Z)g(Y, X)
\]

(1.4)

where and \(A_i, B_i \& D_i\) are non-zero \(1\)-forms defined by \(A_i(X) = g(X, \sigma_i)\), \(B_i(X) = g(X, \tilde{\sigma}_i)\), and \(D_i(X) = g(X, \pi_i)\), for \(i = 1, 2\). The beauty of generalized weakly Ricci symmetric manifold is that it has the flavour of Ricci symmetric, Ricci recurrent, generalized Ricci recurrent, pseudo Ricci symmetric, generalized pseudo Ricci symmetric, semi-pseudo Ricci symmetric, generalized semi-pseudo Ricci symmetric, almost pseudo Ricci symmetric, almost generalized pseudo Ricci symmetric and weakly Ricci symmetric space as special cases.

Now, if the vectors associated to the \(1\)-forms \(A_1, B_1 \& D_1\) are respectively co-directional with that of \(A_2, B_2 \& D_2\) that is \(A_1(X) = \phi A_2(X), B_1(X) = \phi B_2(X) \& D_1(X) = \phi D_2(X) \forall X\), where \(\phi\) being a non-zero constant function, then the relation (1.4) turns into

\[
(\nabla_X Z)(Y, U) = A_1(X)Z(Y, U) + B_1(Y)Z(X, U) + D_1(U)Z(X, U)
\]

where \(Z(X, Y) = S(X, Y) + \phi g(X, Y)\) is well known \(Z\)-tensor introduced in ([15, 18]).

This leads to the following

**Proposition 1.1.** Every generalized weakly Ricci symmetric manifold is a weakly \(Z\)-symmetric manifold provided the vector fields associated to the \(1\)-forms \(A_1, B_1 \& D_1\) are co-directional with that of \(A_2, B_2 \& D_2\) respectively.
Our work is structured as follows. Section 2 is concerned with \((LCS)_n\)-manifolds and some known results. In section 3, we have investigated a generalized weakly symmetric \((LCS)_n\)-manifold and it is observed that such a space is an \(\eta\)-Einstein manifold provided \(B^*(\xi) \neq -\alpha\). We also tabled different type of curvature restrictions for which \((LCS)_n\)-manifolds are sometimes Einstein and some other time remain \(\eta\)-Einstein. Section 4, is concerned with a generalized weakly Ricci-symmetric \((LCS)_n\)-manifold which is found to be an \(\eta\)-Einstein space. Finally, we have constructed an example of a generalized weakly symmetric \((LCS)_n\)-manifold.

2. \((LCS)_n\)-manifolds and some known results

An \(n\)-dimensional Lorentzian manifold \(M\) is a smooth connected paracompact Hausdorff manifold with a Lorentzian metric \(g\), that is, \(M\) admits a smooth symmetric tensor field \(g\) of type \((0, 2)\) such that for each point \(p \in M\), the tensor \(g_p : T_pM \times T_pM \rightarrow R\) is a non-degenerate inner product of signature \((- +, \ldots, +)\), where \(T_pM\) denotes the tangent vector space of \(M\) at \(p\) and \(R\) is the real number space. A non-zero vector \(v \in T_pM\) is said to be timelike (resp., non-spacelike, null, spacelike) if it satisfies \(g_p(U, U) < 0\) (resp, \(\leq 0\), \(= 0, > 0\)), [20]. The category to which a given vector falls is called its causal character.

Let \(M^n\) be a Lorentzian manifold admitting a unit timelike concircular vector field \(\xi\), called the characteristic vector field of the manifold. Then we have

\[
g(\xi, \xi) = -1. \tag{2.1}\]

Since \(\xi\) is a unit concircular vector field, there exists a non-zero 1-form \(\eta\) such that for

\[
g(X, \xi) = \eta(X) \tag{2.2}\]

the equation of the following form holds

\[
(\nabla_X \eta)(Y) = \alpha \{g(X, Y) + \eta(X)\eta(Y)\} \quad (\alpha \neq 0) \tag{2.3}
\]

for all vector fields \(X, Y\) where \(\nabla\) denotes the operator of covariant differentiation with respect to the Lorentzian metric \(g\) and \(\alpha\) is a non-zero scalar function satisfies

\[
\nabla_X \alpha = (X\alpha) = \alpha(X) = \rho \eta(X), \tag{2.4}
\]

\(\rho\) being a certain scalar function. If we put

\[
\phi X = \frac{1}{\alpha} \nabla_X \xi, \tag{2.5}
\]

then from (2.3) and (2.5), we have

\[
\phi X = X + \eta(X)\xi, \tag{2.6}
\]

from which it follows that \(\phi\) is a symmetric \((1, 1)\) tensor. Thus the Lorentzian manifold \(M^n\) together with the unit timelike concircular vector field \(\xi\), its associated 1-form \(\eta\) and \((1, 1)\) tensor field \(\phi\) is said to be a Lorentzian concircular structure manifold (briefly \((LCS)_n\)-manifold) [5]. In a \((LCS)_n\)-manifold, the following relations hold [25]:

\[
\eta(\xi) = -1, \quad \phi \circ \xi = 0, \tag{2.7}
\]

\[
\eta(\phi X) = 0, \quad g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{2.8}
\]

\[
\eta(R(X, Y)Z) = (\alpha^2 - \rho)[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)]; \tag{2.9}
\]

\[
R(X, Y)\xi = (\alpha^2 - \rho)[\eta(Y)X - \eta(X)Y], \tag{2.10}
\]

\[
S(X, \xi) = (n - 1)(\alpha^2 - \rho)\eta(X) \tag{2.11}
\]
for any vector fields \( X, Y, Z \).

**Lemma 2.1.** Let \((M^n, g)\) be a \((LCS)_n\)-manifold. Then for any \( X; Y; Z \) the following relation holds:

\[
(\nabla_W S)(X, \xi) = (n-1)[\alpha(\alpha^2 - \rho)g(X, W) + 2(\alpha\rho - \beta)\eta(W)\eta(X)] - \alpha S(X, W) \tag{2.12}
\]

In this connection we would like to mention that equation (2.3) is the defining property of concircular or unit time-like torse-forming vector field. Moreover eq (2.4) and the consequent integrability relations (2.10) and (2.11) in [16] ensure that the unit time-like vector is an eigen vector of the Ricci tensor. Also, Proposition 3.7 of [17] ensures that the space-time is a generalized Robertson-Walker space-time, i.e. the metric is written in the form

\[
ds^2 = -dt^2 + f(t, x^\gamma)\hat{g}_{\alpha\beta}dx^\alpha dx^\beta,
\]

if and only if it admits a unit time-like torse-forming vector field. Moreover eq (2.4) and the consequent integrability relations (2.10) and (2.11) in [16] ensure that the unit time-like vector is an eigen vector of the Ricci tensor. Also, Proposition 3.7 of [17] ensures that the space-time is a generalized Robertson-Walker space-time, i.e. the metric is written in the form

\[
ds^2 = -dt^2 + f(t)^2\hat{g}_{\alpha\beta}dx^\alpha dx^\beta,
\]

\(\hat{g}\) being the metric tensor of a \(n - 1\) dimensional Riemannian manifold.

### 3. Generalized weakly symmetric \((LCS)_n\)-manifold

A non-flat \(n\)-dimensional \((LCS)_n\)-manifold \((M^n; g)\) \((n > 2)\), is termed as generalized weakly symmetric manifold, if its Riemannian curvature tensor \(\tilde{R}\) of type \((0; 4)\) is not identically zero and admits the identity

\[
+ D^*(W)\tilde{R}(Y, U, X, W) + \alpha^*(X)\tilde{R}(Y, U, V, W)
+ \beta^*(Y)\tilde{R}(X, U, V, W) + \gamma^*(V)\tilde{R}(Y, U, X, W) \tag{3.1}
\]

where

\[
G(Y, U, V, W) = [g(U, V)g(Y, W) - g(Y, V)g(U, W)] \tag{3.2}
\]

and \(A^*, B^*, D^*, \alpha^*, \beta^* \& \gamma^*\) are non-zero 1-forms which are defined as \(A^*(X) = g(X, \theta_1), B^*(X) = g(X, \phi_1), D^*(X) = g(X, \pi_1), \alpha^*(X) = g(X, \theta_2), \beta^*(X) = g(X, \phi_2)\) and \(\gamma^*(X) = g(X, \pi_2)\).

Now, contracting \(U\) over \(V\) in both sides of (3.1) we find

\[
-B^*(R(Y, X)W) + D^*(R(X, W)Y) + (n - 1)[\alpha^*(X)\eta(W)\eta(Y) + \beta^*(Y)\eta(X)\eta(W) + \gamma^*(W)\eta(X)\eta(W)]
\]

which yields

\[
(n - 1)[\alpha(\alpha^2 - \rho)g(X, W) + 2(\alpha\rho - \beta)\eta(W)\eta(X)] - \alpha S(X, W)
= (\alpha^2 - \rho)[(n - 1)[\alpha^*(X)\eta(W) + D^*(W)\eta(X)] + \eta(W)B^*(X) - g(X, W)B^*(\xi) + \eta(W)D^*(\xi) - \eta(X)D^*(W)] + B^*(\xi)S(X, W)
+ (n - 1)[\alpha\eta(W) + \beta^*(\xi)g(X, W) + \gamma^*(W)\eta(X)]
- \beta^*(\xi)\eta(X) + \gamma^*(\xi)\eta(W) - \gamma^*(W)\eta(X) \tag{3.4}
\]
for \( Y = \xi \). Setting \( X = W = \xi \) in the foregoing equation, we obtain

\[
-(2\alpha \rho - \beta) = (\alpha^2 - \rho)\{A^*(\xi) + B^*(\xi) + D^*(\xi)\}
+ [\alpha^*(\xi) + \beta^*(\xi) + \gamma^*(\xi)].
\]

(3.5)

In a weakly symmetric \((LCS)_n\)-manifold we have the relation (3.4). Setting \( X = \xi \) in (3.4) we get

\[
(n - 2)[(\alpha^2 - \rho)D^*(W) + \gamma^*(W)] = [(n - 1)\{(2\alpha \rho - \beta) + (\alpha^2 - \rho)\{A^*(\xi) + B^*(\xi)\}\}
+ (\alpha^2 - \rho)D^*(\xi)]\eta(W) + [(n - 1)\{\alpha^*(\xi) + \beta^*(\xi)\} + \gamma^*(\xi)]\eta(W).
\]

(3.6)

In view of (3.5), the relation (3.6) reduces to

\[
[(\alpha^2 - \rho)D^*(W) + \gamma^*(W)] = -[(\alpha^2 - \rho)D^*(\xi) + \gamma^*(\xi)]\eta(W).
\]

(3.7)

Again, contracting over \( Y \) and \( W \) in (3.1) we get

\[
(\nabla_X S)(U, V) = A^*(X)S(U, V) + B^*(R(X, U)V) + B^*(U)S(X, V)
+ D^*(V)S(U, X) + D^*(R(X, V)U) + (n - 1)\{\alpha^*(X)g(U, V)
+ \beta^*(U)g(X, V) + \gamma^*(\xi)g(U, W)\} + [\gamma^*(X)\eta(U)
- \gamma^*(\xi)g(U, X) + \beta^*(X)g(U, V) - \beta^*(U)g(X, W)].
\]

(3.8)

Setting \( V = \xi \) in (3.8) and using (2.12), (2.11), we get

\[
(n - 1)[(\alpha^2 - \rho)g(X, U) + (2\alpha \rho - \beta)\eta(U)\eta(X)] - S(X, U)
= (\alpha^2 - \rho)[(n - 1)\{A^*(X)\eta(U) + B^*(U)\eta(X)\} + B^*(X)\eta(U) - B^*(U)\eta(X)
+ D^*(U)\eta(X) - D^*(\xi)g(U, X)] + D^*(\xi)S(U, X) + (n - 1)\{\alpha^*(X)\eta(U)
+ \beta^*(U)\eta(X) + \gamma^*(\xi)g(U, V)\} + [\gamma^*(X)\eta(U)
- \gamma^*(\xi)g(U, X) + \beta^*(X)\eta(U) - \beta^*(U)\eta(X)],
\]

which turns into

\[
[(\alpha^2 - \rho)B(U) + \beta(U)] = -[(\alpha^2 - \rho)B(\xi) + \beta(\xi)]\eta(U)
\]

(3.10)

for \( X = \xi \) and

\[
[(\alpha^2 - \rho)A^*(X) + \alpha^*(X)] = -[(\alpha^2 - \rho)A^*(\xi) + \alpha^*(\xi)]\eta(X)
\]

(3.11)

for \( U = \xi \). In view of (3.5), (3.7), (3.10) and (3.11) we have

\[
(2\alpha \rho - \beta)\eta(X) = (\alpha^2 - \rho)[A^*(X) + B^*(X) + D^*(X)]
+ [\alpha^*(X) + \beta^*(X) + \gamma^*(X)].
\]

(3.12)

Now, making use of (3.10)-(3.12) in (3.4), we find that

\[
-\{\alpha + B^*(\xi)\}S(X, W) = [(n - 2)\beta^*(\xi) - (\alpha^2 - \rho)\{(n - 1)\alpha + B^*(\xi)\}]g(X, W)
- [(n - 2)\{(2\alpha \rho - \beta)\eta(W)\eta(X) + (\alpha^2 - \rho)\{A^*(X)\eta(W) + D^*(W)\eta(X)\} + \{\alpha^*(X)\eta(W) + \gamma^*(W)\eta(X)\}]
\]

(3.13)

which leaves

\[
S(X, W) = \left[ (\alpha^2 - \rho) + (n - 2) \left( \frac{(\alpha^2 - \rho)\alpha - \beta^*(\xi)}{\alpha + B^*(\xi)} \right) \right] g(X, W)
- \frac{(n - 2)\{(\alpha^2 - \rho)B^*(\xi) + \gamma^*(\xi)\} \eta(W)\eta(X)}{[\alpha + B^*(\xi)]}
\]

(3.14)
after a straightforward calculation. Approaching in a different manner, we can also have
\[
S(X,W) = \left[ (\alpha^2 - \rho) + (n-2) \left( \frac{(\alpha^2 - \rho)\alpha - \gamma^*(\xi)}{\alpha + D^*(\xi)} \right) \right] g(X,W) \\
- \frac{(n-2)[(\alpha^2 - \rho)D^*(\xi) + \beta^*(\xi)]}{[\alpha + D^*(\xi)]} \eta(W)\eta(X).
\]
This leads to the followings.

**Theorem 3.1.** A generalized weakly symmetric \((\text{LCS})_n\)-manifold \(M^n(\phi, \xi, \eta, g)(n > 2)\) is an \(\eta\)-Einstein provided that \(B^*(\xi) \neq -\alpha\).

**Theorem 3.2.** In an \((\text{LCS})_n\)-manifold the following table hold good

<table>
<thead>
<tr>
<th>Type of curvature restriction</th>
<th>Nature of the space corresponding to curvature restriction</th>
</tr>
</thead>
<tbody>
<tr>
<td>locally symmetric space</td>
<td>Einstein space</td>
</tr>
<tr>
<td>locally recurrent space</td>
<td>Einstein space</td>
</tr>
<tr>
<td>generalized recurrent space</td>
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</tr>
<tr>
<td>pseudo symmetric space</td>
<td>(\eta)-Einstein space</td>
</tr>
<tr>
<td>generalized pseudo symmetric space</td>
<td>(\eta)-Einstein space</td>
</tr>
<tr>
<td>semi-pseudo symmetric space</td>
<td>(\eta)-Einstein space</td>
</tr>
<tr>
<td>generalized semi-pseudo symmetric space</td>
<td>(\eta)-Einstein space</td>
</tr>
<tr>
<td>almost pseudo symmetric space</td>
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</tr>
<tr>
<td>weakly symmetric space</td>
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</tr>
</tbody>
</table>

Note that if a manifold is locally recurrent, then it is Ricci recurrent, i.e. \(\nabla_k R_{jl} = \beta_k R_{jl}\), for a non-null one form \(\beta_j\) which leaves after transvection \(\nabla_k R = \beta_k R\). Consequently, the manifold is Ricci flat as it is known that the scalar curvature of an Einstein manifold is constant. Thus we can state the following corollary.

**Corollary 3.3.** Every locally recurrent \((\text{LCS})_n\) manifold is Ricci flat.

### 4. Generalized weakly Ricci symmetric \((\text{LCS})_n\)-manifold

A non-flat \(n\)-dimensional \((\text{LCS})_n\)-manifold \((M^n; g)\) \((n > 2)\), is said to be a generalized weakly Ricci symmetric manifold, if its Ricci tensor \(S\) of type \((0, 2)\) is not identically zero and admits the identity
\[
(\nabla_X S)(Y, Z) = A^*_1(X)S(Y, Z) + B^*_1(Y)S(X, Z) + D^*_1(Z)S(Y, X) \\
+ A^*_2(X)g(Y, Z) + B^*_2(Y)g(X, Z) + D^*_2(Z)g(Y, X) \tag{4.1}
\]
where \(A^*_1, B^*_1 \& D^*_1\) are non-zero 1-forms which are defined as \(A^*_1(X) = g(X, \theta_i)\), \(B^*_i(X) = g(X, \phi_i), D^*_1(X) = g(X, \pi_i)\) for \(i = 1, 2\). Setting, \(Y = \xi\) in (4.1) and then making use of (2.12), we have
\[
(n-1)[\alpha(\alpha^2 - \rho)g(X, Z) + (2\alpha\rho - \beta)\eta(Z)\eta(X)] - \alpha S(X, Z) \\
= (\alpha^2 - \rho)(n-1)[A^*_1(X)\eta(Z) + D^*_1(Z)\eta(X)] + B^*_1(\xi)S(X, Z) \\
+ A^*_2(X)\eta(Z) + B^*_2(\xi)g(X, Z) + D^*_2(Z)\eta(X) \tag{4.2}
\]
which yields
\[(\alpha^2 - \rho)(n - 1)[A_1^*(\xi) + B_1^*(\xi) + D_1^*(\xi)] + [A_2^*(\xi) + B_2^*(\xi) + D_2^*(\xi)] \quad = -(n - 1)(2\alpha \rho - \beta), \tag{4.3}\]
for \(X = Z = \xi\).

Setting \(Z = \xi\) in (4.2) we obtain
\[(n - 1)(\alpha^2 - \rho)[A_1^*(X) + A_1^*(\xi)] = -[A_2^*(X) + A_2^*(\xi)\eta(X)]. \tag{4.4}\]

Proceeding in a similar manner we can find
\[(\alpha^2 - \rho)(n - 1)[B_1^*(X) + B_1^*(\xi)] = -[B_2^*(X) + B_2^*(\xi)\eta(X)], \tag{4.5}\]
\[(\alpha^2 - \rho)(n - 1)[D_1^*(X) + D_1^*(\xi)] = -[D_2^*(\xi) + D_2^*(\xi)\eta(X)]. \tag{4.6}\]

**Theorem 4.1.** In a generalized weakly Ricci symmetric (LCS)\(_n\)-manifold \(M^n(\phi, \xi, \eta, g)(n > 2)\) the 1-forms are related by
\[(\alpha^2 - \rho)(n - 1)[A_1^*(X) + B_1^*(X) + D_1^*(X)] + [A_2^*(X) + B_2^*(X) + D_2^*(X)] \quad = (n - 1)(2\alpha \rho - \beta)\eta(X). \tag{4.7}\]

**Proof.** Adding (4.4), (4.5) & (4.6) and then making use of (4.3) in the resultant, one can easily obtain (4.7). \(\square\)

Now, making use of (4.3)-(4.7) in (4.2), we find that
\[S(X, Z) = \left[\frac{(n - 1)\alpha(\alpha^2 - \rho) - B_2^*(\xi)}{\alpha + B_1^*(\xi)}\right]g(X, Z) - \left[\frac{(\alpha^2 - \rho)(n - 1)B_1^*(\xi) + B_2^*(\xi)}{\alpha + B_1^*(\xi)}\right]\eta(X)\eta(Z) \tag{4.8}\]

This leads to the followings

**Theorem 4.2.** A generalized weakly Ricci symmetric (LCS)\(_n\)-manifold \(M^n(\phi, \xi, \eta, g)\) is an \(\eta\)-Einstein provided that \(B_1^*(\xi) \neq -\alpha\).

**Theorem 4.3.** In an (LCS)\(_n\)-manifold the following table holds good

<table>
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</tr>
<tr>
<td>Ricci recurrent space</td>
<td>Einstein space</td>
</tr>
<tr>
<td>generalized Ricci-recurrent space</td>
<td>Einstein space</td>
</tr>
<tr>
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</tr>
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Note that if a manifold is Ricci recurrent, i.e. $\nabla_k R_{jl} = \beta_k R_{jl}$, for a non-null one form $\beta_k$ which leaves after transvection $\nabla_k R = \beta_k R$. Consequently, the manifold is Ricci flat as it is known that the scalar curvature of an Einstein manifold is constant. Thus we can state the following corollary.

**Corollary 4.4.** Every locally Ricci recurrent $(LCS)_n$ manifold is Ricci flat.

5. Existence of generalized weakly symmetric $(LCS)_3$-manifold

**Example 5.1.** Let $M^3(\phi, \xi, \eta, g)$ be an $(LCS)_n$-manifold $(M^3, g)$ with a $\phi$-basis

$$e_1 = e^z \left( x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right), e_2 = \phi e_1 = e^z \frac{\partial}{\partial y}, e_3 = \xi = e^z \frac{\partial}{\partial z}.$$  

Then from Koszul’s formula for Lorentzian metric $g$, we can obtain the Levi-Civita connection as follows

\[
\begin{align*}
\nabla_{e_1} e_3 &= -e^{2z} e_1, \\
\nabla_{e_2} e_3 &= -e^{2z} e_2, \\
\nabla_{e_3} e_3 &= 0,
\end{align*}
\]

From the above it can be easily seen that $(\phi, \xi, \eta, g)$ is an $(LCS)^3$ structure on $M$. Consequently $M^3(\phi, \xi, \eta, g)$ is an $(LCS)^3$-manifold with $\alpha = -e^{2z} \neq 0$ and $\rho = 2e^{4z}$. Using the above relations, one can easily calculate the non-vanishing components of the curvature tensor $\bar{R}$ (up to symmetry and skew-symmetry)

\[
\begin{align*}
\bar{R}(e_1, e_2, e_1, e_2) &= (1 - e^{2z}) e^{2z} \\
\bar{R}(e_1, e_3, e_1, e_3) &= -e^{4z} = \bar{R}(e_2, e_3, e_2, e_3).
\end{align*}
\]

Since $\{e_1, e_2, e_3\}$ forms a basis, any vector field $X, Y, U, V \in \chi(M)$ can be written as

\[
X = \sum_{i=1}^3 a_i e_i, \quad Y = \sum_{i=1}^3 b_i e_i, \quad U = \sum_{i=1}^3 c_i e_i, \quad V = \sum_{i=1}^3 d_i e_i,
\]

Then

\[
\begin{align*}
\bar{R}(X, Y, U, V) &= [(a_1 b_2 - a_2 b_1)(c_1 d_2 - c_2 d_1)](1 - e^{2z}) e^{2z} \\
&\quad -[(a_1 b_3 - a_3 b_1)(c_1 d_3 - c_3 d_1)] e^{4z} \\
&\quad -[(a_2 b_3 - a_3 b_2)(c_2 d_3 - c_3 d_2)] e^{4z} \\
&= T_1 \ (say), \\
\bar{R}(e_1, Y, U, V) &= -b_3(c_1 d_3 - c_3 d_1) e^{4z} + b_2(c_1 d_2 - c_2 d_1)(1 - e^{2z}) e^{2z} \\
&= \lambda_1 \ (say), \\
\bar{R}(e_2, Y, U, V) &= -b_3(c_2 d_3 - c_3 d_2) e^{4z} - b_1(c_1 d_2 - c_2 d_1)(1 - e^{2z}) e^{2z} \\
&= \lambda_2 \ (say), \\
\bar{R}(e_3, Y, U, V) &= b_1(c_1 d_3 - c_3 d_1) e^{4z} + b_2(c_2 d_3 - c_3 d_2) e^{4z} = \lambda_3 \ (say), \\
\bar{R}(X, e_1, U, V) &= a_3(c_1 d_3 - c_3 d_1) e^{4z} - a_2(c_1 d_2 - c_2 d_1)(1 - e^{2z}) e^{2z} \\
&= \lambda_4 \ (say), \\
\bar{R}(X, e_2, U, V) &= a_3(c_2 d_3 - c_3 d_2) e^{4z} + a_1(c_1 d_2 - c_2 d_1)(1 - e^{2z}) e^{2z} \\
&= \lambda_5 \ (say), \\
\bar{R}(X, e_3, U, V) &= -a_1(c_1 d_3 - c_3 d_1) e^{4z} - a_2(c_2 d_3 - c_3 d_2) e^{4z} = \lambda_6 \ (say), \\
\bar{R}(X, Y, e_1, V) &= -d_3(a_1 b_3 - a_3 b_1) e^{4z} + d_2(a_1 b_2 - a_2 b_1)(1 - e^{2z}) e^{2z} \\
&= \lambda_7 \ (say),
\end{align*}
\]
and the components which can be obtained from these by the symmetry properties. Now, we calculate the covariant derivatives of the non-vanishing components of the curvature tensor as follows

\[
(\nabla_{e_1} \bar{R})(X, Y, U, V) = e^{2z}[a_1 \lambda_3 + a_3 \lambda_1 + b_1 \lambda_6 + b_3 \lambda_4 + c_1 \lambda_9 + c_3 \lambda_7 + d_1 \lambda_{12} + b_3 \lambda_{10}],
\]

\[
(\nabla_{e_2} \bar{R})(X, Y, U, V) = e^{2z}[(a_1 + a_3) \lambda_2 + a_2 \lambda_3 + b_1 + b_3) \lambda_5 + b_2 \lambda_6 + c_1 + c_3) \lambda_8 + c_2 \lambda_9 + (d_1 + d_3) \lambda_{11} + d_2 \lambda_{12}]
+ e^{2z}[a_2 \lambda_1 + b_2 \lambda_4 + c_2 \lambda_7 + d_2 \lambda_{10}],
\]

\[
(\nabla_{e_3} \bar{R})(X, Y, U, V) = 2[(a_1 b_2 - a_2 b_1)(c_1 d_2 - c_2 d_1)](1 - 2e^{2z})e^{4z}
- 4[(a_1 b_3 - a_3 b_1)(c_1 d_3 - c_3 d_1)]e^{6z}
- 4[(a_2 b_3 - a_3 b_2)(c_2 d_3 - c_3 d_2)]e^{6z}.
\]

For the following choice of the 1-forms

\[
A_1^*(e_1) = \frac{e^{2z}[a_1 \lambda_3 + a_3 \lambda_1 + b_1 \lambda_6 + b_3 \lambda_4]}{T_1},
\]

\[
A_2^*(e_1) = \frac{c_1 \lambda_9 + c_3 \lambda_7 + d_1 \lambda_{12} + b_3 \lambda_{10}}{T_2},
\]

\[
A_1(e_2) = -\frac{e^{2z}[(a_1 + a_3) \lambda_2 + a_2 \lambda_3 + (b_1 + b_3) \lambda_5 + b_2 \lambda_6(c_1 + c_3) \lambda_8 + c_2 \lambda_9 + d_1]}{T_1},
\]
one can easily verify the relations
\[
\begin{align*}
A_2^*(e_2) &= -e^{2z}\left\{\lambda_1 + d_2\lambda_12\right\} + e^{2z}\left\{a_2\lambda_1 + b_2\lambda_1 + c_2\lambda_2 + d_2\lambda_10\right\}, \\
A_2^*(e_3) &= -4, \\
B_2^*(e_3) &= \frac{1}{a_3\lambda_3 + b_3\lambda_6}, \\
B_2^*(e_3) &= \frac{1}{a_3\lambda_3 + b_3\lambda_6}, \\
D_2^*(e_3) &= \frac{1}{c_3\lambda_9 + d_3\lambda_12}, \\
A_2^*(e_3) &= -2\left(a_1 b_2 - a_2 b_1\right)(c_1 d_2 - c_2 d_1)e^{2z}.
\end{align*}
\]

for \(i = 1, 2, 3\).

From the above, we can state the following theorem.

**Theorem 5.2.** There exists an \((LCS)_3\)-manifold \((M^3, g)\) which is a generalized weakly symmetric.

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**References**


On generalized weakly symmetric \((LCS)_n\)-manifolds


The equivalence of uninorms induced by the $U$-partial order

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Abstract
In this paper, some properties of an order induced by uninorms are investigated. In this aim, the set of incomparable elements with respect to the $U$-partial order for any uninorm is introduced and studied. Also, by defining such an order, an equivalence relation on the class of uninorms is defined and this equivalence is deeply investigated. Finally, another set of incomparable elements with respect to the $U$-partial order for any uninorm is introduced and studied.

Mathematics Subject Classification (2010). 03E72, 03B52

Keywords. uninorm, partial order, equivalence

1. Introduction
Uninorms were introduced by Yager and Rybalov [27]. Uninorms are special aggregation operators which have proven to be useful in many applications like fuzzy logic, expert systems, neural networks, fuzzy system modeling [16,18,26].

In recent years, the notation of the order induced by uninorms (nullnorms, triangular norms) has been studied widely. In this sense, in [22], $T$-partial order, denoted $\lesssim_T$, defined by means of $t$-norms on a bounded lattice has been introduced.

Based on these previous studies, in [1,17], respectively, $U$-partial order and $F$-partial order obtained from the uninorm and nullnorm have been introduced and some properties of these orders have been investigated.

The uninorms, nullnorms and $t$-norms were also studied by many other authors [2–4, 8–12,14,15,19–21,24].

In the present paper, we introduce the set of incomparable elements with respect to the $U$-partial order for any uninorm on $[0, 1]$. The main aim is to investigate some properties of this set. The paper is organized as follows. We shortly recall some basic notions in Section 2. In Section 3, we define the set of incomparable elements with respect to the $U$-partial order for any uninorm on $[0,1]$. Also, we determine the set of incomparable elements w.r.t. $U$-partial order for some special uninorms. So, we obtain general form for $t$-norms and $t$-conorms. Then, we define an equivalence relation on the class of uninorms on $[0,1]$. In section 4, we define the set $I^{(x)}_U$, consisting all incomparable elements with any $x \in (0,1)$ according to $\preceq_U$. Furthermore, we show that even if uninorms are equivalent
under this relation, it need not be the case that their partial orders coincide. Finally, we define and study another set of incomparable elements with respect to the $U$-partial order for any uninorm on $[0, 1]$.

2. Preliminaries

Definition 2.1 ([25]). Let $(L, \leq, 0, 1)$ be a bounded lattice. A triangular norm $T$ (briefly t-norm) is a binary operation on $L$ which is commutative, associative, non-decreasing in each variable and has neutral element $1$.

Definition 2.2 ([25]). Let $(L, \leq, 0, 1)$ be a bounded lattice. A triangular conorm $S$ (briefly t-conorm) is a binary operation on $L$ which is commutative, associative, non-decreasing in each variable and has neutral element $0$.

Example 2.3 ([23]). Well-known triangular norms and triangular conorms on $[0, 1]$ are:

\[
T_M(x, y) = \min(x, y)
\]

\[
T_P(x, y) = x + y
\]

\[
T_D(x, y) = \begin{cases} 
0 & (x, y) \in [0, 1)^2, \\
\min(x, y) & \text{otherwise.}
\end{cases}
\]

\[
S_M(x, y) = \max(x, y)
\]

\[
S_P(x, y) = x + y - x.y
\]

\[
S_D(x, y) = \begin{cases} 
1 & (x, y) \in (0, 1]^2, \\
\max(x, y) & \text{otherwise.}
\end{cases}
\]

Extremal t-norms $T_\wedge$ and $T_W$ on $L$ are defined as follows, respectively:

\[
T_\wedge(x, y) = x \wedge y
\]

\[
T_W(x, y) = \begin{cases} 
x & \text{if } y = 1, \\
y & \text{if } x = 1, \\
0 & \text{otherwise.}
\end{cases}
\]

Similarly, the t-conorms $S_\vee$ and $S_W$ can be defined as above.

Especially we have obtained $T_W = T_D$ and $T_\wedge = T_M$ for $L = [0, 1]$.

Definition 2.4 ([7]). A t-norm $T$ on $L$ is divisible if the following condition holds:

\[\forall x, y \in L \text{ with } x \leq y \text{ there is a } z \in L \text{ such that } x = T(y, z).\]

The infimum t-norm $T_\wedge$ is divisible: $x \leq y$ is equivalent to $x \wedge y = x$. A basic example of a non-divisible t-norm on an arbitrary bounded lattice $L$ (i.e., card $L > 3$) is the t-norm $T_W$. Similarly, t-conorm $S_\vee$ is divisible. $S_W$ is a non-divisible t-conorm on an arbitrary bounded lattice $L$ (i.e., card$L > 3$).

Definition 2.5 ([6]). Let $(L, \leq, 0, 1)$ be a bounded lattice. An operation $U : L^2 \rightarrow L$ is called a uninorm on $L$, if it is commutative, associative, non-decreasing in each variable and has a neutral element $e \in L$.

We denote by $U(e)$ the set of all uninorms on $L$ with the neutral element $e \in L$.

\[
A(e) = ([0, e] \times [e, 1) \cup [e, 1) \times [0, e] \text{ for } e \in (0, 1).
\]

Definition 2.6 ([6]). A uninorm $U$ is called idempotent if $U(x, x) = x$ for all $x \in [0, 1]$.

Definition 2.7 ([22]). Let $L$ be a bounded lattice, $T$ be a t-norm on $L$. The order defined as following is called a $T$- partial order (triangular order) for t-norm $T$:

\[x \preceq_T y :\Leftrightarrow T(\ell, y) = x \text{ for some } \ell \in L.\]
**Definition 2.8** ([17]). Let $L$ be a bounded lattice, $S$ be a t-conorm on $L$. The order defined as following is called a $S$—partial order for t-conorm $S$:

$$x \preceq_S y :\iff S(\ell, x) = y$$

for some $\ell \in L$.

**Definition 2.9** ([17]). Let $(L, \preceq, 0, 1)$ be a bounded lattice and $U$ be a uninorm with neutral element $e$ on $L$. Define the following relation, for $x, y \in L$, as

$$x \preceq_U y \iff \begin{cases} 
\text{if } x, y \in [0, e] \text{ and there exist } k \in [0, e] \text{ such that } U(y, k) = x \text{ or,} \\
\text{if } x, y \in [e, 1] \text{ and there exist } \ell \in [e, 1] \text{ such that } U(x, \ell) = y \text{ or,} \\
\text{if } (x, y) \in L^* \text{ and } x \leq y,
\end{cases}$$

(2.1)

where $I_e = \{x \in L \mid x \parallel e\}$ and $L^* = [0, e] \times [e, 1] \cup [0, e] \times I_e \cup [e, 1] \times I_e \cup [0, e] \cup I_e \times [0, e] \cup I_e \times [e, 1] \cup I_e \times I_e$.

**Proposition 2.10** ([17]). The relation $\preceq_U$ defined in (2.1) is a partial order on $L$.

**Proposition 2.11** ([13]). Let $T$ be a t-norm on $[0, 1]$. $T$ is divisible if and only if $T$ is continuous.

3. **The set $K_U \subset [0, 1]$ consisting of incomparable elements with respect to $\preceq_U$ on $[0, 1]$**

In this section, we study the set of elements being incomparable with some other element with respect to the $U$-partial order $\preceq_U$ with some uninorm $U$ on $[0, 1]$.

Let $U$ be a uninorm on $[0, 1]$ and let $K_U$ be defined by

$$K_U = \{x \in (0, 1) \mid \text{for some } y \in (0, 1), \ [x < y \text{ and } x \not\preceq_U y]$$

or $[y < x \text{ and } y \not\preceq_U x]\}.$$

Note that an element $x \in K_U$ is not necessarily incomparable with all elements $y \in [0, 1] \setminus \{0, 1, x\}$.

We want to determine above introduced set for the smallest and greatest uninorms on $[0, 1]$.

**Proposition 3.1.** Let $e \in [0, 1]$. Consider the uninorm $U_e : [0, 1]^2 \to [0, 1]$ with neutral element $e$ defined by

$$U_e(x, y) = \begin{cases} 
0 & (x, y) \in [0, e]^2, \\
\max(x, y) & (x, y) \in [e, 1]^2, \\
\min(x, y) & \text{otherwise.}
\end{cases}$$

Then, $K_{U_e} = (0, e)$.

**Proof.** Let $x \in (0, e)$ and $y < x < 1$. Let us show that $y \not\preceq_{U_e} x$. We consider $y \not\preceq_{U_e} x$. Then, there exists an element $\ell \in [0, e]$ such that $y = U_e(x, \ell)$. Since $y \neq x$, it is not possible $\ell = e$. Since $(x, \ell) \in [0, e)^2$, it is obtained that

$$y = U_e(x, \ell) = 0,$$

a contradiction. Since for any $x \in (0, e)$, there exists an element $y < x < 1$ such that $y \not\preceq_{U_e} x$. So, we have $x \in K_{U_e}$. Thus, it is obtained that $(0, e) \subseteq K_{U_e}$.

Conversely, let $x \in K_{U_e}$. We need to show that $x \in (0, e)$. Suppose that $x \notin (0, e)$. Since $x \in K_{U_e}$, there exists an element $y \in (0, 1)$ such that $x < y$ and $x \not\preceq_{U_e} y$ or $y < x$ and $y \not\preceq_{U_e} x$. 


We assume that \( e \leq x \).

- Let \( x < y \) and \( y \not\leq_U x \). Since \( e \leq x \), it must be the case that \( e < y \). Since \( x < y \), we have \( \max(x, y) = y \). By the definition of \( U_e \), we obtain that
  \[
  y = \max(x, y) = U_e(x, y).
  \]
  Then, it holds that \( x \not\leq_U y \), a contradiction.

- Let \( y < x \) and \( y \not\leq_U x \). Since \( e \leq x \), we have that \( e < y \). Otherwise, if \( y \leq e \), then it is obtained that \( y \not\leq_U x \), a contradiction by the definition of \( \leq_U \). Since \( y < x \), it must be \( \max(x, y) = x \). By the definition of \( U_e \), we obtain that
  \[
  x = \max(x, y) = U_e(x, y).
  \]
  Then, it holds that \( y \not\leq_U x \), a contradiction.

  We consider that \( x = 0 \). In this case, we have \( 0 \not\leq_U y \), a contradiction. So, it must be the case that \( x \in (0, e) \). Thus, it is obtained that \( K_{U_e} \subseteq (0, e) \). Consequently, we can show that \( K_{U_e} = (e, 1) \).

Corollary 3.2. For the drastic product t-norm \( T_D \) on \([0, 1]\), \( K_{T_D} = (0, 1) \).

Corollary 3.3. For the maximum t-conorm \( S_M \) on \([0, 1]\), \( K_{S_M} = \emptyset \).

Proposition 3.4. Let \( e \in [0, 1] \). Consider the uninorm \( U_e : [0, 1]^2 \to [0, 1] \) with neutral element \( e \) defined by

\[
U_e(x, y) = \begin{cases} 
\min(x, y) & (x, y) \in [0, e]^2, \\
1 & (x, y) \in (e, 1]^2, \\
\max(x, y) & \text{otherwise}.
\end{cases}
\]

Then, \( K_{U_e} = (e, 1) \).

Corollary 3.5. For the minimum t-norm \( T_M \) on \([0, 1]\), \( K_{T_M} = \emptyset \).

Corollary 3.6. For the drastic sum t-conorm \( S_D \) on \([0, 1]\), \( K_{S_D} = (0, 1) \).

Lemma 3.7. Let \( U \) be a uninorm on \([0, 1]\). Then, \( 0 \leq_U x, x \leq_U x \) and \( x \leq_U 1 \) for all \( x \in [0, 1] \).

Theorem 3.8. ([18]). Let \( e \in [0, 1] \). \( U \in \mathcal{U}(e) \) if and only if

\[
U(x, y) = \begin{cases} 
T_U & (x, y) \in [0, e]^2, \\
S_U & (x, y) \in [e, 1]^2, \\
C & (x, y) \in A(e),
\end{cases}
\]

where \( T_U \) and \( S_U \) are operations respectively isomorphic with some triangular norm and triangular conorm and increasing operation \( C : A(e) \to [0, 1] \) fulfills

\[
\min(x, y) \leq C(x, y) \leq \max(x, y) \text{ for } (x, y) \in A(e).
\]

Proposition 3.9. Let \( U \) be a uninorm on \([0, 1]\) with neutral element \( e \) in Theorem 3.8. If \( T_U \) and \( S_U \) are continuous, then \( K_U = \emptyset \).

Corollary 3.10. Let \( e \in [0, 1] \). Consider the uninorms \( U_e^{\text{min}} \) and \( U_e^{\text{max}} \) as unique idempotent uninorm \( U_e^{\text{min}} \) and \( U_e^{\text{max}} \), respectively:

\[
U_e^{\text{min}}(x, y) = \begin{cases} 
\max(x, y) & (x, y) \in [e, 1]^2, \\
\min(x, y) & \text{otherwise}.
\end{cases}
\]

\[
U_e^{\text{max}}(x, y) = \begin{cases} 
\min(x, y) & (x, y) \in [0, e]^2, \\
\max(x, y) & \text{otherwise}.
\end{cases}
\]
Then, it is obtained that $K_{U_{\min}} = \emptyset$ and $K_{U_{\max}} = \emptyset$.

Theorem 3.11. ([18]). Let $U : [0, 1]^2 \rightarrow [0, 1]$ be a uninorm with neutral element $e \in [0, 1]$. Then the sections $x \mapsto U(x, 1)$ and $x \mapsto U(x, 0)$ are continuous in each point except perhaps for $e$ if and only if $U$ is given by one of the following formulas.

(a) If $U(0, 1) = 0$, then

$$U(x, y) = \begin{cases} eT(\frac{x}{e}, \frac{y}{x}) & , (x, y) \in [0, e]^2 \\ e + (1 - e)S(\frac{x - e}{1 - e}, \frac{y - e}{1 - e}) & , (x, y) \in [e, 1]^2 \\ \min(x, y) & , (x, y) \in A(e), \end{cases}$$

(3.1)

where $T$ is a t-norm and $S$ is a t-conorm.

(b) If $U(0, 1) = 1$, then the same structure holds, changing minimum by maximum in $A(e)$.

The class of uninorms as in case (a) will be denoted by $\mathcal{U}_{\min}$ and the class of uninorms as in case (b) by $\mathcal{U}_{\max}$. We will denote a uninorm $U$ in $\mathcal{U}_{\min}$ with underlying t-norm $T$, underlying t-conorm $S$ and neutral element $e$ by $U \equiv \langle T, e, S \rangle_{\min}$ and in a similar way, a uninorm in $\mathcal{U}_{\max}$ by $U \equiv \langle T, e, S \rangle_{\max}$.

Proposition 3.12. Let $U$ be a uninorm such that $U \equiv \langle T, e, S \rangle_{\min}$ or $U \equiv \langle T, e, S \rangle_{\max}$. Then,

$$K_U = eK_T \cup (e + (1 - e)K_S).$$

Proof. For any two elements $x, y \in [0, e]$, the $U$-comparability of $x$ and $y$ is equivalent to the $T$-comparability of $\frac{x}{e}$ and $\frac{y}{e}$. Therefore, $K_U$ contains the set $eK_T$. Similarly, for any two elements $x, y \in [e, 1]$, the $U$-comparability of $x$ and $y$ is equivalent to the $S$-comparability of $\frac{x - e}{1 - e}$ and $\frac{y - e}{1 - e}$.

Therefore, $K_U$ contains the set $e + (1 - e)K_S$.

For any two elements $x, y \in A(e)$, we have that $K_U = \emptyset$ by the definition of $\preceq_U$. On the other hand, if $(x, y) \in [0, e]^2 \cup [e, 1]^2$, then proof is trivial. Also, if $(x, y) \in A(e)$, then proof is trivial by the definition of $\mathcal{U}_{\min}$ or $\mathcal{U}_{\max}$. \hfill \Box

As an example of application of the previous proposition we consider $U = \langle T_D, e, S_M \rangle$ such that drastic product t-norm and maximum t-conorm. Then,

$$K_U = (0, e).$$

Proposition 3.13. Let $U$ be a uninorm such that $U \equiv \langle T, e, S \rangle_{\min}$ or $U \equiv \langle T, e, S \rangle_{\max}$. Then,

i) If $x, y \in [0, e]$, then $x \preceq_U y$ if and only if $\frac{x}{e} \preceq_T \frac{y}{e}$.

ii) If $x, y \in [e, 1]$, then $x \preceq_U y$ if and only if $\frac{x - e}{1 - e} \preceq_S \frac{y - e}{1 - e}$.

Proof. (i) (Necessity) Let $x \preceq_U y$ for $x, y \in [0, e]$. Then, there exists an element $k \in [0, e]$ such that $U(y, k) = x$. By the definition of $U$, it must be the case that $x = eT(\frac{y}{e}, \frac{k}{e})$. Then, we have $\frac{x}{e} = T(\frac{y}{e}, \frac{k}{e})$. Since $\frac{k}{e} \leq 1$, it is obtained that $\frac{x}{e} \preceq_T \frac{y}{e}$.

(Sufficiency) Let $\frac{x}{e} \preceq_T \frac{y}{e}$ for $x, y \in [0, e]$. Then, there exists an element $\ell \in [0, e]$ such that $T(\frac{y}{e}, \ell) = \frac{x}{e}$, that is $x = eT(\frac{y}{e}, \ell)$. Clearly, $x = eT(\frac{y}{e}, \ell)$. This means that $x = U(y, \ell)$. Since $\ell e \leq e$, it is obtained that $x \preceq_U y$.

(ii) (Necessity) It can be shown that the case (i).

(Sufficiency) Let $\frac{x - e}{1 - e} \preceq_S \frac{y - e}{1 - e}$ for $x, y \in [e, 1]$. Then, there exists an element $\ell \in [e, 1]$ such that $S(\frac{x - e}{1 - e}, \ell) = \frac{y - e}{1 - e}$, that is $y = e + (1 - e)S(\frac{x - e}{1 - e}, \ell)$. It is clear that $y = e + (1 - e)S(\frac{x - e}{1 - e}, \ell)$. Since $e \leq \ell - \ell e + e$, it is obtained that $x \preceq_U y$. \hfill \Box

The $\preceq_U$-partial order introduced above allows us to introduce the next equivalence relation on the class of all uninorms on $[0, 1]$. 

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Definition 3.14. Define a relation $\beta$ on the class of all uninorms on $[0, 1]$ by $U_1 \beta U_2$, 
$$U_1 \beta U_2 \iff K_{U_1} = K_{U_2}.$$ 

Lemma 3.15. The relation $\beta$ given in Definition 3.14 is an equivalence relation.

Definition 3.16. For a given uninorm $U$ on $[0, 1]$, we denote by $\overline{U}$ the $\beta$ equivalence class linked to $U$, i.e., 
$$\overline{U} = \{U' \mid U' \beta U\}.$$ 

Proposition 3.17. The set $[0, 1]/\beta$, is uncountably infinite.

Proof. Let $e_1, e_2 \in (0, 1)$ and $e_1 \neq e_2$. Suppose that $e_1 < e_2$.

Consider the functions on $[0, 1]$ defined as follows:
$$U_{e_1}(x, y) = \begin{cases} 0 & (x, y) \in [0, e_1)^2, \\ \max(x, y) & (x, y) \in [e_1, 1)^2, \\ \min(x, y) & \text{otherwise}, \end{cases}$$

and
$$U_{e_2}(x, y) = \begin{cases} 0 & (x, y) \in [0, e_2)^2, \\ \max(x, y) & (x, y) \in [e_2, 1)^2, \\ \min(x, y) & \text{otherwise}. \end{cases}$$

The functions $U_{e_1}$ and $U_{e_2}$ are uninorms on $[0, 1]$ with neutral elements $e_1$ and $e_2$, respectively. Since $K_{U_{e_1}} = (0, e_1)$ and $K_{U_{e_2}} = (0, e_2)$ by Proposition 3.1 and $e_1 < e_2$, then we have that the uninorms $U_{e_1}$ and $U_{e_2}$ are not equivalent under the relation $\beta$. So, we obtain that $\overline{U_{e_1}} \neq \overline{U_{e_2}}$.

Define the mapping $\delta : (0, 1) \to [0, 1]/\beta$ by
$$\delta(e) = \overline{U_e}.$$ 

We showed that if $e_1 \neq e_2$, then it must be $\delta(e_1) \neq \delta(e_2)$. So, $\delta$ is an injective function, it is obtained that $|[0, 1)| \leq |[0, 1]/\beta|$. So, the set $[0, 1]/\beta$ has uncountably infinite cardinality.

Definition 3.18. Let $U_1$ and $U_2$ be two uninorms on $[0, 1]$. If $\preceq_{U_1} \subseteq \preceq_{U_2}$, then we say that $U_2$ is order-stronger than $U_1$, or equivalently, that $U_1$ is order-weaker than $U_2$.

In [5], it was shown that for t-norms $T_W$ and $T_\wedge$ on $L$, $T_W$ is the order-weakest and $T_\wedge$ is the order-strongest t-norm, i.e., $\preceq_{T_W} \subseteq \preceq_T \subseteq \preceq_{T_\wedge}$. But for the uninorms, it need not be that case. Now, let us investigate the following example.

Example 3.19. Let us consider a smallest uninorm $U_e : [0, 1]^2 \to [0, 1]$ in Proposition 3.1 and a greatest uninorm $\overline{U_e} : [0, 1]^2 \to [0, 1]$ in Proposition 3.4 with neutral elements $e$. We claim that $U_e$ is not order-weakest and $\overline{U_e}$ is not order-strongest uninorm. We choose $e = \frac{1}{3}$. Since $U_e(\frac{1}{2}, \frac{2}{3}) = \frac{2}{3}$, it is obtained that $\frac{1}{2} \preceq_{U_e} \frac{2}{3}$.

On the other hand, $\frac{1}{2} \not\preceq_{\overline{U_e}} \frac{2}{3}$. On the condition that $\frac{1}{2} \preceq_{\overline{U_e}} \frac{2}{3}$, there exists an element $\ell \geq \frac{1}{2}$ such that $\overline{U_e}(\frac{1}{2}, \ell) = \frac{2}{3}$. By the definition of $\overline{U_e}$, we have that $\overline{U_e}(\frac{1}{2}, \ell) = \min(\frac{1}{2}, \ell) = \frac{1}{3}$ or $\overline{U_e}(\frac{1}{2}, \ell) = \max(\frac{1}{2}, \ell) = \frac{1}{3}$. In first condition, we have $\ell = \frac{3}{3}$, a contradiction. In second condition, it is obtained that $\ell = \frac{3}{3}$ and $(\frac{1}{2}, \frac{2}{3}) \in (0, 1)^2$, a contradiction. So, $\frac{1}{2} \not\preceq_{\overline{U_e}} \frac{2}{3}$.

4. About the set $J_{U}$ consisting all incomparable elements with any $x \in (0, 1)$ according to $\preceq_U$

Definition 4.1. Let $U$ be a uninorm on $[0, 1]$ and let $J_{U}^{(x)}$ for $x \in (0, 1)$ be defined by
$$J_{U}^{(x)} = \{y \in (0, 1) \mid [x < y \text{ and } x \not\preceq_U y] \text{ or } [y < x \text{ and } y \not\preceq_U x]\}.$$
After that the notation will be \( J_U^{(x)} \) to denote the set of all incomparable elements with \( x \in (0, 1) \) according to \( \preceq_U \). The set \( J_U^{(x)} \) in Definition 4.1 can be defined on a bounded lattice and uninorm acting on this lattice.

We want to determine above introduced set for the smallest and greatest uninorms on \([0, 1]\).

**Proposition 4.2.** Let us consider the smallest uninorm \( U_e \) with neutral element \( e \) in Proposition 3.1. Then,

a) \( J_{U_e}^{(x)} = \{ y \in (0, e) \mid x \neq y \} \) for \( x \in (0, e) \).

b) \( J_{U_e}^{(x)} = \emptyset \) for \( e \leq x \).

a) Let \( y \in (0, e) \) be arbitrary such that \( x \neq y \) for \( x \in (0, e) \). Let us show that \( y \in J_{U_e}^{(x)} \). Suppose that \( y \notin J_{U_e}^{(x)} \). That is, \( y \) is comparable to \( x \) according to \( \preceq_{U_e} \). Then, \( y < x \) and \( y \preceq_{U_e} x \) or \( x < y \) and \( x \preceq_{U_e} y \).

- Let \( y < x \) and \( y \preceq_{U_e} x \). Then there exists an element \( k \in [0, e] \) such that \( y = U_e(x, k) \). Since \( y \neq 0 \), it must be the case that

\[
y = \min(x, k) = U_e(x, k)
\]

or

\[
y = \max(x, k) = U_e(x, k).
\]

If \( y = \min(x, k) = U_e(x, k) \), then we have that \( y = k \) by \( x \neq y \). In this case, we have that \( y = 0 \), a contradiction from \( y, x < e \). If \( y = \max(x, k) = U_e(x, k) \), then we have that \( y = k \) by \( x \neq y \). Then, we have that \( y = 0 \), a contradiction from \( y, x < e \). So, \( y \in J_{U_e}^{(x)} \).

- Let \( x < y \) and \( x \preceq_{U_e} y \). Then, there exists an element \( \ell \in [0, e] \) such that \( x = U_e(y, \ell) \). Since \( x \neq 0 \), it must be the case that

\[
y = \min(x, k) = U(x, k)
\]

or

\[
y = \max(x, k) = U(x, k)
\]

and similar argument can be done for this case. So, \( y \in J_{U_e}^{(x)} \).

Conversely, let \( y \in J_{U_e}^{(x)} \) be arbitrary for \( x \in (0, e) \). By Lemma 3.7 it must be \( x \neq y \). So, we need to show that \( y \notin (0, e) \). Suppose that \( y \notin (0, e) \). First, we assume that \( y = 0 \). In this case \( 0 \preceq_{U_e} x \), a contradiction. So, it must be \( y \neq 0 \). Let \( e \leq y \). Since \( y \in J_{U_e}^{(x)} \), we have \( y < x \) and \( y \notin U_e(x) \) or \( x < y \) and \( x \notin U_e(y) \).

Let \( y < x \) and \( y \notin U_e(x) \). Since \( x = \max(x, y) \), by the definition of \( U_e \),

\[
x = \max(x, y) = U_e(x, y).
\]

It is obtained that \( y \preceq_{U_e} x \), a contradiction. Similarly, it can be shown that \( x \notin U_e(y) \) for \( x < y \). So, it must be the case that \( y \in (0, e) \). Consequently, we can show that \( J_{U_e}^{(x)} = \{ y \in (0, e) \mid x \neq y \} \) for \( x \in (0, e) \).

b) Suppose that \( J_{U_e}^{(x)} \neq \emptyset \). Let \( y \in J_{U_e}^{(x)} \) be arbitrary. So, it must be \( y < x \) and \( y \notin U_e(x) \) or \( x < y \) and \( x \notin U_e(y) \).

Let \( y < x \) and \( y \notin U_e(x) \). If \( y < e \leq x \), by the definition of \( \preceq_{U_e} \), it would be \( y \preceq_{U_e} x \), a contradiction. If \( e < y < x \), then we have that \( x = \max(x, y) \). By the definition of \( U_e \), we have that \( x = \max(x, y) = U_e(x, y) \). It leads to \( y \notin U_e(x) \), a contradiction. Similarly it can be shown that \( x < y \) and \( x \notin U_e(y) \). So, \( J_{U_e}^{(x)} = \emptyset \) for \( e \leq x \).
Proposition 4.3. Let us consider the greatest uninorm \( U_e \) with neutral element \( e \) in Proposition 3.4. Then,

a) \( J_{U_e}^{(x)} = \{ y \in (e, 1) \mid x \neq y \} \) for \( x \in (e, 1) \).

b) \( J_{U_e}^{(x)} = \emptyset \) for \( x \leq e \).

Since we will use the results of the following example in subsequent propositions, it is useful to elaborate on the example.

Example 4.4. The uninorm \( U := U_{\min(T^n M, S_M, \frac{1}{2})} : [0, 1]^2 \rightarrow [0, 1] \) with neutral element \( e = \frac{1}{2} \) defined as follows:

\[
U_{\min(T^n M, S_M, \frac{1}{2})}(x, y) = \begin{cases} 
0 & (x, y) \in [0, \frac{1}{2}]^2 \text{ and } x + y \leq \frac{1}{2}, \\
\max(x, y) & (x, y) \in [\frac{1}{2}, 1]^2, \\
\min(x, y) & \text{otherwise}.
\end{cases}
\]

Then,

a) \( J_{U}^{(x)} = \{ y \in (0, \frac{1}{2} - x) \mid x \neq y \} \) for \( x \in (0, \frac{1}{2}) \).

b) \( J_{U}^{(x)} = \emptyset \) for \( \frac{1}{2} \leq x \).

Now, we want to show this claims.

a) Let \( x < \frac{1}{2} \) and \( y \in (0, \frac{1}{2} - x] \). Let us show that \( y \in J_{U}^{(x)} \). We assume that \( y \notin J_{U}^{(x)} \), i.e., \( y < x \) and \( y \preceq_U x \) or \( x < y \) and \( x \preceq_U y \).

Let \( y < x \) and \( y \preceq_U x \). Then, there exists an element \( k \in [0, \frac{1}{2}] \) such that \( y = U(x, k) \). Since \( y \neq 0 \), it must be the case that

\[
y = \min(x, k) = U(x, k).
\]

Since \( x \neq y \), it is obtained that \( y = k \). Since \( x, y < \frac{1}{2} \) and \( y \in (0, \frac{1}{2} - x] \), it is obtained that \( x + y \leq \frac{1}{2} \), a contradiction. Then, it must be the case that \( y \in J_{U}^{(x)} \).

Let \( x < y \) and \( x \preceq_U y \). Since \( y \in (0, \frac{1}{2} - x] \), it is not possible the case \( x < \frac{1}{2} < y \). So, \( x < y < \frac{1}{2} \). Then, there exists an element \( \ell \in [0, \frac{1}{2}] \) such that \( x = U(y, \ell) \). Since \( x \neq 0 \), it must be

\[
x = \min(y, \ell) = U(y, \ell)
\]

and similar argument can be done for this case. So, \( y \in J_{U}^{(x)} \).

Conversely, let \( y \in J_{U}^{(x)} \). Suppose that \( y \notin (0, \frac{1}{2} - x] \). Since \( y \in J_{U}^{(x)} \), we have \( y < x \) and \( y \preceq_U x \) or \( x < y \) and \( x \preceq_U y \). Let \( y > \frac{1}{2} - x \).

Let \( y < x \) and \( y \preceq_U x \). Since \( y = \min(x, y) \) and \( x + y > \frac{1}{2} \), by the definition of \( U \), we have that

\[
y = \min(x, y) = U(x, y).
\]

It is obtained that \( y \preceq_U x \), a contradiction.

Let \( x < y \). Similarly it can be shown that \( x \preceq_U y \).

If \( y = 0 \), we have that \( 0 \preceq_U x \), a contradiction. So it must be \( y \in (0, \frac{1}{2} - x] \). Consequently, we can show that \( J_{U}^{(x)} = \{ y \in (0, \frac{1}{2} - x) \mid x \neq y \} \) for \( x \in (0, \frac{1}{2}) \).

b) Let \( \frac{1}{2} \leq x \). We assume that \( J_{U}^{(x)} \neq \emptyset \). Let \( y \in J_{U}^{(x)} \) be arbitrary. That is, \( y < x \) and \( y \preceq_U x \) or \( x < y \) and \( x \preceq_U y \).

Let \( y < x \) and \( y \preceq_U x \). If \( y < \frac{1}{2} \leq x \), by the definition of \( \preceq_U \), it would be \( y \preceq_U x \), a contradiction. If \( \frac{1}{2} < y < x \), then we have that \( x = \max(x, y) \). By the definition of \( U \), we have that \( x = \max(x, y) = U(x, y) \). It leads to \( y \preceq_U x \), contradiction. Similarly, if \( x < y \) and \( x \preceq_U y \), then we have similar contradiction. So, it is obtained that \( J_{U}^{(x)} = \emptyset \) for \( \frac{1}{2} \leq x \).
Lemma 4.5. Let \( U \) be a uninorm on \([0, 1]\). Then \( K_U = \bigcup_{x \in [0, 1]} J_U^{(x)} \).

Proposition 4.6. Let \( U_1 \) and \( U_2 \) be two uninorms on \([0, 1]\) with neutral element \( e \). If \( \preceq_{U_1} \subseteq \preceq_{U_2} \), then \( J_{U_1}^{(e)} \subseteq J_{U_2}^{(e)} \) for all \( x \in [0, 1] \).

**Proof.** Let \( U_1 \) and \( U_2 \) be two uninorms on \([0, 1]\) with neutral elements \( e \) and \( \preceq_{U_1} \subseteq \preceq_{U_2} \). We assume that \( J_{U_2}^{(e)} \not\subseteq J_{U_1}^{(e)} \) for some \( x \in (0, 1) \). Let \( y \in J_{U_2}^{(e)} \) and \( y \not\in J_{U_1}^{(e)} \) for some \( y \in (0, 1) \). Since \( y \in J_{U_2}^{(x)} \), then it must be the case that \( x < y \), \( x \not\in U_2 y \) or \( y < x \), \( y \not\in U_2 x \).

Without loss of generality, we assume that \( x < y \) and \( x \not\in U_2 y \). Then, it must be \( x < y < e \) or \( e < x < y \). If \( x < e < y \) or \( y < e < x \), then we have that \( x \preceq_U y \) or \( y \preceq_U x \), a contradiction by the definition of \( \preceq_U \). Without loss of generality, we assume that \( x < y < e \) and \( x \not\in U_2 y \). Since \( y \not\in J_{U_1}^{(e)} \) and \( x < y \), it is obtained that \( x \preceq_{U_1} y \). Since \( \preceq_{U_1} \subseteq \preceq_{U_2} \), then we have that \( x \preceq_{U_2} y \), i.e., \( y \not\in J_{U_2}^{(e)} \), a contradiction. Thus, it is obtained that \( J_{U_2}^{(x)} \subseteq J_{U_1}^{(x)} \) for all \( x \in [0, 1] \). \( \square \)

Remark 4.7. In Proposition 4.6, if \( \preceq_{U_1} \subseteq \preceq_{U_2} \), then we can not say \( J_{U_1}^{(e)} \subseteq J_{U_2}^{(e)} \) for all \( x \in [0, 1] \). To illustrate this claim, the following example can be given:

**Example 4.8.** Consider the functions on \([0, 1]\) defined as follows:

\[
U_1(x, y) = \begin{cases} 
0 & (x, y) \in [0, \frac{1}{2})^2, \\
1 & (x, y) \in (\frac{1}{2}, 1)^2, \\
y & x = \frac{1}{2}, \\
x & y = \frac{1}{2}, \\
\min(x, y) & \text{otherwise}, 
\end{cases}
\]

and

\[
U_2(x, y) = \begin{cases} 
\min(x, y) & (x, y) \in [0, \frac{1}{2})^2, \\
\max(x, y) & \text{otherwise}. 
\end{cases}
\]

\( U_1 \) and \( U_2 \) are uninorms with neutral elements \( \frac{1}{2} \) by [23]. It is clear that \( \preceq_{U_1} \subseteq \preceq_{U_2} \). We claim that \( \frac{1}{3} \in J_{U_2}^{(\frac{1}{2})} \) but \( \frac{1}{3} \notin J_{U_1}^{(\frac{1}{2})} \). On the condition that \( \frac{1}{3} \preceq_{U_1} \frac{1}{2} \), there exists an element \( k \in [0, \frac{1}{2}] \) such that \( U_1(\frac{1}{2}, k) = \frac{1}{3} \). If \( k = \frac{1}{2} \), then we have \( \frac{1}{3} = \frac{1}{3} \), a contradiction. If \( k \in [0, \frac{1}{2}) \), then we have \( \frac{1}{3} = 0 \), a contradiction. So, \( \frac{1}{3} \notin J_{U_1}^{(\frac{1}{2})} \). Thus, \( \frac{1}{3} \notin J_{U_1}^{(\frac{1}{2})} \). On the other hand, \( \frac{1}{3} \notin J_{U_2}^{(\frac{1}{2})} \) by \( U_2(\frac{1}{2}, \frac{1}{2}) = \frac{1}{3} \). We can generalize this set for the uninorms above as follows:

\[
J_{U_1}^{(x)} = \{ y \in (0, 1) \mid x \neq y \text{ and } y \neq \frac{1}{2} \} \text{ for } x \in (0, 1) \text{ and } x \neq \frac{1}{2}.
\]

**Corollary 4.9.** Let \( U_1 \) and \( U_2 \) be two uninorms on \([0, 1]\) with neutral elements \( e \). If \( \preceq_{U_1} \subseteq \preceq_{U_2} \), then \( J_{U_1}^{(x)} = J_{U_2}^{(x)} \) for all \( x \in [0, 1] \).

**Corollary 4.10.** Let \( U_1 \) and \( U_2 \) be two uninorms on \([0, 1]\) with neutral elements \( e \). If \( \preceq_{U_1} \subseteq \preceq_{U_2} \), then \( K_{U_1} = K_{U_2} \).

**Proposition 4.11.** Let \( U_1 \) and \( U_2 \) be two uninorms on \([0, 1]\). If for all \( x \in [0, 1] \), \( J_{U_1}^{(x)} = J_{U_2}^{(x)} \), then the uninorms \( U_1 \) and \( U_2 \) are equivalent under the relation \( \beta \).

**Remark 4.12.** The converse of Proposition 4.11 does not have to be true. Below is an example.
Example 4.13. Let us consider a uninorm $U$ on $[0, 1]$ in Example 4.4 and a uninorm $U_e$ on $[0, 1]$ in Proposition 3.1 with neutral elements $e = \frac{1}{2}$. By Proposition 3.1, we have $K^*_{U_e} = (0, \frac{1}{2})$. Now, we want to show $K_U = (0, \frac{1}{2})$. Let $x \in (0, \frac{1}{2})$ and $y \leq \frac{1}{2}$. By Example 4.4, it is clear that $x < y$, $x \not\leq_U y$ or $y < x$, $y \not\leq_U x$. So, $x \in K_U$. That is, $(0, \frac{1}{2}) \subseteq K_U$. Conversely let $x \in K_U$ be arbitrary. We need to show that $x \in (0, \frac{1}{2})$. suppose that $x \not\in (0, \frac{1}{2})$. Since $x \in K_U$, then there exists an element $y \in (0, 1)$ such that $x < y$, $x \not\leq_U y$ or $y < x$, $y \not\leq_U x$. Without loss of generality, we assume that $x < y$, $x \not\leq_U y$. If $x \geq \frac{1}{2}$, then we have that $y > x \geq \frac{1}{2}$. So, we have $\max(x, y) = y$. By the definition of $U$,

$$U(x, y) = \max(x, y) = y.$$ 

It is obtained that $x \not\leq_U y$, a contradiction. If $x = 0$, then we have $0 \leq_U y$, a contradiction. So, it must be the case that $x \in (0, \frac{1}{2})$. This proves that $K_U = (0, \frac{1}{2})$.

Now, we want to show that this claim. Since $\frac{1}{5} = \frac{U(\frac{2}{3}, \frac{1}{3})}{2} = \frac{1}{3} \not\in \mathcal{U}(\frac{2}{3}).$ On the other hand, $\frac{1}{5} \not\in \mathcal{U}(\frac{2}{3}).$ On the condition that $\frac{1}{5} \leq \mathcal{U}(\frac{2}{3}),$ there exists an element $k \in (0, \frac{1}{2})$ such that $\mathcal{U}(\frac{2}{3}, k) = \frac{1}{5}$. By the definition of $U_e$, we have that $0 = \frac{1}{5} = \mathcal{U}(\frac{2}{3}, k)$, a contradiction. So, $\frac{1}{5} \in \mathcal{U}(\frac{2}{3}).$ Thus, we have $\mathcal{U}(\frac{2}{3}) \neq \mathcal{U}(\frac{2}{3}).$

Corollary 4.14. Although the uninorms $U_1$ and $U_2$ are equivalent under the relation $\beta$, it need not be the case that the $U_1$-partial order coincides with the $U_2$-partial order.

Definition 4.15. Let $U$ be a uninorm on $[0, 1]$ with neutral element $e$. $K^*_U$ is defined by

$$K^*_U = \{ x \in K_U \mid \text{for some } y, y' \in (0, e), \left[ x \leq y \text{ but } x \not\leq_U y \right] \text{ and } \left[ y' \leq x \text{ but } y' \not\leq_U x \right] \}.$$

Remark 4.16. By the definition of $K^*_U$, it is clear that $K^*_U \subseteq K_U$. But the reverse inclusion may not be true. Now, let us investigate the following example.

Example 4.17. Let us consider a uninorm $U$ on $[0, 1]$ with neutral element $e = \frac{1}{2}$ in Example 4.4. Since $K_U = (0, \frac{1}{2})$ by Example 4.13, it is clear that $\frac{1}{2} \in K_U$. But $\frac{1}{2} \not\in K^*_U$. That is, there does not exist some $y \in (0, 1)$, $\frac{1}{2} < y$ and $\frac{1}{2} \not\leq_U y$. It is because that we have $\frac{1}{2} + y > \frac{1}{2}$ since $\frac{1}{2} < y$. So, we have that $U(y, \frac{1}{2}) = \frac{1}{2}$ by the definition of $U$. So, $\frac{1}{2} \not\leq_U y$ for all $\frac{1}{2} < y$.

Remark 4.18. One can wonder which uninorms are provided $K^*_U = K_U$. Consider the smallest uninorm $U_e$ on $[0, 1]$ with neutral element $e$ in Proposition 3.1. It can be shown that, for all $x \in (0, e)$, there exist elements $y, y' \in (0, e)$ such that $x < y$, $x \not\leq_U y$ and $y' < x$, $y' \not\leq_U x$. So, $K^*_U = (0, e)$. This show that $K^*_U = K_{U_e}$. Also, consider the greatest uninorm $U_e$ on $[0, 1]$ with neutral element $e$ in Proposition 3.4. Similarly, $K^*_U = (e, 1)$. So, we have that $K^*_U = K_{U_e}$.

The set, denoted $K^*_U$, allows us to introduce the next equivalence relation on the class of all uninorms on $[0, 1]$.

Definition 4.19. Define a relation $\beta^*$ on the class of all uninorms on $[0, 1]$ by $U_1 \beta^* U_2$, $U_1 \beta^* U_2 :\Rightarrow K^*_U_1 = K^*_U_2$.

Lemma 4.20. The relation $\beta^*$ given in Definition 4.19 is an equivalence relation.

Remark 4.21. Although $K_{U_1} = K_{U_2}$ for the uninorms $U_1$ and $U_2$, it need not be the case that $K^*_U_1 = K^*_U_2$. To illustrate this claim we shall give the following example.
Example 4.22. Let us consider a uninorm $U$ on $[0,1]$ in Example 4.4 and a uninorm $U_e$ on $[0,1]$ in Proposition 3.1 with neutral elements $e = \frac{1}{2}$. We know that $K_{U_e} = K_U = (0, \frac{1}{2})$. So, it is obtained that the uninorms $U_e$ and $U$ are equivalent under the relation $\beta$. Also, we have $K_{U_e}^\beta = (0, \frac{1}{2})$ by Remark 4.18. It is obtained that $K_U^\beta = (0, \frac{1}{4})$ by Example 4.17.

5. Conclusion

We have defined the set of incomparable elements with respect to the $U$-partial order for any uninorm on $[0,1]$. Also we have introduced and studied an equivalence relation $\beta$ defined on the class of all uninorms on $[0,1]$. We have defined that the set $\mathcal{J}_U(x)$, consisting all incomparable elements with any $x \in (0,1)$ accordingly to $\succeq_U$. Furthermore, we have shown that even if uninorms are equivalent under this relation, it need not be the case that their partial orders coincide. Finally, we have defined and studied another set of incomparable elements with respect to the $U$-partial order for any uninorm on $[0,1]$.

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References

Solutions of Some Diophantine Equations in terms of Generalized Fibonacci and Lucas Numbers

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Abstract

In this study, we present some identities involving generalized Fibonacci sequence \((U_n)\) and generalized Lucas sequence \((V_n)\). Then we give all solutions of the Diophantine equations

\[
x^2 - V_n xy + (-1)^n y^2 = \pm(p^2 + 4) U_n^2, \quad x^2 - V_n xy + (-1)^n y^2 = \pm U_n^2, \quad x^2 - (p^2 + 4) U_n xy - (p^2 + 4)(-1)^n y^2 = \pm V_n^2, \quad x^2 - V_n xy + y^2 = \pm 1, \quad x^2 - (p^2 + 4) U_n xy - (p^2 + 4)(-1)^n y^2 = 1,
\]

and

\[
x^2 - V_n xy + (-1)^n y^2 = \pm(p^2+4), \quad x^2 - V_n xy + y^2 = \pm(p^2+4)V_n^2, \quad x^2 - V_n xy + y^2 = (p^2 + 4) U_n^2
\]

and \(x^2 - V_n xy + y^2 = \pm V_n^2\) in terms of the sequences \((U_n)\) and \((V_n)\) with \(p \geq 1\) and \(p^2 + 4\) squarefree.

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1. Introduction

Let \(p \geq 1\) be an integer. The generalized Fibonacci sequence \((U_n) = (U_n(p, 1))\) and the generalized Lucas sequence \((V_n) = (V_n(p, 1))\) are defined by

\[
U_n = pU_{n-1} + U_{n-2}, \quad U_0 = 0, \quad U_1 = 1
\]

and

\[
V_n = pV_{n-1} + V_{n-2}, \quad V_0 = 2, \quad V_1 = p
\]

for \(n \geq 2\). The terms \(U_n\) and \(V_n\) are called the \(n\)th generalized Fibonacci and Lucas numbers, respectively. In general \(U_{-n} = (-1)^{n+1}U_n\), \(V_{-n} = (-1)^n V_n\) and \(V_n = U_{n+1} + U_{n-1}\) for all \(n \in \mathbb{N}\). Properties of these sequences are determined in [7, 8, 11, 12] and [18].

In 1979, Kiss considered the sequence \((R_n)\) with linear recurrence relation \(R_n = AR_{n-1} - BR_{n-2}\), \(R_0 = 0, R_1 = 1\) for some \(n > 1\), where \(A, B\) are integers such that \(A > 0\) and \(B = -1\) or \(A > 3\) and \(B = 1\). Then he proved that for non-negative integers \(x, y\), the equation \(|x^2 - Axy + By^2| = 1\) holds if and only if \(x\) and \(y\) are consecutive terms of sequence \((R_n)\), in [9].

In 1993, Matiyasevich mentioned that the conic \(x^2 - kxy + y^2 = 1\) with \(k \geq 2\) has \((x, y)\) integer points if and only if \((u_n, u_{n+1})\) for some \(n\), where \(u_{n+1} = ku_n - u_{n-1}\), starting with \(u_0 = 0\) and \(u_1 = 1\), in [10].

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In [12], Melham showed that the solutions of the equations \( x^2 - V_nxy \pm y^2 = \pm U_n^2 \) are given by \( (x, y) = \pm(U_{n+m}, U_n) \) for \( m, n \in \mathbb{Z} \). Moreover he showed that if \( m \) is an even integer and \( p^2 + 4 \) is a squarefree integer, then all solutions of the equation \( y^2 - V_nxy + x^2 = \pm(p^2 + 4)U_n^2 \) are given by \( (x, y) = \mp(V_n, V_{n+m}) \) with \( n \in \mathbb{Z} \). These theorems of Melham are generalized forms of the theorems given in [11], by McDaniel. In [8], Kılıç and Ömür examined more general situations of the conics that McDaniel and Melham dealt in [11] and [12], respectively.

In [1], Demirtürk and Keskin determined all solutions of the known Diophantine equations \( x^2 - \frac{L_n}{2}xy - y^2 = \mp 1, x^2 - \frac{L_n}{2}xy + \frac{(-1)^n}{2} y^2 = \mp 5 \) and new Diophantine equations; \( x^2 - 5F_nxy - 5(-1)^n y^2 = \mp 1, x^2 - \frac{L_n}{2}xy + \frac{y^2}{2} = \mp 5F_n^2, x^2 - \frac{L_n}{2}xy + y^2 = \mp L_n^2 \) and \( x^2 - \frac{L_n}{2}xy + y^2 = \mp \frac{1}{2}L_n^2 \). Moreover in [2], the authors give solutions of generalizations of these equations.

In this paper, our main purpose is to determine all \((x, y)\) solutions of some Diophantine equations, mentioned in the abstract.

2. Some identities concerning the sequences \((U_n)\) and \((V_n)\)

In this section, we obtain some identities by using special matrices including generalized Fibonacci and Lucas numbers. From [6,13–15], the following identities are given for all \(m, n \in \mathbb{Z}\) by

\[
\begin{align*}
V_n^2 - pV_nV_{n-1} - V_{n-1}^2 &= (-1)^n\left(p^2 + 4\right), \\
V_mU_n - U_mV_n &= 2(-1)^mU_{n-m}, \\
V_mV_n - \left(p^2 + 4\right)U_mU_n &= 2(-1)^nV_{m-n}, \\
V_mV_n + \left(p^2 + 4\right)U_mU_n &= 2V_{n+m}, \\
V_mU_n + U_mV_n &= 2U_{n+m}, \\
V_{n+1} + U_{n-1} &= V_n, \\
V_{n+1} + V_{n-1} &= \left(p^2 + 4\right)U_n, \\
V_n^2 - \left(p^2 + 4\right)U_n^2 &= 4(-1)^n, \\
V_{m+1}U_n + V_mU_{n-1} &= V_{n+m}.
\end{align*}
\]

**Theorem 2.1.**

\[
V_{n+m} - (p^2 + 4)(-1)^{n+t}U_{t-n}V_{n+m}U_{m+t} - (p^2 + 4)(-1)^{n+t}U_{m+t}^2 = (-1)^{m+t}V_{t-n}^2,
\]

for all \(m, n, t \in \mathbb{Z}\).

**Proof.** Assume that \(A = \begin{bmatrix} V_n/2 & (p^2 + 4)U_n/2 \\ U_t/2 & V_t/2 \end{bmatrix} \). If we consider (2.4) and (2.5), then we have \(A\begin{bmatrix} V_m \\ U_m \end{bmatrix} = \begin{bmatrix} V_{n+m} \\ U_{m+t} \end{bmatrix} \). By using (2.3), we get

\[
\begin{bmatrix} V_m \\ U_m \end{bmatrix} = A^{-1} \begin{bmatrix} V_{n+m} \\ U_{m+t} \end{bmatrix} = \frac{2}{(-1)^nV_{t-n}} \begin{bmatrix} V_t/2 & -(p^2 + 4)U_t/2 \\ -U_t/2 & V_n/2 \end{bmatrix} \begin{bmatrix} V_{n+m} \\ U_{m+t} \end{bmatrix}
\]

since \(\text{det } A = V_mV_t(p^2 + 4)U_tU_m = (-1)^nV_{t-n}^2 \neq 0\). Then it follows that

\[
V_m = \frac{(-1)^n(V_tV_{n+m} - (p^2 + 4)U_tU_{m+t})}{V_{t-n}} \quad \text{and} \quad U_m = \frac{(-1)^n(V_nU_{m+t} - U_tV_{n+m})}{V_{t-n}}.
\]

By using (2.8), we see that

\[
\left(V_tV_{n+m} - (p^2 + 4)U_tU_{m+t}\right) - (p^2 + 4)\left(V_nU_{m+t} - U_tV_{n+m}\right) = 4(-1)^mV_{t-n}^2.
\]
Hence, we obtain 
\[(V^2 - (p^2 + 4)U^2) V^2 - 2(p^2 + 4) (V(U - V) V_{n+m} U_{n+t} - (p^2 + 4) (V^2 - (p^2 + 4)U^2) U^2 = 4(-1)^m V_{l-n}.\] 

Thus, it is seen that 
\[4(-1)^t V^2_n - 4(-1)^n (p^2 + 4) U_{n} V_{n+m} U_{n+t} - 4(-1)^n (p^2 + 4) U^2_{m+t} = 4(-1)^m V^2_{l-n} \]
by (2.2) and (2.8). Then it follows that
\[V^2_{n+m} - (p^2 + 4)(-1)^{n+t} V_{n+m} V_{n+t} - (p^2 + 4)(-1)^{n+t} U^2_{m+t} = (-1)^{m+t} V^2_{l-n}. \]
(2.10) which concludes the proof. \(\square\)

**Theorem 2.2.** Let \(m, n, t \in \mathbb{Z}\) and \(t \neq n\). Then
\[V^2_{n+m} - (p^2 + 4)(-1)^{n+t} V_{n+m} V_{n+t} + (p^2 + 4)(-1)^{n+t} V^2_n = (-1)^{m+t+1}(p^2 + 4) U_{l-n}^2. \]

**Proof.** Assume that \(B = \begin{bmatrix} V_n & (p^2 + 4)U_n/2 \\ V_t & (p^2 + 4)U_t/2 \end{bmatrix} \). By using (2.4), we can write the matrix multiplication \(B \begin{bmatrix} V_{m+n} \\ V_{m+t} \end{bmatrix} = \begin{bmatrix} V_{n+m} \\ V_{n+t} \end{bmatrix} \). Since \(t \neq n\), we get \(\det B = (p^2 + 4)(-1)^n U_{l-n} \neq 0 \) by (2.2). Hence it is seen that \(V_{l-n} = \frac{(-1)^n(U_{l-n} V_{n+m} - U_n V_{m+t})}{(p^2 + 4)U_{l-n}} \) and \(U_{l-n} = \frac{(-1)^n(U_{l-n} V_{n+m} - V_n V_{m+t})}{(p^2 + 4)U_{l-n}} \).

Since \(V^2_{l-n} - (p^2 + 4)U^2_{l-n} = 4(-1)^m \) by (2.8), we get
\[(p^2 + 4)(U_l V_{n+m} - U_n V_{m+t})^2 - (V_n V_{m+t} - V_l V_{n+m})^2 = 4(-1)^m (p^2 + 4)U^2_{l-n}. \]

Hence, it is seen that
\[V^2_{n+m} - (p^2 + 4)(-1)^{n+t} V_{n+m} V_{n+t} + (p^2 + 4)(-1)^{n+t} U^2_{m+t} = (-1)^{m+t+1}(p^2 + 4) U^2_{l-n}. \]
(2.11) by (2.3) and (2.8). \(\square\)

Using (2.5) and the matrix multiplication
\[
\begin{bmatrix} V_n \\ U_t \end{bmatrix} = \begin{bmatrix} V_{n+m} \\ U_{m+t} \end{bmatrix} \]
we can give the following theorem.

**Theorem 2.3.** Let \(m, n, t \in \mathbb{Z}\) and \(t \neq n\). Then
\[U^2_{n+m} - (p^2 + 4) U_{n+m} U_{n+t} + (p^2 + 4)(-1)^{n+t} U^2_{m+t} = (-1)^{m+t} U^2_{l-n}. \]
(2.12)

In this section, we also recall divisibility rules of the sequences \((U_n)\) and \((V_n)\). We omit their proofs, since they are proved in \([3-5,16,17]\).

**Theorem 2.4.** Let \(m, n \in \mathbb{N}\). \(V_n | U_m \) iff \(m = 2kn \) for some \(k \in \mathbb{N}\).

**Theorem 2.5.** Let \(m, n \in \mathbb{N}\) and \(U_n \neq 1\). \(U_n | V_m \) iff \(m = kn \) for some \(k \in \mathbb{N}\).

**Theorem 2.6.** Let \(m, n \in \mathbb{N}\) and \(V_n \neq 1\). \(V_n | V_m \) iff \(m = (2k + 1)n \) for some \(k \in \mathbb{N}\).

**Theorem 2.7.** Let \(m, n \in \mathbb{N}\) and \(n > 1\). \(U_n | V_m \) iff \(n = 2 \) and \(m \) is an odd integer, where \(p \geq 3\).
3. Solutions of some Diophantine equations

In this section, firstly we remind some Diophantine equations with their solutions. These equations are studied in [7,11,18]. We use these equations for determining all solutions of more general Diophantine equations. Throughout this paper, unless otherwise stated, we will take \( p \geq 1 \) and \( p^2 + 4 \) will be a squarefree integer.

**Theorem 3.1.** All solutions of the equation \( x^2 - pxy - y^2 = \pm 1 \) are given by \( (x, y) = \mp(U_{m+1}, U_m) \) with \( m \in \mathbb{Z} \).

**Corollary 3.2.** All solutions of the equations \( x^2 - pxy - y^2 = -1 \) and \( x^2 - pxy - y^2 = 1 \) are given by \( (x, y) = \mp(U_{2m}, U_{2n-1}) \) and \( (x, y) = \mp(U_{2m+1}, U_{2m}) \) with \( m \in \mathbb{Z} \), respectively.

**Theorem 3.3.** All solutions of the equation \( x^2 - (p^2 + 4)U_nxy - (p^2 + 4)(-1)^ny^2 = -V_n^2 \) and \( x^2 - (p^2 + 4)U_nxy - (p^2 + 4)(-1)^ny^2 = V_n^2 \) are given by \( (x, y) = \mp(V_{n+m}, U_m) \) with \( m \) odd and \( m \) even, respectively.

**Proof.** Suppose that \( x^2 - (p^2 + 4)U_nxy - (p^2 + 4)(-1)^ny^2 = -V_n^2 \) for some integers \( x \) and \( y \). By using (2.8) in this equation, we get \((2x - (p^2 + 4)U_ny)^2 - (p^2 + 4)V_n^2y^2 = -4V_n^2\).

Hence it is seen that \( V_n|2x - (p^2 + 4)U_ny \). Then taking

\[
\begin{align*}
    u &= \left(\frac{2x - (p^2 + 4)U_ny}{V_n}\right) + py \\
    v &= y,
\end{align*}
\]

we obtain \( u = (x - V_ny)/V_n \) by (2.7). Thus it follows that

\[
    u^2 - puv - v^2 = \left(x^2 - (p^2 + 4)U_nxy - (p^2 + 4)(-1)^ny^2\right)/V_n^2 = -V_n^2/V_n = -1,
\]

by (2.1) and (2.7). From Corollary 3.2, it is seen that \((u, v) = \mp(U_{m+1}, U_m)\) for some odd \( m \).

Hence

\[
    (x - V_ny)/V_n = \mp U_{m+1} \text{ and } y = \mp U_m.
\]

Now using (2.9), we obtain

\[
    (x, y) = \mp(V_{n+m}, U_m)
\]

for some odd \( m \). Conversely, if \((x, y) = \mp(V_{n+m}, U_m)\) for some odd \( m \), then it can be seen that \( x^2 - (p^2 + 4)U_nxy - (p^2 + 4)(-1)^{n}y^2 = -V_n^2 \) by (2.10).

Now assume that \( x^2 - (p^2 + 4)U_nxy - (p^2 + 4)(-1)^{n}y^2 = V_n^2 \) for some integers \( x \) and \( y \). Then taking \( u = (x - V_ny)/V_n \) and \( v = y \), we obtain

\[
    u^2 - puv - v^2 = 1
\]

by (2.1) and (2.7). From Corollary 3.2, we get \((u, v) = \mp(U_{m+1}, U_m)\) for some even \( m \).

Thus, it follows that \((x, y) = \mp(V_{n+m}, U_m)\) by (2.9), where \( m \) is even. Conversely, if \((x, y) = \mp(V_{n+m}, U_m)\) for some even \( m \), then it can be seen that \( x^2 - (p^2 + 4)U_nxy - (p^2 + 4)(-1)^{n}y^2 = V_n^2 \) by (2.10). \( \square \)

**Theorem 3.4.** All solutions of the equation \( x^2 - (p^2 + 4)U_nxy - (p^2 + 4)(-1)^ny^2 = 1 \) are given by \( (x, y) = \mp \left(V_{(2t+1)n}/V_n, U_{2tn}/V_n\right) \) with \( t \in \mathbb{Z} \).

**Proof.** Assume that \( x^2 - (p^2 + 4)U_nxy - (p^2 + 4)(-1)^{n}y^2 = 1 \) for some integers \( x \) and \( y \). Multiplying both sides of this equation by \( V_n^2 \), we get

\[
    (V_nx)^2 - (p^2 + 4)U_n(V_nx)(V_ny) - (p^2 + 4)(-1)^n(V_ny)^2 = V_n^2.
\]

Thus, it follows that \( V_nx = \mp V_{n+m} \) and \( V_ny = \mp U_m \) for some integer \( m \) by Theorem 3.3. Hence, we get \((x, y) = \mp(V_{n+m}/V_n, U_m/V_n)\). From Theorem 2.4 and Theorem 2.6, it can be seen that \( m = 2tn \) for some \( t \in \mathbb{Z} \). Therefore, we obtain \((x, y) = \mp \left(V_{(2t+1)n}/V_n, U_{2tn}/V_n\right)\).
Conversely, if \((x, y) = \pm \left(\frac{V_{(2t+1)n}}{V_n}, \frac{U_{2tn}}{V_n}\right)\) for some \(t \in \mathbb{Z}\), then it is easy to verify that \(x^2 - (p^2 + 4)U_nxy - (p^2 + 4)(-1)^ny^2 = 1\) by (2.10).

The following corollary can be given from Theorems 3.3, 2.4 and 2.6.

**Corollary 3.5.** The equation \(x^2 - (p^2 + 4)U_nxy - (p^2 + 4)(-1)^ny^2 = -1\) has no solution.

Now we will prove Theorem 3.6, which is stated by Melham in [12].

**Theorem 3.6.** All solutions of the equation \(x^2 - V_nxy + (-1)^ny^2 = -(p^2 + 4)U_n^2\) and \(x^2 - V_nxy + (-1)^ny^2 = (p^2 + 4)U_n^2\) are given by \((x, y) = \mp \left(V_{n+m}, V_m\right)\) with \(m\) even and \(m\) odd, respectively.

**Proof.** Suppose that \(x^2 - V_nxy + (-1)^ny^2 = -(p^2 + 4)U_n^2\) for some integers \(x\) and \(y\). Then using (2.8), we get \((2x - V_ny)^2 - (p^2 + 4)U_n^2y^2 = -4(p^2 + 4)U_n^2\). Thus, it follows that \(U_n|2x - V_ny\). Therefore, there is an integer \(z\) such that \(2x - V_ny = U_nz\). Hence we can write \(x^2 - (p^2 + 4)y^2 = -4(p^2 + 4)\). This implies that \((p^2 + 4)\) is square free.

Then there is an integer \(a\) such that \(z = (p^2 + 4)a\) and we have \(2x - V_ny = (p^2 + 4)U_na\). Thus, it follows that \(y^2 - p^2a^2 = 4 + 4a^2\).

Hence \(y^2 - p^2a^2\) is even. Then we can see that \(y\) and \(pa\) have the same parity. Taking \(u = (y + pa)/2\) and \(v = a\), we obtain

\[
u = \frac{2x - V_ny}{(p^2 + 4)U_n}.
\]

Thus, we get

\[
u^2 - puv + v^2 = -(p^2 + 4) \left(x^2 - V_nxy + (-1)^ny^2\right)/(p^2 + 4)U_n^2 = 1.
\]

Therefore it follows that \((u, v) = \mp (U_{m+1}, U_m)\) for some even \(m\) by Corollary 3.2. Thus, we obtain

\[
(px + V_{n-1}y)/(p^2 + 4)U_n = \mp U_{m+1} \text{ and } (2x - V_ny)/(p^2 + 4)U_n = \mp U_m.
\]

This together with (2.4), (2.6) and (2.7) yields \((x, y) = \mp \left(V_{n+m}, V_m\right)\) for some even \(m\).

Conversely, if \((x, y) = \mp \left(V_{n+m}, V_m\right)\) for some even \(m\), then it follows that \(x^2 - V_nxy + (-1)^ny^2 = -(p^2 + 4)U_n^2\) by (2.11).

Following the similar steps, we obtain the expected solutions of the equation \(x^2 - V_nxy + (-1)^ny^2 = (p^2 + 4)U_n^2\). \(\blacksquare\)

**Theorem 3.7.** If \(n\) is even, then all solutions of the equation \(x^2 - V_{2n}xy + y^2 = -(p^2 + 4)U_n^2\) are given by \((x, y) = \mp \left(V_{(2t+3)n}/V_n, V_{(2t+1)n}/V_n\right)\) with \(t \in \mathbb{Z}\). If \(n\) is odd, then all solutions of the equation \(x^2 - V_{2n}xy + y^2 = (p^2 + 4)U_n^2\) are given by \((x, y) = \mp \left(V_{(2t+3)n}/V_n, V_{(2t+1)n}/V_n\right)\) with \(t \in \mathbb{Z}\).

**Proof.** Assume that \(n\) is even and \(x^2 - V_{2n}xy + y^2 = -(p^2 + 4)U_n^2\) for some integers \(x\) and \(y\). Multiplying both sides of this equation by \(V_n^2\) and considering the fact that \(U_{2n} = U_nV_n\), we get

\[
(V_nx)^2 - V_{2n}(V_nx)(V_ny) + (V_ny)^2 = -(p^2 + 4)U_{2n}^2.
\]

From Theorem 3.6, it follows that \((x, y) = \mp \left(V_{2n+m}/V_n, V_m/V_n\right)\) for some even \(m\). Moreover, since \(x\) and \(y\) are integers, there is an integer \(t\) such that \(m = (2t + 1)\) by Theorem 2.6. Therefore we obtain \((x, y) = \mp \left(V_{(2t+3)n}/V_n, V_{(2t+1)n}/V_n\right)\).
By using Theorems 3.7, 3.6, and 2.6, the following corollary can be proved. So, we omit its proof.

**Corollary 3.8.** If \( n \) is odd, then the equation \( x^2 - V_nxy + y^2 = -(p^2 + 4)U_n^2 \) and if \( n \) is even, then the equation \( x^2 - V_nxy + y^2 = (p^2 + 4)U_n^2 \) has no solution.

**Theorem 3.9.** All solutions of the equation \( x^2 - V_nxy + (-1)^ny^2 = -(p^2 + 4) \) are given by \( x, y = \mp \left( \frac{V_{n+1} + V_n}{U_n}, \frac{V_{n+1} + V_n}{U_n} \right) \) with \( m \) even and \( U_n|V_m \).

**Proof.** Assume that \( x^2 - V_nxy + (-1)^ny^2 = -(p^2 + 4) \) for some integers \( x \) and \( y \). Multiplying both sides of the equation by \( U_n^2 \), we get

\[
(U_nx)^2 - V_n(U_nx)(U_ny) + (-1)^n(U_ny)^2 = -(p^2 + 4)U_n^2.
\]

Hence using Theorem 3.6, we obtain the expected result.

Conversely, if \( m \) is even and \( (x, y) = \mp \left( \frac{V_{n+1} + V_n}{U_n}, \frac{V_{n+1} + V_n}{U_n} \right) \), then it follows that \( x^2 - V_nxy + (-1)^ny^2 = -(p^2 + 4) \) by (2.11). □

The following corollaries can be given from Theorem 3.9 and Theorem 2.7.

**Corollary 3.10.** All solutions of the equation \( x^2 - pxy - y^2 = -(p^2 + 4) \) are given by \( x, y = \mp \left( \frac{V_{2t+1}}{V_2} \right) \) with \( t \in \mathbb{Z} \).

**Corollary 11.** If \( p \geq 3 \), then the equation \( x^2 - (p^2 + 2)xy + y^2 = -(p^2 + 4) \) has no solution.

**Theorem 3.12.** All solutions of the equation \( x^2 - V_nxy + (-1)^ny^2 = p^2 + 4 \) are given by \( x, y = \mp \left( \frac{V_{n+m}}{U_n}, \frac{V_{n+m}}{U_n} \right) \) with \( m \) odd and \( U_n|V_m \).

**Proof.** Assume that \( x^2 - V_nxy + (-1)^ny^2 = p^2 + 4 \) for some integers \( x \) and \( y \). Multiplying both sides of the equation by \( U_n^2 \), we get

\[
(U_nx)^2 - V_n(U_nx)(U_ny) + (-1)^n(U_ny)^2 = (p^2 + 4)U_n^2.
\]

Hence using Theorem 3.6, we have \( (x, y) = \mp \left( \frac{V_{n+m}}{U_n}, \frac{V_{n+m}}{U_n} \right) \) for some odd \( m \) with \( U_n|V_m \).

If \( m \) is odd and \( (x, y) = \mp \left( \frac{V_{n+m}}{U_n}, \frac{V_{n+m}}{U_n} \right) \), then by using (2.11), it is seen that \( x^2 - V_nxy + (-1)^ny^2 = p^2 + 4 \). □

The following corollaries can be given from Theorem 3.12 and Theorem 2.7.

**Corollary 3.13.** All solutions of the equation \( x^2 - pxy - y^2 = p^2 + 4 \) are given by \( x, y = \mp \left( \frac{V_{2t+2}}{V_2}, \frac{V_{2t+2}}{V_2} \right) \) with \( t \in \mathbb{Z} \).

**Corollary 14.** All solutions of the equation \( x^2 - (p^2 + 2)xy + y^2 = p^2 + 4 \) are given by \( x, y = \mp \left( \frac{V_{2t+3}}{p}, \frac{V_{2t+3}}{p} \right) \) with \( t \in \mathbb{Z} \).

Moreover, the following corollary can be proven easily.

**Corollary 3.15.** All solutions of the equation \( x^2 - 6xy + y^2 = 8 \) are given by \( x, y = \mp \left( \frac{V_{2t+3}}{2}, \frac{V_{2t+3}}{2} \right) \) with \( t \in \mathbb{Z} \).

Now we give the following theorem without proof, since it can be proved in the same manner with the proof of Theorem 3.12.
Theorem 3.16. All solutions of the equation \( x^2 - V_{2n}xy + y^2 = -(p^2 + 4)V_n^2 \) are given by \((x, y) = \mp (V_{2n+m}/U_n, V_n/U_n)\) with \(m\) even and \(U_n | V_m\).

The following corollaries can be given by using Theorem 3.16 and Theorem 2.7.

Corollary 3.17. All solutions of the equation \( x^2 - (p^2 + 2)xy + y^2 = -p^2(p^2 + 4) \) are given by \((x, y) = \mp (V_{2t+2}, V_{2t})\) with \(t \in \mathbb{Z}\).

Corollary 3.18. If \(p \geq 3\), then the equation \( x^2 - [p^2(p^2 + 4) + 2]xy + y^2 = -(p^2 + 4)(p^2 + 2)^2 \) has no solutions.

Theorem 3.19. All solutions of the equation \( x^2 - V_{2n}xy + y^2 = (p^2 + 4)V_n^2 \) are given by \((x, y) = \mp (V_{2n+m}/U_n, V_m/U_n)\) with \(m\) odd and \(U_n | V_m\).

Proof. Assume that \( x^2 - V_{2n}xy + y^2 = (p^2 + 4)V_n^2 \) for some integers \(x\) and \(y\). Multiplying both sides of this equation by \(U_n^2\), we have

\[
(U_nx)^2 - 2V_n(U_nx)(U_ny) + (U_ny)^2 = (p^2 + 4)U_{2n}^2.
\]

Then it follows that \((x, y) = \mp (V_{2n+m}/U_n, V_m/U_n)\) for some odd \(m\) with \(U_n | V_m\) by Theorem 3.6.

Conversely, if \(m\) is odd and \((x, y) = \mp (V_{2n+m}/U_n, V_m/U_n)\), then we get \(x^2 - V_{2n}xy + y^2 = (p^2 + 4)V_n^2\) by (2.11) \(\square\).

The following corollaries can be given by using Theorem 2.7 and Theorem 3.19.

Corollary 3.20. All solutions of the equation \( x^2 - (p^2 + 2)xy + y^2 = p^2(p^2 + 4) \) are given by \((x, y) = \mp (V_{2t+3}, V_{2t+1})\) with \(t \in \mathbb{Z}\).

Corollary 3.21. If \(p \geq 2\), then all solutions of the equation \( x^2 - [p^2(p^2 + 4) + 2]xy + y^2 = (p^2 + 4)(p^2 + 2)^2 \) are given by \((x, y) = \mp \left(V_{(2t+5)/p}, V_{(2t+1)/p}\right)\) with \(t \in \mathbb{Z}\).

Now we give the following theorem which is stated by Kılıç and Ömür in [8].

Theorem 3.22. All solutions of the equation \( x^2 - V_{n}xy + (-1)^n y^2 = -U_n^2 \) and \(x^2 - V_{n}xy + (-1)^n y^2 = U_n^2\) are given by \((x, y) = \mp (U_{n+m}, U_m)\) with \(m\) odd and \(m\) even, respectively.

Proof. Suppose that \( x^2 - V_{n}xy + (-1)^n y^2 = -U_n^2 \) for some integers \(x\) and \(y\). Completing the square gives \((2x - V_n y)^2 - (p^2 + 4)U_n^2 y^2 = -4U_n^2\), and it is seen that \(U_n|2x - V_n y\).

Thus, it follows that

\[
((2x - V_n y)/U_n)^2 - (p^2 + 4)y^2 = -4.
\]

Taking \(u = ((2x - V_n y)/U_n) + py)/2 = (x - U_{n-1}y)/U_n\) and \(v = y\), we have \(u^2 - pv^2 - v^2 = -1\). Therefore, from Corollary 3.2, we get \((u, v) = \mp (U_{m+1}, U_m)\) for some odd \(m\).

By using the fact that \(U_{m+1}U_n + U_mU_{n-1} = U_{n+m}\), we get \((x, y) = \mp (U_{n+m}, U_m)\).

Conversely, if \((x, y) = \mp (U_{n+m}, U_m)\) for some odd \(m\), then it can be seen that \(x^2 - V_{n}xy + (-1)^n y^2 = -U_n^2\) by (2.12).

Following the similar steps, we obtain the expected solutions of the equation \(x^2 - V_{n}xy + (-1)^n y^2 = U_n^2\) \(\square\).

Theorem 3.23. All solutions of the equation \( x^2 - V_{2n}xy + y^2 = U_n^2 \) are given by \((x, y) = \mp \left(U_{(2t+2)n}/V_n, U_{2tn}/V_n\right)\) with \(t \in \mathbb{Z}\).

Proof. Assume that \( x^2 - V_{2n}xy + y^2 = U_n^2 \) for some integers \(x\) and \(y\). Multiplying both sides of this equation by \(V_n^2\), we get

\[
(V_nx)^2 - 2V_n(V_nx)(V_ny) + (V_ny)^2 = U_{2n}^2.
\]

Then from Theorem 3.22, it follows that \((x, y) = \mp (U_{2n+m}/V_n, U_m/V_n)\) for some even \(m\). Hence, using Theorem 2.4, it is seen that \(m = 2tn\) for some \(t \in \mathbb{Z}\). Therefore, \((x, y) = \mp \left(U_{(2t+2)n}/V_n, U_{2tn}/V_n\right)\).
Conversely, if \((x, y) = \mp (U_{(2t+2)n}/V_n, U_{2tn}/V_n)\) for some \(t \in \mathbb{Z}\), then it is seen that \(x^2 - V_nxy + y^2 = U_n^2\) by (2.12).

Theorem 3.24. The equation \(x^2 - V_nxy + y^2 = -U_n^2\) has no solution.

Proof. Assume that \(x^2 - V_nxy + y^2 = -U_n^2\) for some integers \(x\) and \(y\). Multiplying both sides of this equation by \(V_n^2\), we get
\[
(V_nx)^2 - V_n(2n/V_n)(V_ny) + (V_ny)^2 = -U_{2n}^2.
\]
From Theorem 3.22, it follows that \((x, y) = \mp (U_{2n+m}/V_n, U_m/V_n)\) for some odd \(m\). This together with Theorem 2.4 yields the result.

Theorem 3.25. If \(n\) is odd, then all solutions of the equation \(x^2 - V_nxy + y^2 = -V_n^2\) are given by \((x, y) = \mp (U_{(2t+3)n}/V_n, U_{(2t+1)n}/U_n)\) with \(t \in \mathbb{Z}\).

Proof. Assume that \(x^2 - V_nxy + y^2 = -V_n^2\) for some integers \(x\) and \(y\). Multiplying both sides of this equation by \(V_n^2\), we get
\[
(U_nx)^2 - V_n(U_nx)(U_ny) + (U_ny)^2 = -U_{2n}^2.
\]
Then from Theorem 3.23, it follows that \((x, y) = \mp (U_{2n+m}/V_n, U_m/V_n)\) for some odd \(m \in \mathbb{Z}\). Hence, using Theorem 2.5 it is seen that \(n|m\). Since \(n\) and \(m\) are odd, we have \(m = (2t + 1)n\) for some \(t \in \mathbb{Z}\). Therefore, \((x, y) = \mp (U_{(2t+3)n}/V_n, U_{(2t+1)n}/U_n)\).

Conversely, if \(n\) is odd and \((x, y) = \mp (U_{(2t+3)n}/V_n, U_{(2t+1)n}/U_n)\) for some \(t \in \mathbb{Z}\), then from (2.12), it follows that \(x^2 - V_nxy + y^2 = -V_n^2\).

Now we can give the following corollaries by using Theorem 3.22, Theorem 2.5, and Equation (2.12).

Corollary 3.26. If \(n\) is even, then the equation \(x^2 - V_nxy + y^2 = -V_n^2\) has no solution.

Corollary 3.27. If \(n\) is even, then all solutions of the equation \(x^2 - V_nxy + y^2 = V_n^2\) are given by \((x, y) = \mp (U_{(t+2)n}/V_n, U_{tn}/U_n)\) with \(t \in \mathbb{Z}\). If \(n\) is odd, then all solutions of the equation \(x^2 - V_nxy + y^2 = V_n^2\) are given by \((x, y) = \mp (U_{(2t+2)n}/V_n, U_{2tn}/U_n)\) with \(t \in \mathbb{Z}\).

Theorem 3.28. If \(n\) is odd, then all solutions of the equations \(x^2 - V_nxy - y^2 = 1\) and \(x^2 - V_nxy - y^2 = -1\) are given by \((x, y) = \mp (U_{(2t+2)n}/V_n, U_{(2t+1)n}/U_n)\) with \(t \in \mathbb{Z}\), respectively. If \(n\) is even, then all solutions of the equation \(x^2 - V_nxy + y^2 = 1\) are given by \((x, y) = \mp (U_{(t+1)n}/V_n, U_{tn}/U_n)\) with \(t \in \mathbb{Z}\).

Proof. Assume that \(n\) is odd and \(x^2 - V_nxy - y^2 = 1\) for some integers \(x\) and \(y\). Multiplying both sides of this equation by \(U_n^2\), we get
\[
(U_nx)^2 - V_n(U_nx)(U_ny) -(U_ny)^2 = -U_n^2.
\]
From Theorem 3.22, it follows that \(x = \mp U_{n+m}/V_n\) and \(y = \mp U_m/V_n\) for some odd \(m\). Since \(x\) and \(y\) are integers, it is clear that \(m = (2t + 1)n\) for some \(t \in \mathbb{Z}\) by Theorem 2.5. Then we obtain
\[
(x, y) = \mp (U_{(2t+2)n}/V_n, U_{(2t+1)n}/U_n).
\]
Conversely, if \(n \geq 3\) is odd and \((x, y) = \mp (U_{(2t+2)n}/V_n, U_{(2t+1)n}/U_n)\) for some \(t \in \mathbb{Z}\), then it follows that \(x^2 - V_nxy - y^2 = -1\) by (2.12).

If \(n\) is odd, then in a similar way, it is easy to see that all solutions of the equation \(x^2 - V_nxy - y^2 = 1\) are given by \((x, y) = \mp (U_{(2t+1)n}/V_n, U_{2tn}/U_n)\) with \(t \in \mathbb{Z}\).
Now assume that $n$ is even and $x^2 - V_n xy + y^2 = 1$ for some integers $x$ and $y$. Multiplying both sides of this equation by $U_n^2$ and using Theorem 3.22, it is seen that $x = \mp U_{n+m}/U_n$ and $y = \mp U_m/U_n$, for some even $m$. Since $x$ and $y$ are integers, it is clear that $m = tn$ for some $t \in \mathbb{Z}$ by Theorem 2.5. Then we obtain

$$(x, y) = \mp \left( U_{(t+1)n}/U_n, U_{tn}/U_n \right).$$

Moreover, if $n$ is even and $(x, y) = \mp \left( U_{(t+1)n}/U_n, U_{tn}/U_n \right)$ with $t \in \mathbb{Z}$, then it follows that $x^2 - V_n xy + y^2 = 1$ by (2.12).

Multiplying both sides of the equation $x^2 - V_n xy + y^2 = -1$ by $U_n^2$ and using Theorem 2.5 and Theorem 3.22, the following corollary can be given.

**Corollary 3.29.** If $n$ is even, then the equation $x^2 - V_n xy + y^2 = -1$ has no solution.

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Kantorovich-Stancu type operators including Boas-Buck type polynomials

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Abstract

The aim of the paper is to introduce a Kantorovich-Stancu type modification of a generalization of Szász operators defined via Boas-Buck type polynomials and to obtain rates of convergence for these operators. Furthermore, we give the figures for comparing approximation properties of the operators $K_n^{(\alpha,\beta)}$ and $B_n$.

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1. Introduction

The Szász–Mirakyan operators are defined by

$$S_n(f; x) := e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right)$$

where $n \in \mathbb{N}$ [23]. We consider $f \in C[0, \infty)$ for which the corresponding series is convergent. Up to now, various operators via special functions, especially generalizations and modifications of Szász operators, have been introduced by many authors and have been studied their approximation properties (see [1, 2, 8, 13–15, 19–22, 24, 25, 27–29]). In 1969, Jakimovski et al. [16] defined a generalization of Szász operators using Appell polynomials. In 1974, Ismail [11] introduced another generalization of Szász operators via Sheffer polynomials. Inspired by the papers [11, 16], $f \in C[0, \infty)$ for which the corresponding series is convergent, Varma et al. [27] studied many properties of the following generalization of Szász operators defined by means of the Brenke type polynomials

$$L_n(f; x) := \frac{1}{A(1)B(nx)} \sum_{k=0}^{\infty} p_k(nx) f\left(\frac{k}{n}\right)$$

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under the assumptions

\[ (i) \quad A (1) \neq 0, \quad \frac{a_k - b_k}{A (1)} \geq 0, \quad 0 \leq r \leq k, \quad k = 0, 1, 2, \ldots, \]
\[ (ii) \quad B : [0, \infty) \to (0, \infty), \]
\[ (iii) \quad (1.4) \text{ and } (1.5) \text{ converge for } |t| < R, \quad (R > 1) \quad (1.3) \]

where

\[ A (t) = \sum_{r=0}^{\infty} a_r t^r, \quad a_0 \neq 0 \quad \text{and} \quad B (t) = \sum_{r=0}^{\infty} b_r t^r, \quad b_r \neq 0 \quad (r \geq 0) \quad (1.4) \]

are analytic functions and the Brenke type polynomials [5] are generated by

\[ A (t) B (xt) = \sum_{k=0}^{\infty} p_k (x) t^k \quad (1.5) \]

where

\[ p_k (x) = \sum_{r=0}^{k} a_{k-r} b_r x^r, \quad k = 0, 1, 2, \ldots. \]

In 2013, Aktas et al. [1] defined the following Kantorovich-Stancu version of the operators given by (1.2) for \( n \in \mathbb{N}, x \geq 0 \) and \( f \in C [0, \infty) \) for which the corresponding series is convergent under the assumptions (1.3)

\[ K_n^{(\alpha, \beta)} (f; x) := \frac{n + \beta}{A (1) B (nx)} \sum_{k=0}^{\infty} p_k (nx) \int_{(k+\alpha+1)/(n+\beta)}^{(k+\alpha+1)/n} f (t) \, dt. \quad (1.6) \]

For \( \alpha = \beta = 0 \), this operator returns to the Kantorovich type of the operators given by (1.2) [24]

\[ K_n (f; x) := \frac{n}{A (1) B (nx)} \sum_{k=0}^{\infty} p_k (nx) \int_{k/n}^{(k+1)/n} f (t) \, dt \]

which gives the Kantorovich version of Szász-Mirakyan operators [4] in the special case of \( B (t) = e^t \) and \( A (t) = 1 \)

\[ K_n (f; x) := n e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} \int_{k/n}^{(k+1)/n} f (t) \, dt. \quad (1.7) \]

The approximation properties of the operators (1.7) can be found in [9, 18, 26, 28, 30] and the references cited therein.

In 2012, Sucu et al. [21] constructed linear positive operators by means of Boas-Buck type polynomials which give the Brenke-type polynomials, Sheffer polynomials, and Appell polynomials in the special cases. In [12], Boas-Buck-type polynomials are generated by

\[ A (t) B (x H (t)) = \sum_{k=0}^{\infty} p_k (x) t^k \quad (1.8) \]

where \( A (t), B (t) \) and \( H (t) \) are analytic functions

\[ A (t) = \sum_{r=0}^{\infty} a_r t^r, \quad (a_0 \neq 0) \quad \text{and} \quad B (t) = \sum_{r=0}^{\infty} b_r t^r, \quad (b_r \neq 0) \]
\[ H (t) = \sum_{r=1}^{\infty} h_r t^r, \quad (h_1 \neq 0). \quad (1.9) \]

These operators defined by Sucu et al. [21] are as follows for \( x \geq 0 \) and \( n \in \mathbb{N} \)

\[ B_n (f; x) := \frac{1}{A (1) B (nx H (1))} \sum_{k=0}^{\infty} p_k (nx) f \left( \frac{k}{n} \right), \quad (1.10) \]
which satisfy
\begin{align}
(i) & \quad A(1) \neq 0, \quad H'(1) = 1, \quad p_k(x) \geq 0, \quad k = 0, 1, 2, \ldots, \\
(ii) & \quad B : \mathbb{R} \to (0, \infty), \\
(iii) & \quad (1.8) \text{ and } (1.9) \text{ converge for } |t| < R, \quad (R > 1).
\end{align}

In the present paper, we consider a Kantorovich-Stancu version of the operators (1.10) as follows
\begin{equation}
\mathcal{K}_n^{(\alpha, \beta)}(f; x) := \frac{n + \beta}{A(1) B(nxH(1))} \sum_{k=0}^{\infty} p_k(nx) \int_{(k+\alpha)/(n+\beta)}^{(k+\alpha+1)/(n+\beta)} f(t) \, dt
\end{equation}
under the assumption (1.11), \( f \in C[0, \infty) \) and \( 0 \leq \alpha \leq \beta \), and we study the approximation properties of these operators. We also present special cases of these operators including Charlier polynomials and Gould-Hopper polynomials.

The case of \( H(t) = t \) in the operators (1.12) gives the Kantorovich-Stancu type operators (1.6) including Brenke-type polynomials. For \( B(t) = e^t \), the operators (1.12) reduce to the Kantorovich-Stancu type of the operators with Sheffer polynomials defined by Ismail [11]. In the special case of \( B(t) = e^t \) and \( H(t) = t \), we have Kantorovich-Stancu type of the operators with Appell polynomials introduced by Jakimovski et al. [16]. Also, for \( A(t) = 1, B(t) = e^t \) and \( H(t) = t \), it turns out the Kantorovich-Stancu type of Szász-Mirakyan operators.

**2. Approximation properties of the operators \( \mathcal{K}_n^{(\alpha, \beta)} \)**

First, for the operators \( \mathcal{K}_n^{(\alpha, \beta)} \) given by (1.12), we shall give some auxiliary results to prove the main theorem.

**Lemma 2.1.** For each \( x \in [0, \infty) \), the Kantorovich-Stancu type operators (1.12) have the following properties
\begin{align}
\mathcal{K}_n^{(\alpha, \beta)}(1; x) &= 1, \quad (2.1) \\
\mathcal{K}_n^{(\alpha, \beta)}(s; x) &= \frac{nB'(nxH(1))}{(n+\beta)B(nxH(1))} x + \frac{2A'(1) + (2\alpha + 1) A(1)}{2(n+\beta) A(1)}, \quad (2.2) \\
\mathcal{K}_n^{(\alpha, \beta)}(s^2; x) &= \frac{n^2B''(nxH(1))}{(n+\beta)^2B(nxH(1))} x^2 + n \frac{[2A'(1) + (2\alpha + 2) A(1) A(1) H''(1)] B'(nxH(1))}{(n+\beta)^2 A(1) B(nxH(1))} x \\
&\quad + \frac{1}{3(n+\beta)^2 A(1)} \left\{ 3(A''(1) + A'(1)) + 3(2\alpha + 1) A'(1) + \left( 3\alpha^2 + 3\alpha + 1 \right) A(1) \right\}. \quad (2.3)
\end{align}

**Proof.** From the generating function of the Boas-Buck-type polynomials given by (1.8), a few calculations reveal that
\begin{align*}
\sum_{k=0}^{\infty} p_k(nx) &= A(1) B(nxH(1)), \\
\sum_{k=0}^{\infty} kp_k(nx) &= A'(1) B(nxH(1)) + nx A(1) B'(nxH(1)), \\
\sum_{k=0}^{\infty} k^2p_k(nx) &= n^2x^2 A(1) B''(nxH(1)) + nx B'(nxH(1)) \{ 2A'(1) + A(1) + A(1) H''(1) \} \\
&\quad + B(nxH(1)) \{ A''(1) + A'(1) \}.
\end{align*}
By using these equalities, we obtain the assertions of the lemma by simple calculation. \( \square \)
Lemma 2.2. For each \( x \in [0, \infty) \), we have
\[
\mathcal{K}_n^{(\alpha, \beta)}((s - x)^2; x) = \left\{ \frac{n^2 B''(nxH(1))}{(n + \beta)^2 B(nxH(1))} - \frac{2nB'(nxH(1))}{(n + \beta) B(nxH(1))} + 1 \right\} x^2
\]
\[
+ \left\{ \frac{2A'(1) + (2\alpha + 2) A(1) + H''(1) A(1)}{(n + \beta)^2 A(1)} - \frac{2A'(1) + (2\alpha + 1) A(1)}{(n + \beta) A(1)} \right\} x
\]
\[
+ \frac{1}{(n + \beta)^2 A(1)} \left\{ A''(1) + (2\alpha + 2) A'(1) + (\alpha^2 + \alpha + 1/3) A(1) \right\}.
\]

Theorem 2.3. Let
\[
E := \left\{ f : x \in [0, \infty), \frac{f(x)}{1 + x^2} \text{ is convergent as } x \to \infty \right\}
\]
and
\[
\lim_{y \to \infty} \frac{B'(y)}{B(y)} = 1 \quad \text{and} \quad \lim_{y \to \infty} \frac{B''(y)}{B(y)} = 1. \quad (2.4)
\]
If \( f \in C[0, \infty) \cap E \), then
\[
\lim_{n \to \infty} \mathcal{K}_n^{(\alpha, \beta)}(f; x) = f(x),
\]
and the operators \( \mathcal{K}_n^{(\alpha, \beta)} \) converge uniformly in each compact subset of \( [0, \infty) \).

Proof. From Lemma 2.1, by considering the equality (2.4), one obtains
\[
\lim_{n \to \infty} \mathcal{K}_n^{(\alpha, \beta)}(s^i; x) = x^i, \ i = 0, 1, 2,
\]
where the convergence is satisfied uniformly in each compact subset of \( [0, \infty) \). Then, using the universal Korovkin-type property (vi) of Theorem 4.1.4 in [3] completes the proof. □

Now, we compute the rates of convergence of the operators \( \mathcal{K}_n^{(\alpha, \beta)}(f) \) to \( f \) by means of a classical approach, the second modulus of continuity and Peetre’s \( K \)-functional.

Let \( f \in \tilde{C}[0, \infty) \). Then for \( \delta > 0 \), the modulus of continuity of \( f \) which is denoted by \( w(f; \delta) \) is defined by
\[
w(f; \delta) := \sup_{x, y \in [0, \infty), |x-y| \leq \delta} |f(x) - f(y)|
\]
where \( \tilde{C}[0, \infty) \) denotes the space of uniformly continuous functions on \( [0, \infty) \). Then, for any \( \delta > 0 \) and each \( x \in [0, \infty) \),
\[
|f(x) - f(y)| \leq w(f; \delta) \left( \frac{|x - y|}{\delta} + 1 \right)
\]
holds.

One can estimate the rate of convergence of the sequence \( \mathcal{K}_n^{(\alpha, \beta)}(f) \) to \( f \) via the modulus of continuity as follows.

Theorem 2.4. If \( f \in \tilde{C}[0, \infty) \cap E \), then we have
\[
|\mathcal{K}_n^{(\alpha, \beta)}(f; x) - f(x)| \leq 2w(f; \sqrt[n]{\lambda_n(x)}).
\]
Using linearity of the operators $\lambda = \lambda_n (x) = \mathcal{K}_n^{(\alpha, \beta)} \left( (s - x)^2 ; x \right)$

$$
\lambda = \lambda_n (x) = \mathcal{K}_n^{(\alpha, \beta)} \left( (s - x)^2 ; x \right)
= \left\{ \frac{n^2 B'' (nxH (1))}{(n + \beta)^2 B (nxH (1))} - \frac{2n B' (nxH (1))}{(n + \beta) B (nxH (1))} + 1 \right\} x^2
+ \left\{ \frac{[2A' (1) + (2\alpha + 2) A (1) + H'' (1) A (1)] n B' (nxH (1))}{(n + \beta)^2 A (1)} \right\} x
+ \frac{1}{(n + \beta)^2 A (1)} \left\{ A'' (1) + (2\alpha + 2) A' (1) + \left( \alpha^2 + \alpha + 1/3 \right) A (1) \right\}.
$$

**Proof.** Using linearity of the operators $\mathcal{K}_n^{(\alpha, \beta)}$, (2.1) and (2.5), we get

$$
\left| \mathcal{K}_n^{(\alpha, \beta)} (f ; x) - f (x) \right| \leq \frac{n + \beta}{A (1) B (nxH (1))} \sum_{k=0}^{\infty} p_k (nx)^{(k+\alpha+1)/(n+\beta)} \int_{(k+\alpha)/(n+\beta)}^{(k+\alpha+1)/(n+\beta)} \left( |s - x| + 1 \right) w (f ; \delta) \, ds
\leq \left\{ 1 + \frac{n + \beta}{A (1) B (nxH (1)) \delta} \sum_{k=0}^{\infty} p_k (nx)^{(k+\alpha+1)/(n+\beta)} \int_{(k+\alpha)/(n+\beta)}^{(k+\alpha+1)/(n+\beta)} |s - x| \, ds \right\} w (f ; \delta).
$$

If we apply the Cauchy-Schwarz inequality for integration, it follows

$$
\int_{(k+\alpha)/(n+\beta)}^{(k+\alpha+1)/(n+\beta)} |s - x| \, ds \leq \frac{1}{\sqrt{n + \beta}} \left( \int_{(k+\alpha)/(n+\beta)}^{(k+\alpha+1)/(n+\beta)} |s - x|^2 \, ds \right)^{1/2},
$$

which gives

$$
\sum_{k=0}^{\infty} p_k (nx) \int_{(k+\alpha)/(n+\beta)}^{(k+\alpha+1)/(n+\beta)} |s - x| \, ds
\leq \frac{1}{\sqrt{n + \beta}} \sum_{k=0}^{\infty} p_k (nx) \left( \int_{(k+\alpha)/(n+\beta)}^{(k+\alpha+1)/(n+\beta)} |s - x|^2 \, ds \right)^{1/2}.
$$
Considering Cauchy-Schwarz inequality for summation on the right hand side of (2.8), one can easily obtain
\[
\sum_{k=0}^{\infty} p_k(nx) \frac{(k+\alpha+1)/(n+\beta)}{(k+\alpha)/(n+\beta)} \int_{(k+\alpha)/(n+\beta)}^{|s-x|} ds 
\leq \sqrt{A(1) B(nxH(1))} \left( \frac{A(1) B(nxH(1))}{n+\beta} \right)^{1/2} \mathcal{K}_{n}^{(\alpha,\beta)} \left( \frac{(s-x)^2}{x} \right)^{1/2} 
\]
\[
= \frac{A(1) B(nxH(1))}{n+\beta} \left( \mathcal{K}_{n}^{(\alpha,\beta)} \left( \frac{(s-x)^2}{x} \right)^{1/2} \right) 
= \frac{A(1) B(nxH(1))}{n+\beta} \left( \lambda_n(x) \right)^{1/2} 
\]  
(2.9)
where \( \lambda_n(x) \) is given by (2.6). Taking into account this inequality in (2.7) leads to
\[
\left| \mathcal{K}_{n}^{(\alpha,\beta)} (f; x) - f(x) \right| \leq \left\{ 1 + \frac{1}{\delta} \sqrt{\lambda_n(x)} \right\} w(f; \delta). 
\]
If we get \( \delta = \sqrt{\lambda_n(x)} \), we obtain the desired. \( \Box \)

Now, we give the rates of convergence of the operators \( \mathcal{K}_{n}^{(\alpha,\beta)} \) to \( f \) by means of the second modulus of continuity and Peetre’s \( K \)-functional.

We remind that the second modulus of continuity of \( f \in C_B[0, \infty) \) is defined by
\[
w_2(f; \delta) := \sup_{0 < t \leq \delta} \| f(\cdot + 2t) - 2f(\cdot + t) + f(\cdot) \|_{C_B} 
\]
where \( C_B[0, \infty) \) denotes the class of real valued functions defined on \([0, \infty)\) that are bounded and uniformly continuous with the norm \( \| f \|_{C_B} = \sup_{x \in [0, \infty)} |f(x)| \).

Peetre’s \( K \)-functional of the function \( f \in C_B[0, \infty) \) is defined by
\[
K(f; \delta) := \inf_{g \in C_B[0, \infty)} \left\{ \| f - g \|_{C_B} + \delta \left\| \frac{g}{\sqrt{\delta}} \right\|_{C_B[0, \infty)} \right\} 
\]  
(2.10)
where
\[
C_B[0, \infty) := \{ g \in C_B[0, \infty) : g', g'' \in C_B[0, \infty) \} 
\]
and the norm \( \left\| \frac{g}{\sqrt{\delta}} \right\|_{C_B[0, \infty)} := \| g \|_{C_B} + \| g' \|_{C_B} + \| g'' \|_{C_B} \) (see [7]). Also, in [6] we have the following inequality:
\[
K(f; \delta) \leq M \left\{ w_2(f; \sqrt{\delta}) + \min(1, \delta) \| f \|_{C_B} \right\} 
\]  
(2.11)
for all \( \delta > 0 \) where \( M \) is a constant which is independent of the function \( f \) and \( \delta \).

**Theorem 2.5.** Let \( f \in C_B^2[0, \infty) \). For the operators \( \mathcal{K}_{n}^{(\alpha,\beta)} \) defined by (1.12), we have
\[
\left| \mathcal{K}_{n}^{(\alpha,\beta)} (f; x) - f(x) \right| \leq \zeta \| f \|_{C_B^2}, 
\]
where
\[ \zeta = \zeta_n(x) = \left\{ \frac{n^2 B''(nxH(1))}{2(n + \beta)^2 B(nxH(1))} - \frac{nB'(nxH(1))}{(n + \beta) B(nxH(1))} + \frac{1}{2} \right\} x^2 \]
\[ + \left\{ \frac{2A'(1) + (2\alpha + 2) A(1) + H''(1) A(1)}{2(n + \beta)} nB'(nxH(1)) \right\} x \]
\[ - \frac{2A'(1) + (2\alpha + 1) A(1)}{2(n + \beta) A(1)} + \frac{nB'(nxH(1))}{(n + \beta) B(nxH(1))} - 1 \}
\[ + \frac{1}{6(n + \beta)} \frac{3A''(1) + (6\alpha + 6) A'(1) + (3\alpha^2 + 3\alpha + 1) A(1)}{2(n + \beta) A(1)} \] .

**Proof.** From the Taylor expansion of \( f \), the linearity of the operators \( \mathcal{K}_n^{(\alpha, \beta)} \) and the equality (2.1), we may write for \( \eta \in (x, s) \)
\[ \mathcal{K}_n^{(\alpha, \beta)}(f; x) - f(x) = f'(x) \mathcal{K}_n^{(\alpha, \beta)}(s - x; x) + \frac{f''(\eta)}{2} \mathcal{K}_n^{(\alpha, \beta)}((s - x)^2; x) . \] (2.12)
Using the results in Lemma 2.1, we have
\[ \mathcal{K}_n^{(\alpha, \beta)}(s - x; x) = \left\{ \frac{nB'(nxH(1))}{(n + \beta) B(nxH(1))} - 1 \right\} x + \frac{2A'(1) + (2\alpha + 1) A(1)}{2(n + \beta) A(1)} \geq 0 \]
for \( s \geq x \). Thus, by considering Lemmas 2.1 and 2.2 in (2.12), we obtain
\[ \left| \mathcal{K}_n^{(\alpha, \beta)}(f; x) - f(x) \right| \leq \mathcal{K}_n^{(\alpha, \beta)}(s - x; x) \|f\|_{C_B} + \frac{1}{2} \mathcal{K}_n^{(\alpha, \beta)}((s - x)^2; x) \|f''\|_{C_B} \]
\[ \leq \left\{ \frac{n^2 B''(nxH(1))}{2(n + \beta)^2 B(nxH(1))} - \frac{nB'(nxH(1))}{(n + \beta) B(nxH(1))} + \frac{1}{2} \right\} x^2 \]
\[ + \left\{ \frac{2A'(1) + (2\alpha + 2) A(1) + H''(1) A(1)}{2(n + \beta)} nB'(nxH(1)) \right\} x \]
\[ - \frac{2A'(1) + (2\alpha + 1) A(1)}{2(n + \beta) A(1)} + \frac{nB'(nxH(1))}{(n + \beta) B(nxH(1))} - 1 \}
\[ + \frac{1}{6(n + \beta)} \frac{3A''(1) + (6\alpha + 6) A'(1) + (3\alpha^2 + 3\alpha + 1) A(1)}{2(n + \beta) A(1)} \|f\|_{C_B} \]
which completes the proof. \( \square \)

**Theorem 2.6.** If \( f \in C_B[0, \infty) \), then
\[ \left| \mathcal{K}_n^{(\alpha, \beta)}(f; x) - f(x) \right| \leq 2M \left\{ w_2 \left( f; \sqrt{\delta} \right) + \min (1, \delta) \|f\|_{C_B} \right\} \]
holds where
\[ \delta := \delta_n(x) = \frac{1}{2} \zeta_n(x) \]
and the constant \( M > 0 \) is independent of \( f \) and \( \delta \). Also, \( \zeta_n(x) \) is given as in Theorem 2.5.
**Proof.** We assume that $g \in C_B^2 [0, \infty)$. From Theorem 2.5, we can get

$$
\left| \mathcal{K}_n^{(\alpha, \beta)} (f; x) - f (x) \right| \leq 2 \| f - g \|_{C_B^2} + \zeta \| g \|_{C_B^2} + \delta \| g \|_{C_B^2}.
$$

(2.13)

Since the left-hand side of inequality (2.13) does not depend on the function $g \in C_B^2 [0, \infty)$, it follows from Peetre’s $K$-functional $K (f; \delta)$ defined by (2.10)

$$
\left| \mathcal{K}_n^{(\alpha, \beta)} (f; x) - f (x) \right| \leq 2K (f; \delta).
$$

By using the relation (2.11) in the last inequality, we obtain

$$
\left| \mathcal{K}_n^{(\alpha, \beta)} (f; x) - f (x) \right| \leq 2M \left\{ w_2 (f; \sqrt{\delta}) + \min (1, \delta) \| f \|_{C_B^2} \right\}.
$$

This concludes the proof. \qed

We note that $\lambda_n, \zeta_n, \delta_n \to 0$ when $n \to \infty$ under the assumption (2.4) in Theorems 2.4-2.6.

**Remark 2.7.** For $\alpha = \beta = 0$, the operators (1.12) reduces to the Kantorovich type operators including Boas-Buck-type polynomials given by

$$
\mathcal{K}_n (f; x) := \frac{n}{A (1) B (nxH (1))} \sum_{k=0}^{\infty} p_k (nx) \int_{k/n}^{(k+1)/n} f (t) \, dt.
$$

For $\alpha = \beta = 0$, the results given above are satisfied by the Kantorovich type operators including Boas-Buck-type polynomials.

**Remark 2.8.** In the case of $H (t) = t$, the results obtained in the paper capture the results obtained for Kantorovich-Stancu type operators (1.6) including Brenke-type polynomials in [1].

### 3. Special cases of the operators $\mathcal{K}_n^{(\alpha, \beta)}$

**Case 1.** Gould-Hopper polynomials $g_k^{d+1} (x, h)$ are defined through the identity

$$
g_k^{d+1} (x, h) = \sum_{m=0}^{\left\lfloor \frac{k}{d+1} \right\rfloor} \frac{k!}{m! (k - (d + 1) m)!} h^m x^{k-(d+1)m}
$$

where, as usual, $\left\lfloor . \right\rfloor$ denotes the integer part [10], and they have generating function of the form

$$
e^{ht^{d+1}} \exp (xt) = \sum_{k=0}^{\infty} g_k^{d+1} (x, h) \frac{t^k}{k!}.
$$

(3.1)

Gould-Hopper polynomials are Boas-Buck-type polynomials with for the special case of $A (t) = e^{ht^{d+1}}$, $B (t) = e^t$ and $H (t) = t$ in (1.8). From (1.12), Kantorovich-Stancu type operators including the Gould-Hopper polynomials are as follows:

$$
\mathcal{K}_n^{(\alpha, \beta)} (f; x) := (n + \beta) e^{-nx-h} \sum_{k=0}^{\infty} g_k^{d+1} (nx, h) \frac{(k+\alpha+1)/(n+\beta)}{(k+\alpha)/(n+\beta)} \int_{k/n}^{(k+1)/n} f (t) \, dt
$$

where $x \in [0, \infty)$ and $h \geq 0$ in [1].

**Case 2.** The Charlier polynomials $C_k^{(a)} (x)$ are generated by

$$
e^t \left( 1 - \frac{t}{a} \right)^x = \sum_{k=0}^{\infty} C_k^{(a)} (x) \frac{t^k}{k!}, \quad |t| < a.
$$
Charlier polynomials are the Boas-Buck-type polynomials for the choice $A(t) = e^t$, $B(t) = e^t$ and $H(t) = \ln(1 - \frac{t}{a})$ in (1.8). In order to ensure the restrictions (1.11) and the assumption (2.4), we get the generating function as

$$e^t e^{-(a-1)x \ln(1-t/a)} = \sum_{k=0}^{\infty} C_k^{(a)} \frac{(- (a-1)x)^k}{k!} , \quad |t| < a, \ a > 1.$$  

In this case, the operator (1.12) turns to

$$T_n^{(\alpha,\beta)}(f; x) := (n + \beta) e^{-1} \left(1 - \frac{1}{a}\right)^{(a-1)nx} \sum_{k=0}^{\infty} C_k^{(a)} \frac{(- (a-1) nx)^k}{k!} \int_{(k+\alpha)/(n+\beta)}^{(k+\alpha+1)/(n+\beta)} f(t) \ dt.$$  

For $\alpha = \beta = 0$, we have Szász-Kantorovich type operators based on Charlier polynomials in [17].

4. Some graphical representations

In this section, we give the graphs to compare approximation properties of the operators $K_n^{(\alpha,\beta)}$ with $B_n$.

Firstly, we use $f(x) = e^{-x}$, $\alpha = 1$, $\beta = 2$, $H(t) = t$, $A(t) = 1$, $B(t) = e^t$ and $n = 10, 20, 50, 100$ for operators $K_n^{(\alpha,\beta)}$. In Figure 1, red color line for $n = 10$, green color line for $n = 20$, brown color line for $n = 50$, purple color line for $n = 100$ and black color line for $f(x)$.

![Figure 1](image)

Now, we use $f(x) = \sin(2\pi x)$, $\alpha = 1$, $\beta = 2$, $H(t) = t$, $A(t) = 1$, $B(t) = e^t$ and $n = 20, 50, 100, 200, 500$ for operators $K_n^{(\alpha,\beta)}$. In Figure 2, navy color line for $n = 20$, light green color line for $n = 50$, blue color line for $n = 100$, pink color line for $n = 200$, green color line for $n = 500$ and black color line for $f(x)$ in $x \in [0.5, 1]$. 

Finally, we compare the operators $\mathcal{K}_n^{(\alpha,\beta)}$ with $B_n$ for $f(x) = e^{-x}$, $n = 100$, $\alpha = 1$, $\beta = 2$, $H(t) = t$, $A(t) = 1$, $B(t) = e^t$. In Figure 3, red color line for $\mathcal{K}_n^{(\alpha,\beta)}$, blue color line for $B_n$, black color line for $f(x)$.

References


On generalized autocommutativity degree of finite groups

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Abstract
Let $H$ be a subgroup of a finite group $G$ and $\text{Aut}(G)$ be the automorphism group of $G$. In this paper we introduce and study the probability that the autocommutator of a randomly chosen pair of elements, one from $H$ and the other from $\text{Aut}(G)$, is equal to a given element of $G$.

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1. Introduction

Throughout the paper $H$ denotes a subgroup of a finite group $G$ and $\text{Aut}(G)$ denotes automorphism group of $G$. The autocommutativity degree of $G$, denoted by $\Pr(G, \text{Aut}(G))$, is the probability that an automorphism fixes an element of $G$. In other words,

$$\Pr(G, \text{Aut}(G)) = \frac{|\{(x, \alpha) \in G \times \text{Aut}(G) : [x, \alpha] = 1\}|}{|G||\text{Aut}(G)|}$$

where $[x, \alpha] = x^{-1}\alpha(x)$ is the autocommutator of $x$ and $\alpha$. The study of autocommutativity degree of finite groups was initiated by Sherman [10] in 1975. Many results on $\Pr(G, \text{Aut}(G))$, including some characterizations of $G$ in terms of $\Pr(G, \text{Aut}(G))$, can be found in [1, 3]. In the year 2015, Rismanchian and Sepehrizadeh [9] generalized the concept of autocommutativity degree and studied relative autocommutativity degree of $H$, that is the probability that an automorphism of $G$ fixes an element of $H$. However in the year 2011, Moghaddam et al. [8] also studied this notion. We write $\Pr(H, \text{Aut}(G))$ to denote the relative autocommutativity degree of $H$. Recently, we have obtained several new results on $\Pr(H, \text{Aut}(G))$ in [2]. In this paper, we introduce a new probability concept called the generalized relative autocommutativity degree of $H$ given by the following ratio

$$\Pr_g(H, \text{Aut}(G)) = \frac{|\{(x, \alpha) \in H \times \text{Aut}(G) : [x, \alpha] = g\}|}{|H||\text{Aut}(G)|}$$

(1.1)

where $g$ is an element of $G$. In other words $\Pr_g(H, \text{Aut}(G))$ is the probability that the autocommutator of a randomly chosen pair of elements, one from $H$ and the other from $\text{Aut}(G)$, is equal to a given element $g \in G$. Clearly, if $g = 1$ (the identity element of $G$) then $\Pr_g(H, \text{Aut}(G)) = \Pr(H, \text{Aut}(G))$. In the forthcoming sections, we obtain some computing

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formulae and bounds for $\Pr_g(H, \Aut(G))$. We also obtain some characterizations of groups through $\Pr_g(H, \Aut(G))$. 

Let $S(H, \Aut(G)) = \{\langle x, \alpha \rangle : x \in H \text{ and } \alpha \in \Aut(G) \}$ and $[H, \Aut(G)]$ be the subgroup generated by $S(H, \Aut(G))$. Let $L(H, \Aut(G)) = \{x \in H : \langle x, \alpha \rangle = 1 \text{ for all } \alpha \in \Aut(G) \}$ and $L(G) = L(G, \Aut(G))$, the absolute center of $G$ defined in [5]. Clearly, $L(H, \Aut(G))$ is a normal subgroup of $H$ contained in $H \cap Z(G)$. Let $C_{\Aut(G)}(x) = \{\alpha \in \Aut(G) : \alpha(x) = x\}$ for $x \in G$ and $C_{\Aut(G)}(H) = \{\alpha \in \Aut(G) : \alpha(x) = x \text{ for all } x \in H\}$. Then $C_{\Aut(G)}(x)$ is a subgroup of $\Aut(G)$ and $C_{\Aut(G)}(H) = \cap_{x \in H} C_{\Aut(G)}(x)$. Note that if $g \notin S(H, \Aut(G))$ then $\Pr_g(H, \Aut(G)) = 0$, therefore throughout the paper we consider $g \in S(H, \Aut(G))$.

2. Some computing formulae

We begin with the following results.

**Proposition 2.1.** Let $H$ be a subgroup of $G$. If $g \in G$ then

$$\Pr_{g^{-1}}(H, \Aut(G)) = \Pr_g(H, \Aut(G)).$$

**Proof.** Let $A = \{(x, \alpha) \in H \times \Aut(G) : [x, \alpha] = g\}$ and $B = \{(y, \beta) \in H \times \Aut(G) : [y, \beta] = g^{-1}\}$. Then $(x, \alpha) \mapsto (\alpha(x), \alpha^{-1})$ gives a bijection between $A$ and $B$. Therefore, $|A| = |B|$ and hence the result follows from (1.1). □

**Proposition 2.2.** Let $G_1$ and $G_2$ be two finite groups such that $\gcd(|G_1|, |G_2|) = 1$. Let $H_1$ and $H_2$ be subgroups of $G_1$ and $G_2$ respectively. If $(g_1, g_2) \in G_1 \times G_2$ then

$$\Pr_{(g_1, g_2)}(H_1 \times H_2, \Aut(G_1 \times G_2)) = \Pr_{g_1}(H_1, \Aut(G_1)) \Pr_{g_2}(H_2, \Aut(G_2)).$$

**Proof.** Let

$$X = \{(x, y) : \alpha_{G_1 \times G_2} \in (H_1 \times H_2) \times \Aut(G_1 \times G_2) : [x, y, \alpha_{G_1 \times G_2}] = (g_1, g_2)\},$$

$$Y = \{(x, \alpha_{G_1}) \in H_1 \times \Aut(G_1) : [x, \alpha_{G_1}] = g_1\}$$

and

$$Z = \{(y, \alpha_{G_2}) \in H_2 \times \Aut(G_2) : [y, \alpha_{G_2}] = g_2\}.$$

Since $\gcd(|G_1|, \ |G_2|) = 1$, by [6, Lemma 2.1], we have $\Aut(G_1 \times G_2) = \Aut(G_1) \times \Aut(G_2)$. Therefore, for every $\alpha_{G_1 \times G_2} \in \Aut(G_1 \times G_2)$ there exist unique $\alpha_{G_1} \in \Aut(G_1)$ and $\alpha_{G_2} \in \Aut(G_2)$ such that $\alpha_{G_1 \times G_2} = \alpha_{G_1} \times \alpha_{G_2}$, where $\alpha_{G_1 \times G_2}((x, y)) = (\alpha_{G_1}(x), \alpha_{G_2}(y))$ for all $(x, y) \in H_1 \times H_2$. Also, for all $(x, y) \in H_1 \times H_2$, we have $[x, y, \alpha_{G_1 \times G_2}] = (g_1, g_2)$ if and only if $[x, \alpha_{G_1}] = g_1$ and $[y, \alpha_{G_2}] = g_2$. These lead to show that $X = Y \times Z$. Therefore

$$\frac{|X|}{|H_1 \times H_2| |\Aut(G_1 \times G_2)|} = \frac{|Y|}{|H_1| |\Aut(G_1)|} \cdot \frac{|Z|}{|H_2| |\Aut(G_2)|}.$$ 

Hence, the result follows from (1.1). □

In the year 1940, Hall [4] introduced the concept of isoclinism between two groups. Following Hall, Moghaddam et al. [7] have defined autoisoclinism between two groups, in the year 2013. Recently in [2], we generalize the notion of autoisoclinism between two groups. Let $H_1$ and $H_2$ be subgroups of the groups $G_1$ and $G_2$ respectively. The pairs $(H_1, G_1)$ and $(H_2, G_2)$ are said to be autoisoclinic if there exist isomorphisms $\psi : \frac{H_1}{L(H_1, \Aut(G_1))} \rightarrow \frac{H_2}{L(H_2, \Aut(G_2))}$, $\beta : [H_1, \Aut(G_1)] \rightarrow [H_2, \Aut(G_2)]$ and $\gamma : \Aut(G_1) \rightarrow \Aut(G_2)$ such that the following diagram commutes

$$\begin{array}{c}
\frac{H_1}{L(H_1, \Aut(G_1))} \times \Aut(G_1) \xrightarrow{\psi \times \gamma} \frac{H_2}{L(H_2, \Aut(G_2))} \times \Aut(G_2) \\
\downarrow \alpha_{(H_1, \Aut(G_1))} \quad \downarrow \alpha_{(H_2, \Aut(G_2))} \\
[H_1, \Aut(G_1)] \quad \rightarrow \quad [H_2, \Aut(G_2)]
\end{array}$$
where the maps \( a_{(H_i, \text{Aut}(G_i))} : \frac{H_i}{L(H_i, \text{Aut}(G_i))} \times \text{Aut}(G_i) \to [H_i, \text{Aut}(G_i)] \), for \( i = 1, 2 \), are given by
\[
a_{(H_i, \text{Aut}(G_i))}(x_i L(H_i, \text{Aut}(G_i)), \alpha_i) = [x_i, \alpha_i].
\]
Such a pair \((\psi \times \gamma, \beta)\) is said to be an autoisoclinism between the pairs of groups \((H_1, G_1)\) and \((H_2, G_2)\). We have the following generalization of [3, Theorem 5.1] and [9, Lemma 2.5].

**Theorem 2.3.** Let \( G_1 \) and \( G_2 \) be two finite groups with subgroups \( H_1 \) and \( H_2 \) respectively. If \((\psi \times \gamma, \beta)\) is an autoisoclinism between the pairs \((H_1, G_1)\) and \((H_2, G_2)\) then, for \( g \in G_1 \),
\[
\text{Pr}_{g, \beta(g)}(H_2, \text{Aut}(G_2)) = \text{Pr}_g(H_1, \text{Aut}(G_1)).
\]

**Proof.** Let \( S_g = \{(x_1 L(H_1, \text{Aut}(G_1)), \alpha_1) \in \frac{H_1}{L(H_1, \text{Aut}(G_1))} \times \text{Aut}(G_1) : [x_1, \alpha_1] = g\} \) and \( T_{\beta(g)} = \{(x_2 L(H_2, \text{Aut}(G_2)), \alpha_2) \in \frac{H_2}{L(H_2, \text{Aut}(G_2))} \times \text{Aut}(G_2) : [x_2, \alpha_2] = \beta(g)\} \). Since \((H_1, G_1)\) is autoisoclinic to \((H_2, G_2)\) we have \( |S_g| = |T_{\beta(g)}| \). Again, it is clear that
\[
|\{(x_1, \alpha_1) \in H_1 \times \text{Aut}(G_1) : [x_1, \alpha_1] = g\}| = |L(H_1, \text{Aut}(G_1))||S_g| \quad (2.1)
\]
and
\[
|\{(x_2, \alpha_2) \in H_2 \times \text{Aut}(G_2) : [x_2, \alpha_2] = \beta(g)\}| = |L(H_2, \text{Aut}(G_2))||T_{\beta(g)}| \quad (2.2)
\]
Hence, the result follows from (1.1), (2.1) and (2.2).

Note that \( \text{Aut}(G) \) acts on \( G \) by the action \((\alpha, x) \mapsto \alpha(x)\) where \( \alpha \in \text{Aut}(G) \) and \( x \in G \). Let \( \text{orb}(x) = \{\alpha(x) : \alpha \in \text{Aut}(G)\} \) be the orbit of \( x \in G \). Then by orbit-stabilizer theorem, we have
\[
|\text{orb}(x)| = \frac{\text{order}(\text{Aut}(G))}{\text{order}(\text{C}_{\text{Aut}(G)}(x))}.
\]
Now we obtain the following computing formula for \( \text{Pr}_g(H, \text{Aut}(G)) \) in terms of the order of \( \text{C}_{\text{Aut}(G)}(x) \) and \( \text{orb}(x) \).

**Theorem 2.4.** Let \( H \) be a subgroup of \( G \). If \( g \in G \) then
\[
\text{Pr}_g(H, \text{Aut}(G)) = \frac{1}{|H||\text{Aut}(G)|} \sum_{x \in H} |\text{C}_{\text{Aut}(G)}(x)| = \frac{1}{|H|} \sum_{x \in H} \frac{1}{|\text{orb}(x)|}.
\]

**Proof.** Let \( T_{x,g}(H, G) = \{\alpha \in \text{Aut}(G) : x \alpha = g\} \) for any \( x \in H \). Then \( T_{x,g}(H, G) \neq \emptyset \) if and only if \( x g \in \text{orb}(x) \). We also have
\[
\{(x, \alpha) \in H \times \text{Aut}(G) : x \alpha = g\} = \biguplus_{x \in H} \{(x) \times T_{x,g}(H, G)\},
\]
where \( \biguplus \) represents the union of disjoint sets. Therefore, by (1.1), we have
\[
|H||\text{Aut}(G)||\text{Pr}_g(H, \text{Aut}(G))| = \biguplus_{x \in H} |\{x \times T_{x,g}(H, G)\}| = \sum_{x \in H} |T_{x,g}(H, G)|. \quad (2.3)
\]

Let \( \sigma \in T_{x,g}(H, G) \) and \( \beta \in \sigma \text{C}_{\text{Aut}(G)}(x) \). Then \( \beta = \sigma \alpha \) for some \( \alpha \in \text{C}_{\text{Aut}(G)}(x) \). We have
\[
[x, \beta] = [x, \sigma \alpha] = x^{-1} \sigma(\alpha(x)) = [x, \sigma] = g.
\]
Therefore, \( \beta \in T_{x,g}(H, G) \) and so \( \sigma \text{C}_{\text{Aut}(G)}(x) \subseteq T_{x,g}(H, G) \). Again, let \( \gamma \in T_{x,g}(H, G) \) then \( \gamma(x) = x g \). We have \( \sigma^{-1} \gamma(x) = \sigma^{-1}(x g) = x \) and so \( \sigma^{-1} \gamma \in \text{C}_{\text{Aut}(G)}(x) \). Therefore, \( \gamma \in \sigma \text{C}_{\text{Aut}(G)}(x) \) which gives \( T_{x,g}(H, G) \subseteq \sigma \text{C}_{\text{Aut}(G)}(x) \). Thus, \( \sigma \text{C}_{\text{Aut}(G)}(x) = T_{x,g}(H, G) \) and hence
\[
|T_{x,g}(H, G)| = |\text{C}_{\text{Aut}(G)}(x)| = \frac{|\text{Aut}(G)|}{|\text{orb}(x)|}. \quad (2.4)
\]
Therefore, the result follows from (2.3) and (2.4).

Putting \( g = 1 \) in Theorem 2.4 we get the following corollary.
Corollary 2.5. Let $H$ be a subgroup of $G$. Then
\[
\Pr(H, \Aut(G)) = \frac{1}{|H|} \sum_{x \in H} |C_{\Aut(G)}(x)| = \frac{|\orb(H)|}{|H|}
\]
where $\orb(H) = \{\orb(x) : x \in H\}$.

As an application of Theorem 2.4 we have the following result.

Proposition 2.6. Let $H$ be a subgroup of $G$. If $\orb(x) = x[H, \Aut(G)]$ for all $x \in H \setminus L(H, \Aut(G))$ then
\[
\Pr_g(H, \Aut(G)) = \begin{cases}
\frac{1}{|H|} \left( 1 + \frac{|H, \Aut(G)| - 1}{|H : L(H, \Aut(G))|} \right), & \text{if } g = 1 \\
\frac{1}{|H|} \left( 1 - \frac{1}{|H : L(H, \Aut(G))|} \right), & \text{if } g \neq 1.
\end{cases}
\]

Proof. If $g = 1$ then the result follows from [2, Proposition 3.4]. If $g \neq 1$, we have $xg \notin \orb(x)$ for all $x \in L(H, \Aut(G))$. Again, since $g \in S(H, \Aut(G)) \subseteq [H, \Aut(G)]$ therefore $xg \in x[H, \Aut(G)] = \orb(x)$ for all $x \in H \setminus L(H, \Aut(G))$. Now from Theorem 2.4 we have
\[
\Pr_g(H, \Aut(G)) = \frac{1}{|H|} \sum_{x \in H \setminus L(H, \Aut(G))} \frac{1}{|\orb(x)|} \sum_{xg \in \orb(x)} 1
\]
\[
= \frac{1}{|H|} \sum_{x \in H \setminus L(H, \Aut(G))} \frac{1}{|H, \Aut(G)|}
\]
\[
= \frac{1}{|H, \Aut(G)|} \left( 1 - \frac{1}{|H : L(H, \Aut(G))|} \right).
\]

3. Various bounds

In this section, we obtain various bounds for $\Pr_g(H, \Aut(G))$. We begin with the following lower bounds.

Proposition 3.1. Let $H$ be a subgroup of $G$. Then, for $g \in G$, we have
\[
\Pr_g(H, \Aut(G)) \geq \begin{cases}
\frac{|L(H, \Aut(G))|}{|H|} + \frac{|C_{\Aut(G)}(H)||H - |L(H, \Aut(G))||}{|H||\Aut(G)|}, & \text{if } g = 1 \\
\frac{|L(H, \Aut(G))||C_{\Aut(G)}(H)||}{|H||\Aut(G)|}, & \text{if } g \neq 1.
\end{cases}
\]

Proof. Let $\mathcal{C}$ denotes the set $\{(x, \alpha) \in H \times \Aut(G) : [x, \alpha] = g\}$.

If $g = 1$ then $(L(H, \Aut(G)) \times \Aut(G)) \cup (H \times C_{\Aut(G)}(H))$ is a subset of $\mathcal{C}$ and $|(L(H, \Aut(G)) \times \Aut(G)) \cup (H \times C_{\Aut(G)}(H))| = |L(H, \Aut(G))||\Aut(G)| + |C_{\Aut(G)}(H)||H - |L(H, \Aut(G))||C_{\Aut(G)}(H)|$. Hence, the result follows from (1.1).

If $g \neq 1$ then $\mathcal{C}$ is non-empty since $g \in S(H, \Aut(G))$. Let $(y, \beta) \in \mathcal{C}$ then $(y, \beta) \notin (L(H, \Aut(G)) \times C_{\Aut(G)}(H))$ otherwise $[y, \beta] = 1$. It is easy to see that the coset $(y, \beta)(L(H, \Aut(G)) \times C_{\Aut(G)}(H))$ having order $|L(H, \Aut(G))||C_{\Aut(G)}(H)|$ is a subset of $\mathcal{C}$. Hence, the result follows from (1.1). \qed

Proposition 3.2. Let $H$ be a subgroup of $G$. If $g \in G$ then
\[
\Pr_g(H, \Aut(G)) \leq \Pr(H, \Aut(G)).
\]
The equality holds if and only if $g = 1$. 
Proof. By Theorem 2.4, we have
\[ \text{Pr}_g(H, \text{Aut}(G)) = \frac{1}{|H||\text{Aut}(G)|} \sum_{x \in H} |\text{C}_{\text{Aut}(G)}(x)| \]
\[ \leq \frac{1}{|H||\text{Aut}(G)|} \sum_{x \in H} |\text{C}_{\text{Aut}(G)}(x)| = \text{Pr}(H, \text{Aut}(G)). \]
Clearly the equality holds if and only if \( g = 1 \).

**Proposition 3.3.** Let \( H \) be a subgroup of \( G \). Let \( g \in G \) and \( p \) the smallest prime dividing \( |\text{Aut}(G)| \). If \( g \neq 1 \) then
\[ \text{Pr}_g(H, \text{Aut}(G)) \leq \frac{|H| - |L(H, \text{Aut}(G))|}{p|H|} < \frac{1}{p}. \]

**Proof.** By Theorem 2.4, we have
\[ \text{Pr}_g(H, \text{Aut}(G)) = \frac{1}{|H|} \sum_{x \in H \setminus L(H, \text{Aut}(G))} \frac{1}{|\text{orb}(x)|} \]
noting that for \( x \in L(H, \text{Aut}(G)) \) we have \( xg \notin \text{orb}(x) \). Also, for \( x \in H \setminus L(H, \text{Aut}(G)) \) and \( xg \in \text{orb}(x) \) we have \( |\text{orb}(x)| > 1 \). Since \( |\text{orb}(x)| \) is a divisor of \( |\text{Aut}(G)| \) we have \( |\text{orb}(x)| \geq p \). Hence, the result follows from (3.1).

**Proposition 3.4.** Let \( H_1 \) and \( H_2 \) be two subgroups of \( G \) such that \( H_1 \subseteq H_2 \). Then
\[ \text{Pr}_g(H_1, \text{Aut}(G)) \leq |H_2 : H_1| \text{Pr}_g(H_2, \text{Aut}(G)). \]
The equality holds if and only if \( xg \notin \text{orb}(x) \) for all \( x \in H_2 \setminus H_1 \).

**Proof.** By Theorem 2.4, we have
\[ |H_1||\text{Aut}(G)||\text{Pr}_g(H_1, \text{Aut}(G))| = \sum_{x \in H_1, xg \notin \text{orb}(x)} |\text{C}_{\text{Aut}(G)}(x)| \]
\[ \leq \sum_{x \in H_2, xg \notin \text{orb}(x)} |\text{C}_{\text{Aut}(G)}(x)| \]
\[ = |H_2||\text{Aut}(G)||\text{Pr}_g(H_2, \text{Aut}(G))|. \]
Hence, the result follows.

We conclude this section with the following result.

**Proposition 3.5.** Let \( H \) be a subgroup of \( G \). If \( g \in G \) then
\[ \text{Pr}_g(H, \text{Aut}(G)) \leq |G : H| \text{Pr}(G, \text{Aut}(G)) \]
with equality if and only if \( g = 1 \) and \( H = G \).

**Proof.** By Proposition 3.2, we have
\[ \text{Pr}_g(H, \text{Aut}(G)) \leq \text{Pr}(H, \text{Aut}(G)) \]
\[ = \frac{1}{|H||\text{Aut}(G)|} \sum_{x \in H} |\text{C}_{\text{Aut}(G)}(x)| \]
\[ \leq \frac{1}{|H||\text{Aut}(G)|} \sum_{x \in G} |\text{C}_{\text{Aut}(G)}(x)| \]
\[ = |G : H| \text{Pr}(G, \text{Aut}(G)). \]
Hence, the result follows from Corollary 2.5.
4. Characterizations through $\Pr_g(H, \text{Aut}(G))$

In this section, we obtain some characterizations of groups through $\Pr_g(H, \text{Aut}(G))$. The following lemma is useful in this regard.

**Lemma 4.1.** Let $H$ be a subgroup of $G$. If $p$ is the smallest prime divisor of $|\text{Aut}(G)|$ and $|[H, \text{Aut}(G)]| = p$ then $\text{orb}(x) = x[H, \text{Aut}(G)]$ for all $x \in H \setminus L(H, \text{Aut}(G))$.

**Proof.** We have $\text{orb}(x) \subseteq x[H, \text{Aut}(G)]$ for all $x \in H$. Also, $|\text{orb}(x)|$ is a divisor of $|\text{Aut}(G)|$ for all $x \in H$. Therefore, $|\text{orb}(x)| \geq p$ for all $x \in H \setminus L(H, \text{Aut}(G))$. Hence, $|\text{orb}(x)| = |x[H, \text{Aut}(G)]| = p$ for all $x \in H \setminus L(H, \text{Aut}(G))$ and the result follows. \hfill $\square$

Now we derive the following characterizations.

**Theorem 4.2.** Let $H$ be a subgroup of a finite group $G$ and $g \in G$. Let $p$ be the smallest prime dividing $|\text{Aut}(G)|$ and $|[H, \text{Aut}(G)]| = p$. If $g \neq 1$ and $\Pr_g(H, \text{Aut}(G)) = \frac{n-1}{np}$ or $g = 1$ and $\Pr_g(H, \text{Aut}(G)) = \frac{n+p-1}{np}$ (where $n$ is a positive integer) then $\frac{H}{L(H, \text{Aut}(G))}$ is isomorphic to a group of order $n$. In particular,

1. if $n = q$ or $q^2$ for some prime $q$ then $\frac{H}{L(H, \text{Aut}(G))} \cong \mathbb{Z}_q^q$, $\mathbb{Z}_q^2$ or $\mathbb{Z}_q \times \mathbb{Z}_q$.
2. if $H$ is abelian and $n = q_1^{k_1} q_2^{k_2} \cdots q_m^{k_m}$, where $q_i$'s are primes not necessarily distinct, then $\frac{H}{L(H, \text{Aut}(G))} \cong \mathbb{Z}_{q_1}^{k_1} \times \mathbb{Z}_{q_2}^{k_2} \times \cdots \times \mathbb{Z}_{q_m}^{k_m}$.

**Proof.** If $g \neq 1$ and $\Pr_g(H, \text{Aut}(G)) = \frac{n-1}{np}$ then, by Lemma 4.1 and Proposition 2.6, we have

$$\frac{n-1}{np} = \frac{1}{p} \left( 1 - \frac{1}{|H : L(H, \text{Aut}(G))|} \right)$$

which gives $|H : L(H, \text{Aut}(G))| = n$. If $g = 1$ and $\Pr_g(H, \text{Aut}(G)) = \frac{n+p-1}{np}$ then, by Lemma 4.1 and Proposition 2.6, we have

$$\frac{n+p-1}{np} = \frac{1}{p} \left( 1 - \frac{p-1}{|H : L(H, \text{Aut}(G))|} \right)$$

which also gives $|H : L(H, \text{Aut}(G))| = n$.

Hence, $\frac{H}{L(H, \text{Aut}(G))}$ is isomorphic to a group of order $n$.

1. If $n = q$ or $q^2$ for some prime $q$ then $|H : L(H, \text{Aut}(G))| = q$ or $q^2$. Therefore $\frac{H}{L(H, \text{Aut}(G))}$ is abelian. Hence, the result follows from fundamental theorem of finite abelian groups.
2. If $H$ is abelian and $n = q_1^{k_1} q_2^{k_2} \cdots q_m^{k_m}$, where $q_i$'s are primes not necessarily distinct then $\frac{H}{L(H, \text{Aut}(G))}$ is an abelian group of order $q_1^{k_1} q_2^{k_2} \cdots q_m^{k_m}$. Hence, the result follows from fundamental theorem of finite abelian groups. \hfill $\square$

Putting $H = G$, in Theorem 4.2, we have the following corollary.

**Corollary 4.3.** Let $G$ be a finite group and $g \in G$. Let $p$ be the smallest prime dividing $|\text{Aut}(G)|$ and $|[G, \text{Aut}(G)]| = p$. If $g \neq 1$ and $\Pr_g(G, \text{Aut}(G)) = \frac{n-1}{np}$ or $g = 1$ and $\Pr_g(G, \text{Aut}(G)) = \frac{n+p-1}{np}$ (where $n$ is a positive integer) then $\frac{G}{L(G)}$ is isomorphic to a group of order $n$. In particular,

1. if $n = q$ or $q^2$ for some prime $q$ then $\frac{G}{L(G)} \cong \mathbb{Z}_q^q$, $\mathbb{Z}_q^2$ or $\mathbb{Z}_q \times \mathbb{Z}_q$.
2. if $G$ is abelian and $n = q_1^{k_1} q_2^{k_2} \cdots q_m^{k_m}$, where $q_i$'s are primes not necessarily distinct, then $\frac{G}{L(G)} \cong \mathbb{Z}_{q_1}^{k_1} \times \mathbb{Z}_{q_2}^{k_2} \times \cdots \times \mathbb{Z}_{q_m}^{k_m}$.

We conclude the paper with the following result which gives converse of Theorem 4.2.
Theorem 4.4. Let $H$ be a subgroup of a finite group $G$ and $g \in G$. Let $p$ be the smallest prime dividing $|\text{Aut}(G)|$ and $|[H, \text{Aut}(G)]|=p$. If $\frac{H}{L(H, \text{Aut}(G))}$ is isomorphic to a group of order $n$ then

$$\text{Pr}_g(H, \text{Aut}(G)) = \begin{cases} 
\frac{n-1}{np}, & \text{if } g \neq 1 \\
\frac{n+p-1}{np}, & \text{if } g = 1.
\end{cases}$$

Proof. If $p$ is the smallest prime dividing $|\text{Aut}(G)|$ and $|[H, \text{Aut}(G)]| = p$ then, by Lemma 4.1, we have $\text{orb}(x) = x[H, \text{Aut}(G)]$ for all $x \in H \setminus L(H, \text{Aut}(G))$. Therefore, by Proposition 2.6, we have

$$\text{Pr}_g(H, \text{Aut}(G)) = \begin{cases} 
\frac{1}{p} \left( 1 + \frac{p-1}{|H : L(H, \text{Aut}(G))|} \right), & \text{if } g = 1 \\
\frac{1}{p} \left( 1 - \frac{1}{|H : L(H, \text{Aut}(G))|} \right), & \text{if } g \neq 1.
\end{cases}$$

If $\frac{H}{L(H, \text{Aut}(G))}$ is isomorphic to a group of order $n$ then $|H : L(H, \text{Aut}(G))| = n$ and hence the result follows. 

Note that putting $H = G$ in Theorem 4.4, we get the converse of Corollary 4.3.

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References

$n$-Hopfian and $n$-co-Hopfian Abelian Groups

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Abstract

For any natural number $n$ we define and study the two notions of $n$-Hopfian and $n$-co-Hopfian abelian groups. These groups form proper subclasses of the classes of Hopfian and co-Hopfian groups, respectively, and some of their exotic properties are established as well. We also consider and investigate $\omega$-Hopfian and $\omega$-co-Hopfian modules over the formal matrix ring.

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Keywords. $n$-Hopfian groups, Hopfian groups, $n$-co-Hopfian groups, co-Hopfian groups

1. Introduction and background

All groups into consideration in this paper, unless specified something else, are assumed to be abelian. The used notions and notations are classical as the unexplained ones follow those from [4], [5] and [10]. For instance, for a group $G$, the symbol $t(G)$ denotes its torsion part.

Recall that a group $G$ is said to be Hopfian if each epimorphism $G \to G$ is an automorphism. Also, it is well known that a group is Hopfian if, and only if, it does not have proper isomorphic quotient groups.

Some obvious examples of such groups are these:

- Finite groups.
- All torsion-free groups of finite rank.
- Every group $G$ with endomorphism ring $E(G) \cong \mathbb{Z}$; in particular, the group of integers $\mathbb{Z}$.

In [4, Problem 75] was asked to explore Hopfian groups. Our strategy here is devoted to the comprehensive investigation of Hopfian groups with torsion automorphism group. In regard to that, a new way to sharp somewhat the concept of Hopficity is the following one, in which there is some part of novelty:

**Definition 1.1.** A group $G$ is called $n$-Hopfian if there exists a natural $n$ such that for each epimorphism $\varphi$ of $G$ the equality $\varphi^n = 1$ holds. If, however, for every epimorphism $\varphi$ there is a positive integer $n(\varphi)$ with $\varphi^{n(\varphi)} = 1$, then $G$ is said to be $\omega$-Hopfian.

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It is self-evident that \( n \)-Hopfian groups are \( \omega \)-Hopfian for any \( n \in \mathbb{N} \), but as it will be shown in the sequel the converse is manifestly untrue. Moreover, a Hopfian group is \( \omega \)-Hopfian (respectively, \( n \)-Hopfian for some \( n \in \mathbb{N} \)) if, and only if, its automorphism group is torsion (respectively, bounded by \( n \)).

Some obvious examples of such groups are these:

- Finite groups are \( n \)-Hopfian, where \( n \) is the LCM of the orders of its automorphisms.
- Any group \( G \) with \( E(G) \cong \mathbb{Z} \) is 2-Hopfian. In particular, \( \mathbb{Z} \) is 2-Hopfian.

In fact, each endomorphism acts as a multiple of some integer \( n \). If \( nG = G \), then \( E(G) \) will also be divisible by \( n \). Thus all epimorphisms are multiple of \( \pm 1 \). Hence \( G \) is 2-Hopfian. Notice also that for a torsion-free group \( G \) of rank 1 it is true that \( E(G) \cong \mathbb{Z} \) if, and only if, \( G \neq pG \) for each prime \( p \).

Some other non-trivial constructions of \( n \)-Hopfian groups will be given below.

Recall that a group \( G \) is said to be co-Hopfian if each monomorphism \( G \rightarrow G \) is an automorphism. Also, it is well known that a group is co-Hopfian if, and only if, it does not have proper isomorphic subgroups to itself.

Some obvious examples of such groups are these:

- Finite groups.
- The quasi-cyclic (Prüfer group) group \( \mathbb{Z}(p^\infty) \).
- Every group whose non-zero endomorphisms are epimorphisms.

A new way to sharpen somewhat the concept of co-Hopficity is the following one:

**Definition 1.2.** A group \( G \) is called \( n \)-co-Hopfian if there exists a natural \( n \) such that for each monomorphism \( \varphi \) of \( G \) the equality \( \varphi^n = 1 \) holds. If, however, for every monomorphism \( \varphi \) there is a positive integer \( n(\varphi) \) with \( \varphi^n(\varphi) = 1 \), then \( G \) is said to be \( \omega \)-co-Hopfian.

It is self-evident that \( n \)-co-Hopfian groups are \( \omega \)-co-Hopfian for all \( n \in \mathbb{N} \), but as it will be shown in the sequel the converse is manifestly untrue. However, a co-Hopfian group is \( \omega \)-co-Hopfian (respectively, \( n \)-co-Hopfian for some \( n \in \mathbb{N} \)) if, and only if, its automorphism group is torsion (respectively, bounded by \( n \)).

Some obvious examples of such groups are these:

- Finite groups are \( n \)-co-Hopfian, where \( n \) is the LCM of the orders of its automorphisms.
- Every reduced group whose non-zero endomorphisms are epimorphisms, that is, \( \mathbb{Z}(p) \).

Some other non-trivial constructions of \( n \)-co-Hopfian groups will be given below.

The leitmotif of the present article is to explore in all details the two new notions of \( n \)-Hopficity and \( n \)-co-Hopficity and to compare the obtained results with these principally known for Hopfian and co-Hopfian groups, respectively, by giving up their discrepancies. It is worthwhile noticing that some common generalizations to both Hopficity and co-Hopficity are presented in [2] and [3], respectively.

2. Examples

**Example 2.1.** A torsion-free group of rank 1 is \( \omega \)-Hopfian if, and only if, it is 2-Hopfian. In particular, \( \mathbb{Q} \) is not \( \omega \)-Hopfian.

**Proof.** One way being elementary, let \( A \) be such a group and assuming \( pA = A \) for any prime \( p \), then \( p \cdot 1_A \) is an automorphism of \( A \) with \((p \cdot 1_A)^n = p^n \cdot 1_A \neq 1_A \) for all \( n \). Hence \( A \neq pA \) and, as we already commented, \( E(A) \cong \mathbb{Z} \). But as we have seen in the fifth bullet above, \( A \) is 2-Hopfian, as asserted. \( \square \)
Example 2.2. Let $k \in \mathbb{N}$ and $p$ a prime. Then the group $\mathbb{Z}(p^k)$, where $p$ is an odd prime, is both $(p^k - p^{k-1})$-Hopfian and $(p^k - p^{k-1})$-co-Hopfian. The group $\mathbb{Z}(2^k)$ is both $2^{k-1}$-Hopfian and $2^{k-2}$-co-Hopfian whenever $k \geq 3$, whereas $\mathbb{Z}(2)$ is both 1-Hopfian and 1-co-Hopfian, and $\mathbb{Z}(4)$ is both 2-Hopfian and 2-co-Hopfian.

Proof. Knowing with the aid of [10] that $\text{Aut}(\mathbb{Z}(p^k)) \cong U(\mathbb{Z}_{p^k})$, then we differ the subsequent cases:

Case 1: $p$ is odd. As it is well-known $U(\mathbb{Z}_{p^k})$ is a cyclic group of order $p^k - p^{k-1}$, which implies that $\mathbb{Z}(p^k)$ is simultaneously $(p^k - p^{k-1})$-Hopfian and $(p^k - p^{k-1})$-co-Hopfian.

Case 2: $p = 2$. If $k = 1$, then $U(\mathbb{Z}_2)$ is cyclic of order 1, whence $\mathbb{Z}(2)$ is simultaneously 1-Hopfian and 1-co-Hopfian.

If $k = 2$, then $U(\mathbb{Z}_4)$ is cyclic of order 2, whence $\mathbb{Z}(4)$ is simultaneously 2-Hopfian and 2-co-Hopfian.

If $k \geq 3$, then $U(\mathbb{Z}_{2^k})$ is the direct product of a cyclic group of order $2^{k-2}$ and a cyclic group of order 2, whence $\mathbb{Z}(2^k)$ is simultaneously $2^{k-2}$-Hopfian and $2^{k-2}$-co-Hopfian. □

Various other examples could be exhibited taking into account the basic results alluded to below (cf. Corollary 3.17, Proposition 3.20, Corollary 3.21, etc.).

3. Main results
3.1. $n$-Hopfian groups

Subgroups of $n$-Hopfian (respectively, $\omega$-Hopfian) groups need not be again the same. However, some subgroups inherit this property. Specifically, the following is valid:

Proposition 3.1. If $G$ is an $n$-Hopfian (an $\omega$-Hopfian) group, then $kG$ is an $n$-Hopfian (an $\omega$-Hopfian) group for any $k \in \mathbb{N}$.

Proof. If $f : kG \to kG$ is an epimorphism of $kG$, then in view of [4, Proposition 113.3] there exists an epimorphism $\phi$ of $G$ whose restriction $\phi|kG = f$. Since $\phi^n = 1_G$, we conclude that $\phi^n|kG = f^n|kG = 1_{kG}$, as required. □

It is worthwhile noticing that the converse implication is not, however, true: Indeed, any infinite $k$-bounded group is not necessarily $\omega$-Hopfian.

The next result completely settles when a torsion group is $\omega$-Hopfian. What can be offered is the following one:

Theorem 3.2. Every torsion $\omega$-Hopfian group is finite.

Proof. Suppose first that $G$ is a $p$-torsion $\omega$-Hopfian group. Utilizing Proposition 3.6 and Corollary 3.13, $G$ should be reduced. If we assume in a way of contradiction that $G$ is infinite, then it has an unbounded basic subgroup. Therefore, appealing to Example 2.2, for any prime $q \neq p$, the automorphism $q \cdot 1_G$ has an infinite order and thus $G$ is manifestly not $\omega$-Hopfian. That is why, $G$ must be finite. Furthermore, in the general case, since an $\omega$-Hopfian group cannot have an infinite number of non-zero $p$-primary components, we are done. □

Remark. Another approach for proving up the last statement could be as follows: If $G$ is an $\omega$-Hopfian $p$-group, then it is Hopfian and hence both reduced and semi-standard. Supposing $G$ is unbounded, we may unambiguously say that its automorphism group has center isomorphic to the group of units of the ring $\mathbb{Z}_p$ of $p$-adic integers, which group is known to be not torsion – a contradiction. So, $G$ has to be simultaneously bounded and semi-standard, whence it is finite, as pursued.

On the other vein, as it is well-known (see, for instance, [6]), there exist unbounded Hopfian separable $p$-groups of cardinality not exceeding $2^{\aleph_0}$ and finite Ulm-Kaplansky invariants. This group is, certainly, not torsion-complete as the next assertion illustrates.
Theorem 3.3. Any Hopfian direct sum of torsion-complete $p$-groups is finite.

Proof. Suppose first that $G$ is a Hopfian torsion-complete $p$-group. Assuming contradictiously that the basic subgroup $B$ of such a group $G$ is infinite, then would exist an epimorphism $f : B \to B$ which is not an automorphism (otherwise $B$ will be a Hopfian direct sum of cyclic groups and thus by [7, Theorem 2] it must be finite, a contradiction). But it is well known that $G = \overline{B}$, where $\overline{B}$ is the torsion completion of $B$ in the $p$-adic topology (see [5]). The map $f$ is then extendible to an epimorphism $\overline{f} : G \to G$ like this $\overline{f}(G) = \overline{f}(\overline{B}) = \overline{f}(\overline{B}) = \overline{B} = G$, thus contradicting Hopficity of $G$ because $\overline{f}$ is an epimorphism of $G$ which is not automorphism (as the restriction $f$ is not so). Finally, $B$ is finite guaranteeing that $G$ is bounded. Since we have seen above that bounded Hopfian $p$-groups have to be finite, we conclude that so is $G$, as expected.

Turning out to the general case, suppose now that $G$ is a Hopfian group which is a direct sum of torsion-complete groups. If we assume that this sum is infinite, we can separate a cyclic direct summand for each torsion-complete direct summand, so that we will obtain an infinite Hopfian direct sum of cyclic $p$-groups, which is hardly true. Therefore, it must be that the direct sum of torsion-complete groups is finite, and thus it is torsion-complete. We henceforth apply the preceding case to get the pursued claim after all. \qed

Since any separable $p$-group can be embedded as a pure and dense subgroup in a torsion-complete $p$-group (e.g., vol. II of [4]), the last statement also shows that there is no abundance of Hopfian separable $p$-groups.

In the case of torsion-free groups, the $n$-torsion exponent can be calculated explicitly like this:

Proposition 3.4. If $G$ is a torsion-free group with torsion commutative group $\text{Aut}(G)$ and $n$ is the order of some its automorphism, then $n \in \{2, 3, 4, 6, 12\}$.

Proof. It follows from the corresponding properties of Section 116 in [4]. \qed

Proposition 3.5. If $A = B \oplus C$, where $B$ and $C$ are fully invariant $n$-Hopfian and $m$-Hopfian groups, respectively, then $A$ is an $[m,n]$-Hopfian group, where $[m,n]$ is the LCM($m,n$).

Proof. It follows immediately from the fact that in this situation the ring isomorphism $E(A) \cong E(B) \times E(C)$ holds. \qed

It is worth noticing that, however, there exists a 2-Hopfian group, say $Z$, such that $Z \oplus Z$ is Hopfian but even not $\omega$-Hopfian. Indeed, the multiplicative group (i.e., the group of units) of the ring $E(Z \oplus Z)$ is not torsion, as required.

Proposition 3.6. A direct summand of an $\omega$-Hopfian group (respectively, of an $n$-Hopfian group) is again $\omega$-Hopfian (respectively, $n$-Hopfian).

Proof. Let $G = H \oplus K$ be $\omega$-Hopfian. We claim that $H$ is $\omega$-Hopfian too. To that purpose, assuming $\varphi$ is an epimorphism of $H$, we then have that $\varphi + 1_K$ is obviously an epimorphism of $G$. Therefore, $\varphi^n + 1_K = 1_G$ which enables us that $\varphi^n = 1_H$, as required.

The same idea is workable for $\omega$-Hopfian groups such that the automorphism $\varphi$ becomes $n(\varphi)$-torsion, as required. Certainly, it may occur that $G$ is $\omega$-Hopfian but $H$ is $n$-Hopfian. \qed

In the torsion-free case we can say even a little more:

Lemma 3.7. Each direct summand of an $\omega$-Hopfian torsion-free group is fully invariant.

Proof. Otherwise this group will have nilpotent endomorphisms, but this contradicts property a) of Section 116 from [4]. \qed
Proposition 3.8. If $A = B \oplus C$ is a group, where $B$ is fully invariant in $A$, $B, C$ are $\omega$-Hopfian and the group $\text{Hom}(C, B)$ is torsion, then $A$ is $\omega$-Hopfian.

**Proof.** Since both $B$ and $C$ are Hopfian, then $A$ is of necessity also Hopfian. Moreover, one observes that the semi-direct product $G = H \rtimes K$ of the torsion groups $H$ and $K$ remains a torsion group. To that goal, if $g \in G$, then $g = xy$, where $x \in H$ and $y \in K$. Furthermore, it follows that $g^n = x^nz$ for some $z \in K$. If now $x^n = 1$, then $g^{mn} = 1$, where $z^m = 1$, which argues the claim. Using this, periodicity of the group $\text{Aut}(A)$ now follows from the formula

$$\text{Aut}(A) = [\text{Hom}(C, B)] \rtimes [\text{Aut}(B) \times \text{Aut}(C)],$$

accomplished with [4, § 113].

As an immediate consequence, we yield:

**Corollary 3.9.** If $A = B \oplus C$ is a group for which $B$ is an $\omega$-Hopfian torsion-free subgroup and $C$ is a finite subgroup, then $A$ is an $\omega$-Hopfian group.

**Proposition 3.11.** A direct sum of cyclic groups $A$ is $\omega$-Hopfian if, and only if, $A = A_0 \oplus (\bigoplus_{i=1}^k A_{p_i})$, where either $A_0 = \{0\}$ or $A_0 \cong \mathbb{Z}$ and $A_{p_i}$ are finite $p_i$-groups for some different primes $p_i; 1 \leq i \leq k$. In particular, $A$ is $n$-Hopfian for some suitable natural $n$.

**Proof.** It follows by a combination of Lemma 3.7 and Theorem 3.2.

**Proposition 3.12.** A torsion-free group $G$ of rank 1 is $\omega$-Hopfian if, and only if, $G \neq pG$ for all primes $p$.

**Proof.** If we assume that $G = pG$ for any prime $p$, then it is not too hard that $p \cdot 1_G$ is an automorphism of $G$ with $p^n \cdot 1_G \neq 1_G$, as required.

**Corollary 3.13.** A non-zero divisible group is not $\omega$-Hopfian.

**Proof.** Knowing that a divisible group is Hopfian if it is torsion-free of finite rank, we need apply Lemma 3.7 in combination with Example 2.1 to get the claim.

**Corollary 3.14.** Torsion-free $\omega$-Hopfian groups are reduced.

**Proof.** This follows from Proposition 3.6 accomplished with Corollary 3.13.

**Corollary 3.15.** Non-zero algebraically compact torsion-free groups are not $\omega$-Hopfian.

**Proof.** This follows from the fact that the automorphism group of such a group is not torsion.

The last statement can be extended to the following.

**Theorem 3.16.** A direct sum of algebraically compact groups is $\omega$-Hopfian if, and only if, it is finite.

**Proof.** Since finite groups are always $n$-Hopfian for some appropriate positive integer $n$, and thus $\omega$-Hopfian, the sufficiency follows.

To treat the necessity, suppose we first deal with an algebraically compact $\omega$-Hopfian group. We will use the complete description of algebraically compact groups from Section 40 in [4] as well as from Theorem 3.2 of Chapter 6 in [5]. With the aid of Proposition 3.6, each component in the direct decomposition of an algebraically compact group must be $\omega$-Hopfian. So, with Corollary 3.13 at hand, we now know that any algebraically compact
ω-Hopfian group is reduced. Moreover, any \( p \)-adic algebraically compact group is the direct sum of a torsion-free group and an adjusted algebraically compact group (see \cite[Theorem 55.5]{4}). However, by Corollary 3.15, the torsion-free part must be zero. Finally, \cite[Theorem 1]{8} tells us that a Hopfian \( p \)-adic adjusted algebraically compact group has to be finite, and in view of Example 2.2 the number of these \( p \)-adic components is also finite, thus giving the desired assertion.

To turn out to the general case, suppose we now have an \( \omega \)-Hopfian group which is an arbitrary direct sum of algebraically compact groups. In accordance with Corollary 3.15, each direct summand has to be an adjusted algebraically compact group in which we separate a cyclic direct summand. But owing to Proposition 3.11, the direct sum is necessarily finite and hence an algebraically compact group. We hereafter employ the previous case to conclude the wanted claim after all. \( \square \)

**Corollary 3.17.** A separable torsion-free group or a vector torsion-free group are \( \omega \)-Hopfian if, and only if, their endomorphism ring is commutative, and every rank one direct summand is 2-Hopfian. In particular, these groups are 2-Hopfian, too.

**Proof.** We again need to combine Lemma 3.7 with Example 2.1 in order to infer the claim. \( \square \)

Every rank 1 torsion-free group of type which is equal to \((k_1, k_2, \ldots)\), where all \( k_i \) are finite \( (i \in \mathbb{N}) \), has automorphism group isomorphic to \( \mathbb{Z}(2) \). Hence, for each cardinal satisfying \( 0 < \alpha \leq 2^{\aleph_0} \), there is a decomposable group having group of automorphisms isomorphic to an elementary 2-group of power \( \alpha \).

Recall that a \( sp \)-group \( A \) is a reduced mixed group with an infinite number of non-zero \( p \)-components \( A_p \) such that the natural embedding \( \bigoplus_p A_p \to A \) can be extended to a pure embedding \( A \to \prod_p A_p \). In \cite{1} was established the following criterion for a group to be a \( sp \)-group. Specifically, the following is valid:

**Theorem 3.18.** The following three conditions are equivalent for a reduced mixed group \( A \) with an infinite number of non-zero \( p \)-components \( A_p \):

1. \( A \) is a \( sp \)-group, i.e., the pure embeddings \( \oplus_p A_p \subset A \subset \prod_p A_p \) hold;
2. The embeddings \( \oplus_p A_p \subset A \subset \prod_p A_p \) hold and \( A/(\oplus_p A_p) \) is a divisible torsion-free group;
3. For each prime \( p \) there is a group \( B_p \) such that \( A = A_p \oplus B_p \) with \( pB_p = B_p \).

We now arrive at the following result.

**Theorem 3.19.** Any \( sp \)-group is not \( \omega \)-Hopfian.

**Proof.** Every epimorphism \( \phi \) of such a group \( A \) can be written as \( \phi = (\ldots, \phi_p, \ldots) \), where \( \phi_p \) is an epimorphism of the \( p \)-component \( A_p \). Since the number of these \( A_p \) is infinite, for each natural \( n \) there exists a prime \( p \) with the property that if \( \phi^n_p = 1 \) for some \( n_p \in \mathbb{N} \), then \( n_p > n \). Certainly, \( \phi^n \neq 1 \) for every \( n \in \mathbb{N} \), which substantiates our claim. \( \square \)

The following considers certain (homological) extensions of \( \omega \)-Hopficity.

**Proposition 3.20.** Let \( 0 \to H \to G \to K \to 0 \) be an exact sequence. If \( H, K \) are both \( \omega \)-Hopfian groups and if \( H \) is invariant under each surjection \( \varphi : G \to G \), then \( G \) is \( \omega \)-Hopfian. In particular, extensions of \( \omega \)-Hopfian torsion groups by torsion-free \( \omega \)-Hopfian groups are again \( \omega \)-Hopfian.

**Proof.** Letting \( \varphi : G \to G \) be a surjection, then by assumption, \( \varphi(H) \subset H \) and so we get an induced map \( \overline{\varphi} : G/H \to G/H \) giving the following commutative diagram:
Since \( \varphi \) is onto, \( \overline{\varphi} \) is also onto and so \( K \), being Hopfian, assures that \( \overline{\varphi} \) is an automorphism. If we show that \( (\varphi | H) : H \rightarrow H \) is onto, then as \( H \) is Hopfian, \( \varphi | H \) will also be an automorphism and the result will follow by an appeal to the well-known 'Five Lemma'. However, the fact that \( \varphi | H \) is onto follows immediately from the commutativity of the first square of the diagram above. If now \( (\varphi | H)^k = 1 \) and \( (\overline{\varphi})^m = 1 \), then \( \varphi^k = 1 \), where \( k = [n, m] \).

The next consequence somewhat characterizes mixed \( \omega \)-Hopfian groups.

**Corollary 3.21.** A group \( G \) is \( \omega \)-Hopfian if both \( t(G) \) and \( G/t(G) \) are \( \omega \)-Hopfian groups. In addition, if \( G \) splits, then the converse is also true.

**Proof.** The 'if' part follows directly from Proposition 3.20.

As for the other part, since \( G \cong t(G) \oplus G/t(G) \), we just employ Proposition 3.6 to conclude the claim. \( \square \)

### 3.2. \( \omega \)-Hopfian modules over the formal matrix ring

We will here consider some results concerning \( \omega \)-Hopfian modules over the ring of formal matrix extending somewhat the corresponding assertions from [9]. Before doing that, we need some background material from [11].

To that aim, suppose that \( R, S \) are associative unital rings and \( M, N \) are \( R\)-\( S \)-bimodules. Suppose also that are given the bimodule homomorphisms \( \varphi : M \otimes_S N \rightarrow R \) and \( \psi : N \otimes_S M \rightarrow S \), which satisfy the conditions: \( (mn)m' = m(nm') \) and \( (nm)n' = n(mn') \) for all \( m, m' \in M \) and \( n, n' \in N \). So, \( mn = \varphi(m \otimes n) \) and \( nm = \psi(n \otimes m) \). The set of all matrix of the kind \( \begin{pmatrix} r & m \\ n & s \end{pmatrix} \), where \( r \in R, s \in S, m \in M, n \in N \), endowed with the usual matrix operations, is called the ring of formal matrix (or the formal matrix ring). The so-defined ring will be denoted by \( K = \begin{pmatrix} R & M \\ N & S \end{pmatrix} \).

If \( I \) and \( J \) are the images of the homomorphisms \( \varphi \) and \( \psi \) respectively, one may write \( I = MN, J = NM \), where \( MN \) (respectively \( NM \)) means the set of all finite sums of elements of the sort \( mn \) (respectively \( nm \)). The ideals \( I \) and \( J \) are said to be trace ideals for the ring \( K \). In the case when \( I = 0 = J \), we will say that \( K \) is a ring with zero trace ideals.

Now, let \( X \) and \( Y \) be left \( R \)-module and \( S \)-module, respectively. Let also exist the homomorphisms of \( R \)-module \( f : M \otimes_S Y \rightarrow X \) and \( S \)-module \( g : N \otimes_R X \rightarrow Y \), respectively, which satisfy the equalities \( m(nx) = (mn)x, n(my) = (nm)y \) for all \( m \in M, n \in N, x \in X, y \in Y \). Here the element \( nx \) is identified by \( g(n \otimes x) \), and the element \( my \) by \( f(m \otimes y) \). The vector-column group \( (X \ Y) \) forms a left \( K \)-module under the standard multiplication of matrix columns. The converse is also valid. Every left \( K \)-module is a naturally isomorphic to some column module. For simplicity, every module of the type \( (X \ Y) \), along with its elements, will be hereafter written as rows. Right \( K \)-module means a vector-row module, in which the module multiplication is defined as a production of rows and matrices. Homomorphisms \( f \) and \( g \) are also often called homomorphisms of module multiplication. Let \( MY \) (respectively \( NX \)) denotes the set of all finite sums of elements of the sort \( my \) (respectively \( nx \)). Certainly, \( MY = \text{Im} f \) and \( NX = \text{Im} g \). The map \( \Phi : (X, Y) \rightarrow (X_1, Y_1) \) will be a \( K \)-homomorphism if and only if there are an \( R \)-homomorphism \( \alpha : X \rightarrow X_1 \), an \( S \)-homomorphism \( \beta : Y \rightarrow Y_1 \) with the properties
\[ \alpha(my) = m\beta(y), \beta(nx) = n\alpha(x) \text{ and } \Phi(x, y) = (\alpha(x), \beta(y)) \text{ for all } \alpha \in M, \beta \in N, x \in X, y \in Y. \]

With this at hand, the \( K \)-module homomorphisms will be henceforth written as the \( \alpha(\beta) \).

As in the case of groups, a \( K \)-module \( V \) is called \( \omega \)-Hopfian provided each its epimorphism \( \varphi \) is \( n \)-torsion for some natural \( n \geq 1 \) depending on \( \varphi \).

We thus come to

**Proposition 3.22.** Suppose \( V = (A, B) \) is a \( K \)-module. If both \( A \) and \( B \) are \( \omega \)-Hopfian modules, then \( V \) is an \( \omega \)-Hopfian module.

**Proof.** Given an epimorphism \( \Phi : V \to V \), we can write that \( \Phi = (\alpha, \beta) \), where \( \alpha \) is an endomorphism of the module \( A \), \( \beta \) is an endomorphism of the module \( B \) and \( \Phi(a, b) = (\alpha(a), \beta(b)) \) for \( (a, b) \in V \).

It is clear that both \( \alpha \) and \( \beta \) are epimorphisms and hence automorphisms. Therefore, \( \Phi \) is also an automorphism. If now \( \alpha^n = 1 \) and \( \beta^m = 1 \), then \( \Phi^k = 1 \) for some \( k = LCM(n, m) \).

Furthermore, for any \( R \)-module \( X \) and \( S \)-module \( Y \) one can define \( K \)-modules \( (X, T(X)) \) and \( (T(Y), Y) \), where \( T(X) = N \otimes_R X \) and \( T(Y) = M \otimes_S Y \).

The following five consequences are helpful.

**Corollary 3.23.** The \( K \)-module \( (X, T(X)) \) is \( \omega \)-Hopfian if, and only if, \( X \) is an \( \omega \)-Hopfian \( R \)-module. Similarly for the \( K \)-module \( (T(Y), Y) \).

**Proof.** The claim follows by a combination of the next four crucial facts: Firstly, all homomorphisms of the \( K \)-modules act coordinate-wise. Secondly, any endomorphism of the module \( (X, T(X)) \) equals to \( (\alpha, 1_N \otimes \alpha) \) for the unique endomorphism \( \alpha \) of the module \( X \) (see [11, Lemma 2.2]). Thirdly, one sees that if \( \alpha \) is an epimorphism, then \( 1_N \otimes \alpha \) is also an epimorphism. Fourthly, if \( \alpha^n = 1 \), then \( (1_N \otimes \alpha)^n = 1_N \otimes \alpha^n = 1_{T(X)} \).

**Corollary 3.24.** If the ring \( K \) has trace ideals \( I, J \) satisfying the equalities \( I = R, J = S \), then the \( \omega \)-Hopficity of the \( K \)-module \( (A, B) \) is equivalent to the \( \omega \)-Hopficity of the \( R \)-module \( A \) and is equivalent to the \( \omega \)-Hopficity of the \( S \)-module \( B \).

**Proof.** Utilizing [11, Corollary 8.2] there are isomorphisms of \( K \)-modules \( (A, B) \cong (A, T(A)) \cong (T(B), B) \). We next just employ Corollary 3.23.

**Corollary 3.25.** Suppose \( K \) is a ring with zero trace ideals, i.e., \( I = 0 = J \). Then the following two items hold:

1. If all indecomposable projective \( R \)-modules and \( S \)-modules are \( \omega \)-Hopfian, then any indecomposable projective \( K \)-module is also \( \omega \)-Hopfian.

2. The assertion in (1) remains true replacing "projective" by "flat".

**Proof.** Assume that \( (A, B) \) is a projective \( K \)-module. Appealing to [11, Theorem 7.3] there exist projective \( R \)-module \( X \) and projective \( S \)-module \( Y \) for which the isomorphism \( (A, B) \cong (X, T(X)) \oplus (T(Y), Y) \) is fulfilled. If the module \( (A, B) \) is indecomposable, then it is isomorphic to either module \( (X, T(X)) \) or to module \( (T(Y), Y) \), as moreover both \( X \) and \( Y \) are indecomposable modules. We next apply Corollary 3.23.

Point (2) can be proved analogously.

It is worthwhile noticing that in [11] are also introduced \( K \)-modules of the type \( (X, H(X)) \) and \( (H(Y), Y) \), where \( H(X) = \text{Hom}_R(M, X), H(Y) = \text{Hom}_S(N, Y) \).

**Corollary 3.26.** Suppose that \( M \) is a projective \( R \)-module. Then the \( K \)-module \( (X, H(X)) \) is \( \omega \)-Hopfian if, and only if, \( X \) is an \( \omega \)-Hopfian \( R \)-module. A similar assertion is valid for the \( K \)-module \( (H(Y), Y) \), provided projectivity of the \( S \)-module \( N \).

**Proof.** It imitates the same idea as that in Corollary 3.23.
Corollary 3.27. Let $M$ and $N$ be a projective $R$-module and $S$-module, respectively. If all indecomposable injective $R$-modules and $S$-modules are $\omega$-Hopfian, then any indecomposable injective $K$-module is also $\omega$-Hopfian.

Proof. For an arbitrary injective $K$-module $(A, B)$ there are an injective $R$-module $X$ and an injective $S$-module $Y$ having the property $(A, B) \cong (X, H(X)) \oplus (H(Y), Y)$ (cf. [11, Corollary 5.7]). If now $(A, B)$ is an indecomposable module, then it is isomorphic to either one of modules $(X, H(X))$ or $(H(Y), Y)$. Likewise, modules $X$ and $Y$ are indecomposable as well. Furthermore, Corollary 3.26 applies to get the claim. □

3.3. $n$-co-Hopfian groups

Proposition 3.28. A direct summand of an $\omega$-co-Hopfian group (respectively, of an $n$-co-Hopfian group) is again $\omega$-co-Hopfian (respectively, $n$-co-Hopfian).

Proof. It is identical to that in Proposition 3.6 stated above. □

Proposition 3.29. A non-zero torsion-free group is not $\omega$-co-Hopfian.

Proof. Since any torsion-free co-Hopfian group must be divisible of finite rank, for any integer $k \neq 0$ the map $k \cdot 1$ is its monomorphism and $(k \cdot 1)^n = k^n \cdot 1 \neq 1$ for all naturals $n$, so that the group is not $\omega$-co-Hopfian. □

Proposition 3.30. A non-zero divisible group is not $\omega$-co-Hopfian. In addition, $\omega$-co-Hopfian groups are reduced.

Proof. Using the structure theorem for divisible groups (e.g., cf. [5]) accomplished with Propositions 3.28 and 3.29, we need just consider the $(p)$-torsion case. However, the group $\mathbb{Z}(p^{\infty})$ is co-Hopfian but has an automorphism group which is not torsion being isomorphic to the unit group of the ring of $p$-adic integers. Thus $\mathbb{Z}(p^{\infty})$ is not $\omega$-co-Hopfian, which substantiates our initial claim.

The second part is now immediate by taking into account Proposition 3.28. □

We remark that it follows from this statement that $\mathbb{Z}(p^{\infty})$ is an example of a co-Hopfian group which is not $\omega$-co-Hopfian.

Proposition 3.31. If $A = B \oplus C$, where $B$ and $C$ are fully invariant $n$-co-Hopfian and $m$-co-Hopfian groups, respectively, then $A$ is an $[m,n]$-co-Hopfian group, where $[m,n]$ is the LCM$(m, n)$.

Proof. Since we have that $E(A) \cong E(B) \times E(C)$, the result follows without any difficulty. □

Proposition 3.32. A direct sum of cyclic groups is $\omega$-co-Hopfian if, and only if, it is finite. In particular, such a group is $n$-co-Hopfian for some $n \in \mathbb{N}$.

Proof. In virtue of Propositions 3.28 and 3.29, we may restrict our attention on $p$-groups. But the co-Hopfian direct sum of cyclic $p$-groups is finite. The finiteness of the number of $p$-components now follows from Example 2.2. □

Theorem 3.33. Any sp-group is not $\omega$-co-Hopfian.

Proof. Every monomorphism $\phi$ of such a group $A$ can be written as $\phi = (\ldots, \phi_p, \ldots)$, where $\phi_p$ is a monomorphism of the $p$-component $A_p$. Since the number of these $A_p$ is infinite, for each natural $n$ there exists a prime $p$ with the property that if $\phi_p^{n_p} = 1$ for some $n_p \in \mathbb{N}$, then $n_p > n$. Certainly, $\phi^n \neq 1$ for every $n \in \mathbb{N}$, which substantiates our claim. □

Proposition 3.34. If $G$ is an $n$-co-Hopfian (an $\omega$-co-Hopfian) group, then $kG$ is an $n$-co-Hopfian (an $\omega$-co-Hopfian) group for any $k \in \mathbb{N}$.
Proof. If \( f : kG \to kG \) is a monomorphism of \( kG \), then in view of [4, Proposition 113.3] there exists a monomorphism \( \varphi \) of \( G \) whose restriction \( \varphi | kG = f \). Since \( \varphi^n = 1_G \), we conclude that \( \varphi^n | kG = f^n | kG = 1_{kG} \), as required.

It is worthwhile noticing that the converse implication is not, however, true: Indeed, any infinite \( k \)-bounded group is not necessarily \( \omega \)-co-Hopfian.

Theorem 3.35. Any torsion \( \omega \)-co-Hopfian group is finite.

Proof. Our argumentation is similar to that from Theorem 3.2.

Proposition 3.36. If \( A = B \oplus C \) is a group, where \( B \) is fully invariant in \( A \), \( B, C \) are \( \omega \)-co-Hopfian and the group \( \text{Hom}(C, B) \) is torsion, then \( A \) is \( \omega \)-co-Hopfian.

Proof. Since \( B \) and \( C \) are both co-Hopfian groups, then \( A \) is co-Hopfian as well. In fact, every endomorphism of \( A \) can be presented as \( f = \begin{pmatrix} \varphi & \psi \\ 0 & \eta \end{pmatrix} \), where \( \varphi \in \text{E}(B) \), \( \psi \in \text{Hom}(C, B) \), \( \eta \in \text{E}(C) \). If \( f \) is a monomorphism, then \( \varphi \) is a monomorphism, and hence an automorphism. If \( \eta(c) = 0 \) for some \( 0 \neq c \in C \), then \( \psi(c) \neq 0 \) and \( \varphi(b) = \psi(c) \) for some \( b \in B \). Consequently, \( f(b-c) = \varphi(b) - \psi(c) = 0 \) for \( b-c \neq 0 \), a contradiction. Thus \( \eta \) is also a monomorphism, whence, an automorphism. Therefore, any monomorphism of \( A \) is its automorphism, which gives our claim about co-Hopficity of \( A \). Now, the periodicity of \( \text{Aut}(A) \) follows directly from the formula \( \text{Aut}(A) = \text{Hom}(C, B) \times [\text{Aut}(B) \times \text{Aut}(C)] \), as required.

We emphasize that Hopfian algebraically compact groups are described in ([7], [8]). However, to the authors’ knowledge, the complete description of co-Hopfian algebraically compact groups is not known to principally exist in the literature, so we offer a weaker version of it at the next statement.

Proposition 3.37. An algebraically compact \( \omega \)-co-Hopfian group is finite, and vice versa.

Proof. According to Propositions 3.28 and 3.30, such a group is reduced. We hereafter may adapt the idea for proof from Theorem 3.16.

The converse part is trivial.

Proposition 3.38. Let \( 0 \to H \to G \to K \to 0 \) be an exact sequence. If \( H, K \) are both \( \omega \)-co-Hopfian groups and if \( H \) is invariant under each injection \( \psi : G \to G \), then \( G \) is \( \omega \)-co-Hopfian.

Proof. The proof is essentially dual to that of Proposition 3.20, so we omit it and leave to the interested reader.

3.4. \( \omega \)-co-Hopfian modules over the formal matrix ring

There are some analogies for \( \omega \)-co-Hopfian modules to statements stated above. In fact, for \( \omega \)-co-Hopfian modules one can deduce analogous assertions to Proposition 3.22 and Corollaries 3.23, 3.24 and 3.25, respectively. Just we need additionally to assume that the modules \( N_R \) and \( M_S \) are flat.

Besides, there are analogies to Corollaries 3.26 and 3.27, where instead of above, no conditions on \( M \) and \( N \) are needed.

4. Left-open problems

As a concluding discussion, it is worthwhile noticing that we may locate our work within the context of groups with torsion automorphism group and relate it to the work of A.L.S. Corner on groups with finite automorphism group. This is possible because, in other terms, a group is \( n \)-Hopfian (respect., \( n \)-co-Hopfian) provided the multiplicative semigroup of epimorphisms (respect., monomorphisms) is \( n \)-bounded.
It is well known that (cf. [4], v. II, Chapter 116, Exercise 3) any elementary 2-group $G_\beta$ of power $2^\beta$, where $\beta$ is a cardinal strictly less than the first strongly intangible cardinal number, can be realized as the group of the automorphisms of some torsion-free group. That is why, one can pose the following (compare with Corollary 3.17 alluded to above):

**Problem 1.** For which cardinals $\beta$ there exists a Hopfian group $A_\beta$ with the property $\text{Aut}(A_\beta) \cong G_\beta$?

It is clear that such groups $A_\beta$ have to be 2-Hopfian.

**Problem 2.** Does there exist a Hopfian (respectively, an $\omega$-Hopfian, an $n$-Hopfian) group $A$ whose $p$-components and the factor-group $A/t(A)$ are not Hopfian?

We close the considerations with our final query:

**Problem 3.** If $G$ is an $\omega$-Hopfian group (or an $n$-Hopfian group for some $n \in \mathbb{N}$), is it true that its automorphism group (respectively its endomorphism group) is also $\omega$-Hopfian (or $m$-Hopfian for some $m \in \mathbb{N}$)?

**References**


ST₂, ΔT₂, ST₃, ΔT₃, Tychonoff, compact and ∂-connected objects in the category of proximity spaces

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Abstract

In this paper, an explicit characterization of the separation properties ST₂, ΔT₂, ST₃, ΔT₃ and Tychonoff objects are given in the topological category of proximity space. Furthermore, the (strongly) compact object and ∂-connected object are also characterized in the category of proximity space. Moreover, we investigate the relationships among ST₂, ΔT₂, ST₃, ΔT₃, the separation properties at a point p, the generalized separation properties Tᵢ, i = 0, 1, 2, T₀, T₁, T₂ and Tychonoff objects in this category. Finally, we investigate the relationships between ∂-connected object and (strongly) connected object in the topological category of proximity space.

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1. Introduction

The notion of proximity on a set X was introduced in 1950 by Efremovich [18]. He characterized the proximity relation “A is close to B” as a binary relation on subsets of a set X. In the meanwhile, in 1941, a study was made by Wallace [39, 40] regarding “separation of sets”. This study can be considered as the primordial version of the proximity concept. A large part of the early work in proximity spaces was done by Smirnov [37] and [38].

All our preliminary information on proximity spaces and more information can be found in [32].

In later years, some authors such as Leader [28], Lodato [29] and Pervin [33] have worked with weaker axioms than Efremovich’s proximity axioms.

Various generalizations of the usual separation properties of topology and for an arbitrary topological category over sets separation properties at a point p are given in [2]. Baran [2] defined separation properties first at a point p, i.e., locally (see [3, 5, 6, 10, 13, 24, 25]), then they are generalized this to point free definitions by using the generic element, [22, p. 39], method of topos theory.

One of the uses of local separation properties is to define the notions of closedness and strong closedness on arbitrary topological categories in set based topological categories.
These notions are introduced by Baran [2, 4, 9] and they are used in [2, 7, 11, 15, 24] to generalize each of the notions of compactness, connectedness, Hausdorffness, and perfectness to arbitrary set based topological categories. Also, it is shown in [10, 11, 13] that closedness and strong closedness form an appropriate closure operator in the sense of Dikranjan and Giuli [17] in some well-known topological categories. Moreover, the notions of each of (strongly) closed morphisms and (strongly) compact objects in a topological category $\mathcal{E}$ over SET are introduced in [7].

The main goal of this paper is

1. to give the characterization of the separation properties $ST_2$, $\Delta T_2$, $ST_3$, $\Delta T_3$ and Tychonoff objects in the topological category of proximity space,
2. to characterize the (strongly) compact object and $\partial$-connected object in the topological category of proximity space,
3. to show that the relationships among $ST_2$, $\Delta T_2$, $ST_3$, $\Delta T_3$ and the separation properties at a point $p$, the generalized separation properties $T_i$, $i = 0, 1, 2$, $T_0$, $T_1$, $T_2$ and Tychonoff objects in this category, and between $\partial$-connected object and (strongly) connected object in the topological category of proximity space.

2. Preliminaries

The following are some basic definitions and notations which we will use throughout the paper.

Let $\mathcal{E}$ and $\mathcal{B}$ be any categories. The functor $\mathcal{U} : \mathcal{E} \rightarrow \mathcal{B}$ is said to be topological or that $\mathcal{E}$ is a topological category over $\mathcal{B}$ if $\mathcal{U}$ is concrete (i.e., faithful and amnestic), has small (i.e., sets) fibers, and for which every $\mathcal{U}$-source has an initial lift or, equivalently, for which each $\mathcal{U}$-sink has a final lift [1].

Note that a topological functor $\mathcal{U} : \mathcal{E} \rightarrow \mathcal{B}$ is said to be normalized if constant objects, i.e., subterminals, have a unique structure, [1, 5, 12, 30, 34].

Recall in [1] or [34], that an object $X \in \mathcal{E}$ (where $X \in \mathcal{E}$ stands for $X \in \text{Ob}(\mathcal{E})$), a topological category, is discrete iff every map $\mathcal{U}(X) \rightarrow \mathcal{U}(Y)$ lifts to a map $X \rightarrow Y$ for each object $Y \in \mathcal{E}$ and an object $X \in \mathcal{E}$ is indiscrete iff every map $\mathcal{U}(Y) \rightarrow \mathcal{U}(X)$ lifts to a map $Y \rightarrow X$ for each object $Y \in \mathcal{E}$.

Let $\mathcal{E}$ be a topological category and $X \in \mathcal{E}$. $A$ is called a subspace of $X$ if the inclusion map $i : A \rightarrow X$ is an initial lift (i.e., an embedding) and we denote it by $A \subset X$.

**Definition 2.1.** ([32]). An (Efremovich) proximity space is a pair $(X, \delta)$, where $X$ is a set and $\delta$ is a binary relation on the powerset of $X$ such that

(P1) $A\delta B$ iff $B\delta A$;
(P2) $A\delta(B \cup C)$ iff $A\delta B$ or $A\delta C$;
(P3) $A\delta B$ implies $A, B \neq \emptyset$;
(P4) $A \cap B \neq \emptyset$ implies $A\delta B$;
(P5) $A\delta B$ implies there is an $E \subseteq X$ such that $A\delta E$ and $(X - E)\delta B$;

where $A\delta B$ means it is not true that $A\delta B$.

A function $f : (X, \delta) \rightarrow (Y, \delta')$ between two proximity spaces is called a proximity mapping (or a $p$-map) iff $f(A)\delta' f(B)$ whenever $A\delta B$. It can easily be shown that $f$ is a $p$-map iff $f^{-1}(C)\delta f^{-1}(D)$ whenever $C\delta D$.

In a (quasi-)proximity space $(X, \delta)$, we write $A \ll B$ if and only if $A \exists (X - B)$. The relation $\ll$ is called $p$-neighborhood relation or the strong inclusion. When $A \ll B$, we say that $B$ is a $p$-neighborhood of $A$ or $A$ is strongly contained in $B$ [20] or [32].

We denote the category of proximity spaces and proximity mappings by $\text{Prox}$. Hunsaker and Sharma [21] showed that the functor $\mathcal{U} : \text{Prox} \rightarrow \text{Set}$ is topological.

**Definition 2.2.** ([35]). Let $X$ be a nonempty set. A proximity-base on $X$ is a binary relation $\mathcal{B}$ on $P(X)$ satisfying the axioms $(B1)$ through $(B5)$ given below:
(B1) \((\emptyset, X) \notin \mathfrak{B}\);
(B2) If \(A \cap B \neq \emptyset\) implies \((A, B) \in \mathfrak{B}\);
(B3) \((A, B) \in \mathfrak{B}\) iff \((B, A) \in \mathfrak{B}\);
(B4) If \((A, B) \in \mathfrak{B}\) and \(A \subseteq A^{\ast}, B \subseteq B^{\ast}\) then \((A^{\ast}, B^{\ast}) \in \mathfrak{B}\);
(B5) If \((A, B) \notin \mathfrak{B}\) then there exists a set \(E \subseteq X\) such that \((A, E) \notin \mathfrak{B}\) and \((X - E, B) \notin \mathfrak{B}\).

2.3 Let \(\mathfrak{B}\) be a proximity-base on a set \(X\) and let a binary relation \(\delta\) on \(P(X)\) be defined as follows: \((A, B) \in \delta\) if, given any finite covers \(\{A_i : 1 \leq i \leq n\}\) and \(\{B_j : 1 \leq j \leq m\}\) of \(A\) and \(B\) respectively, then there exists a pair \((i, j)\) such that \((A_i, B_j) \in \mathfrak{B}\). \(\delta\) is a proximity on \(X\) finer than the relation \(\mathfrak{B}\) [21] or [35].

2.4 Let \(X\) be a non-empty set, for each \(i \in I\), \((X_i, \delta_i)\) be a proximity space and \(f_i : X \rightarrow (X_i, \delta_i)\) be a source in \(\text{Prox}\). Define a binary relation \(\mathfrak{B}\) on \(P(X)\) as follows: for \(A, B \in P(X), A \mathfrak{B} B\) iff \(f_i(A)\delta_i f_i(B)\), for all \(i \in I\). \(\mathfrak{B}\) is a proximity-base on \(X\) [35, Theorem 3.8]. The initial proximity structure \(\delta\) on \(X\) generated by the proximity base \(\mathfrak{B}\) is given by \(\delta = \bigcup \{A \mathfrak{B} B : A, B \in \mathfrak{B}\}\).

2.5 Let \((X, \delta)\) be a proximity space, \(Y\) a non-empty set and \(f\) a function from a proximity space \((X, \delta)\) onto \(Y\). The strong inclusion \(\subseteq^\ast\) induced by the finest proximity \(\delta^\ast\) on \(Y\) making \(f\) proximally continuous is given by: for every \(A, B \subseteq Y\), \(A \subseteq^\ast B\) if and only if, for each binary rational \(s \in [0, 1]\), there is some \(X \subseteq Y\) such that \(C_0 = A, C_1 = B\) and \(s < t\) implies \(f^{-1}(C_s) \subseteq f^{-1}(C_t)\) [20] or [41, p. 276], where \(\subseteq\) represents the strong inclusion induced by the proximity \(\delta\) on \(X\). In addition, if \(f : (X, \delta) \rightarrow (X, \delta^\ast)\) be a one-to-one \(p\) map, then \(A \delta^\ast B\) if and only if \(f^{-1}(A) \delta f^{-1}(B)\) [20, p. 591].

2.6 We write \(\Delta\) for the diagonal in \(X^2\), for \(X \in \text{Prox}\). In \(\text{Prox}\) we define the wedge \(X^2 \vee_\Delta X^2\), as the final structure, with respect to the map \(X^2 \sqcup X^2 \rightarrow X^2 \vee_\Delta X^2\), that is the identification of the two copies of \(X^2\) along the diagonal \(\Delta\). An epi sink \(\{(i_1, i_2 : (X^2, \delta) \rightarrow (X^2 \vee_\Delta X^2, \delta'))\}\), where \(i_1, i_2\) are the canonical injections, in \(\text{Prox}\) is a final lift if and only if the following statement holds. For each pair \(A, B\) in the different component of \(X^2 \vee_\Delta X^2\), \(A\delta' B\) iff there exist sets \(C, D\) in \(X^2\) such that \(C\delta\{\{x, y\}\}\) and \((x, y)\) \(\delta D\) with \(i_k^{-1}(A) = C\) and \(i_k^{-1}(B) = D\) for \(k = 1, 2\), \(j = 1, 2, k \neq j\). If \(A, B\) are in the same component of wedge, then \(A\delta' B\) iff there exist sets \(C, D\) in \(X^2\) such that \(C\delta D\) and \(i_k^{-1}(A) = C\) and \(i_k^{-1}(B) = D\) for some \(k = 1, 2\). Specially, if \(i_k(E) = A\) and \(i_k(F) = B\), then \((i_k(E), i_k(F)) \in \delta'\) iff \((i_k^{-1}(i_k(E)), i_k^{-1}(i_k(F))) = (E, F) \in \delta\). This is a special case of 2.5.

2.7 Let \(X\) be a non-empty set. The discrete proximity structure \(\delta\) on \(X\) is given by for \(A, B \subseteq X\), \(A\delta B\) iff \(A \cap B = \emptyset\) [32, p. 9].

2.8 Let \(X\) be a non-empty set. The indiscrete proximity structure \(\delta\) on \(X\) is given by for \(A, B \subseteq X\), \(A\delta B\) iff \(A \neq \emptyset\) and \(B \neq \emptyset\) [19, p. 5].

3. \(ST_2, \Delta T_2, ST_3\) and \(\Delta T_3\) objects in proximity spaces

In this section, the characterization of \(ST_2, \Delta T_2\) \(ST_3\) and \(\Delta T_3\) objects in this category are given. Furthermore, we investigate the relationships among \(ST_2, \Delta T_2, ST_3, \Delta T_3\), the separation properties at a point \(p\), the generalized separation properties \(T_i, i = 0, 1, 2, T_0, T_1\) and \(T_2\) in the topological category of (Efremovich) proximity spaces.

Let \(B\) be set and \(p \in B\). The infinite wedge product \(\vee_p^{\infty} B\) is formed by taking countably many disjoint copies of \(B\) and identifying them at the point \(p\). Let \(B^{\infty} = B \times B \times \ldots\) be the countable cartesian product of \(B\). Define \(A_p^{\infty} : \vee_p^{\infty} B \rightarrow B^{\infty}\) by \(A_p^{\infty}(x_i) = (p, p, \ldots, p, x, p, \ldots)\), where \(x_i\) is in the \(i\)-th component of the infinite wedge and \(x\) is in the \(i\)-th place in \((p, p, \ldots, p, x, p, \ldots)\) (infinite principal \(p\)-axis map), and \(\vee_p^{\infty} B \rightarrow B\) by \(\vee_p^{\infty}(x_i) = x\) for all \(i \in I\) (infinite fold map), [2, 4].
Note, also, that $A_p^\infty$ is the unique map arising from the multiple pushout of $p : 1 \to B$ for which $A_{ij}^\infty(p, p, ..., p, id, p, ...) : B \to B^\infty$, where the identity map, $id$, is in the $j$-th place [11].

**Definition 3.1.** (cf. [2,4]). Let $\mathcal{U} : \mathcal{E} \longrightarrow \mathbf{Set}$ be a topological functor, $X$ an object in $\mathcal{E}$ with $\mathcal{U}(X) = B$. Let $F$ be a nonempty subset of $B$. We denote by $X/F$ the final lift of the epi $\mathcal{U}$-sink $q : \mathcal{U}(X) = B \to B/F = (B \setminus F) \cup \{\ast\}$, where $q$ is the epi map that is the identity on $B/F$ and identifying $F$ with a point $\{\ast\}$.

Let $p$ be a point in $B$.

1. $p$ is closed iff the initial lift of the $\mathcal{U}$-source $\{A_p^\infty \circ \triangledown_p^\infty B \to \mathcal{U}(X^\infty) = B^\infty\}$ and $\triangledown_p^\infty : \triangledown_p^\infty B \to \triangledown(B) = B$ is discrete.
2. $F \subset X$ is closed iff $\{\ast\}$, the image of $F$, is closed in $X/F$ or $F = \emptyset$.
3. $C \subset X$ is strongly closed iff $X/F$ is $T_1$ at $\{\ast\}$ or $F = \emptyset$.
4. If $B = F = \emptyset$, then we define $F$ to be both closed and strongly closed.
5. $X$ is $ST_2$ iff $\Delta$, the diagonal, is strongly closed in $X^2$, [4].
6. $X$ is $\Delta T_2$ iff $\Delta$, the diagonal, is closed in $X^2$, [4].
7. $X$ is $\Delta T_3$ iff $X$ is $T_1$ and $X/F$ is $\Delta T_2$ if it is $T_1$, where $F \neq \emptyset$ in $U(X)$, [8].
8. $X$ is $ST_3$ iff $X$ is $T_1$ and $X/F$ is $ST_2$ if it is $T_1$, where $F \neq \emptyset$ in $U(X)$, [8].

Recall that a prebornological space is a pair $(B, \mathfrak{F})$, where $\mathfrak{F}$ is a family of subsets of $B$ that is closed under nonempty finite union and contains all finite nonempty subsets of $B$. A morphism $(B, \mathfrak{F}) \to (B_1, \mathfrak{F}_1)$ of such spaces is a function $f : B \to B_1$ such that $f(C) \in \mathfrak{F}_1$ if $C \in \mathfrak{F}$. We denote by $\mathbf{PBorn}$, the category thus obtained. This category is topological category over $\mathbf{Set}$, [9].

The category $\mathbf{Prord}$ of preordered spaces has as objects the pairs $(B, R)$, where $B$ is a set and $R$ is a reflexive and transitive relation on $B$ and has as morphism $(B, R) \to (B_1, R_1)$ those functions $f : B \to B_1$ such that $aRb$, then $f(a)R_1f(b)$ for all $a, b \in B$. This category is topological category over $\mathbf{Set}$, [13].

**Lemma 3.2.** ([13, Theorem 3.6]). Let $(B, R)$ be a preordered set (i.e., $R$ is a reflexive and transitive relation on $B$), and $\emptyset \neq F \subset B$. Then,

1. $F$ is a closed subset of $B$ iff for any $x \in B$, if there exists $a, b \in F$ such that $xRa$ and $bRx$, then $x \in F$.
2. $F$ is a strongly closed subset of $B$ iff for each $x \in B$, if there exists $a \in F$ such that $xRa$ or $aRx$, then $x \in F$.

**Lemma 3.3.** ([4, Theorem 3.9 and 3.10]). Let $(B, \mathfrak{F})$ be a prebornological space. Then,

1. A subset $F \subset B$ is closed iff $F = B$ or $F = \emptyset$.
2. All subsets of $B$ are strongly closed.

**Remark 3.4.**

1. In $\mathbf{Top}$, the notion of closedness coincides with the usual one [2] and $F$ is strongly closed iff $F$ is closed and for each $x \notin F$ there exists a neighbourhood of $F$ missing $x$, [2]. If a topological space is $T_1$, then the notions of closedness and strong closedness coincide, [2].

2. In general, for an arbitrary topological category, the notions of closedness and strong closedness are independent of each other. To see this, let $B = \{-1, 1\}$, $R = \{(-1, 1), (-1, -1), (1, 1)\}$, and $F = \{1\}$. Then $(B, R)$ is a preordered set and by 3.2, $F$ is closed, but $F$ is not strongly closed. On the other hand, let $B = \mathbb{R}$, the set of real numbers, and $\mathfrak{F} = \mathcal{P}(\mathbb{R}) \setminus \{\emptyset\}$, the set of all nonempty subsets of $\mathbb{R}$. Note, [9] Remark 3.2, that $(B, \mathfrak{F})$ is a prebornological space and by 3.3, $Q$, the set of rational numbers, is strongly closed, but $Q$ is not closed.

**Theorem 3.5.** (cf. [26]). Let $(X, \delta)$ be a (Efremovich) proximity space and $p \in X$.

1. $\{p\}$ is closed in $X$ iff for any $B \subset X$, if $\{p\}\delta B$, then $p \in B$. 

Let \( \emptyset \neq F \subset X \) be closed iff \( x \in F \) whenever \( \{x\} \delta F \) for all \( x \in X \).

(3) \( \emptyset \neq F \subset X \) is strongly closed iff \( x \in F \) whenever \( \{x\} \delta F^c \) for all \( x \in X \).

**Definition 3.6.** Let \( E \) be a topological category over \( \text{Set}, X \) an object in \( E \) and \( F \) be a nonempty subset of \( X \).

1. \( F \subset X \) is open iff \( F^c \), the complement of \( F \), is closed in \( X \).
2. \( F \subset X \) is strongly open iff \( F^c \), the complement of \( F \), is strongly closed in \( X \), [15].

Note that in \( \text{Top} \) the notion of openness coincides with the usual one, [15]. If a topological space is \( T_1 \), then the notions of openness and strong openness coincide, [15].

**Theorem 3.7.** ([26]). Let \( (X, \delta) \) be a (Efremovich) proximity space. \( \emptyset \neq F \subset X \) is (strongly) open iff \( x \in F \) whenever \( \{x\} \delta F^c \) for all \( x \in X \).

**Definition 3.8.** ([41, p. 268]). Let \( (X, \delta) \) be a (Efremovich) proximity space and \( A \subset X \). Define \( \bar{A} = \{x|x \delta A\} \) and if \( \bar{A} = A \), then \( A \) is said to be closed.

**Remark 3.9.**

1. Let \( (X, \delta) \) be a (Efremovich) proximity space and \( A \subset X \). \( A \) is closed (in the usual above sense) iff for each \( x \in X \), if \( x \delta A \), then \( x \in A \).
2. Let \( (X, \delta) \) be a (Efremovich) proximity space. It follows from 3.5 and Definition 3.8 that the notions of closedness (in our sense) and strong closedness coincide with the notion of closedness in the usual sense, [26].

**Theorem 3.10.** Let \( (X, \delta) \) be a (Efremovich) proximity space. Then \( (X, \delta) \) is \( ST_2 \) or \( DT_2 \) iff \( \delta \) is separated (Hausdorff) (Efremovich) proximity i.e., if \( \{x\} \delta \{y\} \), then \( x = y \).

**Proof.** \( (X, \delta) \) is \( ST_2 \) or \( DT_2 \) iff by Definition 3.1 (5) ((6)) \( \Delta \) is strongly closed (closed) iff by Theorem 3.5 (3) (Theorem 3.5 (2)), letting \( F = \Delta \) for each \( (x, y) \in X^2 \) if there exists \( (a, a) \in \Delta \) such that \( \{(x, y)\} \delta^2 \{(a, a)\} \) (6) is the product proximity structure on \( X^2 \), then \( (x, y) \in \Delta \) i.e., \( x = y \). We will show that if \( (X, \delta) \) is \( ST_2 \) or \( DT_2 \), then \( \delta \) is separated (Hausdorff) proximity. If \( \{x\} \delta \{y\} \), then we have clearly \( \{(x, y)\} \delta^2 \{(y, y)\} \) or \( \{(x, x)\} \delta^2 \{(y, y)\} \) and consequently \( (x, y) \in \Delta \) i.e., \( x = y \) since \( (X, \delta) \) is \( ST_2 \) or \( DT_2 \). Hence \( \delta \) is separated (Hausdorff) (Efremovich) proximity.

Conversely if \( \delta \) is separated (Hausdorff) (Efremovich) proximity, then clearly \( \Delta \) is strongly closed (closed) i.e., \( (X, \delta) \) is \( ST_2 \) or \( DT_2 \). \( \square \)

**Example 3.11.** Let \( X = \{a, b\} \) and \( \delta = \{(X, X), \{(a), \{a\}\}, \{(b), \{b\}\}, (X, \{a\}), (a), X, (X, \{b\}), (\{b\}, X)\} \). Then \( (X, \delta) \) is \( ST_2 \) or \( DT_2 \) since \( \{a\} \delta \{b\} \), then \( a = b \).

**Theorem 3.12.** Let \( (X, \delta) \) be a (Efremovich) proximity space. If \( (X, \delta) \) is \( ST_2 \) or \( DT_2 \), then \( (X/F, \delta^F) \) is \( ST_2 \) or \( DT_2 \).

**Proof.** Suppose \( (X, \delta) \) is \( ST_2 \) or \( DT_2 \). Let \( x \) and \( y \) be any distinct pair of points in \( X/F \). By Theorem 3.10, we only need to show that \( \{\{x\}, \{y\}\} \notin \delta^* \), where \( \delta^* \) is the structure on \( X/F \) induced by \( q \).

Suppose that \( x \neq * \). By definition of \( q \) map, there exist \( x \in X \) and \( F \subset X \) such that \( q(x) = x \) and \( q(z) = * \) for any \( z \in F \). Since \( x \neq * \) for any \( z \in F \) \( (x \notin F) \) and \( (X, \delta) \) is \( ST_2 \) or \( DT_2 \), then \( \{x\} \delta \{z\} \). By the condition (P2) of Definition 2.1 we obtain \( \{x\} \delta^* F \). Then we have \( \{x\} \delta^* F = q^{-1}(\{x\}) \delta^* q^{-1}(\{\ast\}) \). It follows that by \( \delta \)-neighborhood relation definition and 2.5, for each binary rational \( s \) in \( [0, 1] \) there is some \( C_s \subset X \) \( F \) such that \( C_0 = \{x\}, C_1 = \{\ast\}^2 \) and \( s < t \) implies \( q^{-1}(C_s) \ll \delta q^{-1}(C_t) = q^{-1}(\{x\}) \ll \delta q^{-1}(\{\ast\})^2 \) if and only if \( \{x\} \ll \delta^* \{\ast\}^2 \). Hence \( \{x\} \delta^* \{\ast\}^2 \), i.e., \( \{\{x\}, \{\ast\}\} \notin \delta^* \).

Let \( x \neq y \neq * \). By definition of \( q \) map, there exists a pair \( x, y \in X \) such that \( q(x) = x \) and \( q(y) = y \). In this case \( q \) map can be considered as one-to-one map. Suppose that \( \{x\} \delta^* \{y\} \). By definition of \( q \) map and 2.5, we have \( \{x\} \delta^* \{y\} \) if and only if \( q^{-1}(\{x\}) \delta q^{-1}(\{y\}) = \{x\} \delta \{y\} \). But \( \{x\} \delta^* \{y\} \) since \( (X, \delta) \) is \( ST_2 \) or \( DT_2 \). Hence \( \{x\} \delta^* \{y\} \) i.e., \( \{\{x\}, \{y\}\} \notin \delta^* \).
Consequently for each distinct points \(x\) and \(y\) in \(X/F\), we have \(\{x\}, \{y\}\) \(\notin \delta^*\). Hence by Theorem 3.10, \((X/F, \delta^*)\) is \(ST_2\) or \(\Delta T_2\).

**Theorem 3.13.** (cf. [26, 27]). Let \((X, \delta)\) be a (Efremovich) proximity space and \(p \in X\).

1. \((X, \delta)\) is \(T_1\) at \(p\) if and only if for each \(x \neq p\), \(\{x\}, \{p\}\) \(\notin \delta\).
2. \((X, \delta)\) is \(\overline{T_0}\) at \(p\) if and only if for each \(x \neq p\), \(\{x\}, \{p\}\) \(\notin \delta\).
3. All (Efremovich) proximity spaces are \(T_0\) at \(p\).
4. \((X, \delta)\) is \(T_0\) if and only if, for each distinct pair \(x\) and \(y\) in \(X\), \(\{x\}, \{y\}\) \(\notin \delta\).
5. An Efremovich proximity space is \(T'_0\).
6. \((X, \delta)\) is \(T_1\) if and only if, for each distinct pair \(x\) and \(y\) in \(X\), \(\{x\}, \{y\}\) \(\notin \delta\).
7. An Efremovich proximity space is \(PreT_2\), ([12]).
8. \((X, \delta)\) is \(PreT'_2\) ([12]) if and only if, for each distinct pair \(x\) and \(y\) in \(X\), \(\{x\}, \{y\}\) \(\notin \delta\).
9. \((X, \delta)\) is \(T_2\) ([11]) if and only if, for each distinct pair \(x\) and \(y\) in \(X\), \(\{x\}, \{y\}\) \(\notin \delta\).
10. \((X, \delta)\) is \(T'_2\) ([11]) if and only if, for each distinct pair \(x\) and \(y\) in \(X\), \(\{x\}, \{y\}\) \(\notin \delta\).

**Definition 3.14.** (cf. [32, 36]). An Efremovich proximity space \((X, \delta)\) is said to be a

- **\(T_0\)-space** if \(x \neq y\) for \(x, y \in X\) implies that \(x \delta y\).
- **\(T_1\)-space** if \(x \neq y\) for \(x, y \in X\) implies that \(x \delta y\).
- **\(T_2\)-space (Hausdorff)** if \(x \delta y\) for \(x, y \in X\) implies that \(x = y\).

**Theorem 3.15.** An (Efremovich) proximity space \((X, \delta)\) is \(ST_3\) or \(\Delta T_3\) if and only if, \(\delta\) is separated (Hausdorff) (Efremovich) proximity i.e., if \(\{x\} \delta \{y\}\), then \(x = y\).

**Proof.** It follows from Definition 3.1 (7), (8) and Theorems 3.12, 3.13 (6).

We give explicit relationships among the generalized separation properties \(ST_2\), \(\Delta T_2\), \(ST_3\), \(\Delta T_3\), the separation properties at a point \(p\), the generalized separation properties \(T_i\), \(i = 0, 1, 2, T_0\), \(T_1\) and \(T_2\) in the topological category of (Efremovich) proximity spaces.

**Remark 3.16.** Let \((X, \delta)\) be a (Efremovich) proximity space and \(p \in A\).

(i) By Theorems 3.10, 3.13 and 3.15, then the followings are equivalent:

1. \((X, \delta)\) is \(\overline{T_0}\) at \(p\) for all \(p \in A\).
2. \((X, \delta)\) is \(T_1\) at \(p\) for all \(p \in A\).
3. \((X, \delta)\) is \(ST_i\), \(i = 2, 3\).
4. \((X, \delta)\) is \(\Delta T_i\), \(i = 2, 3\).

(ii) By Theorems 3.10, 3.13 and 3.15, if \((X, \delta)\) is \(ST_i\) or \(\Delta T_i\), \(i = 2, 3\), then \((X, \delta)\) is \(T'_0\) at \(p\) for all \(p \in A\). But the reverse of implication is not true, in general. For example, let \(X = \{a, b\}\) and \(\delta = \{(X, X), ((a), \{a\}), ((b), \{b\}), (X, \{a\}), (\{a\}, X), (X, \{b\}), ((b), X), ((a), \{b\}), ((b), \{a\})\}\). Then \((X, \delta)\) is \(T'_0\) at \(a\) but it is not \(ST_i\) or \(\Delta T_i\), \(i = 2, 3\), at \(a\) since \((\{a\}, \{b\})\) \(\notin \delta\) but \(a \neq b\).

(iii) By Theorems 3.10, 3.13, 3.15, and Definition 3.14, then the followings are equivalent:

1. \((X, \delta)\) is \(\overline{T_0}\).
2. \((X, \delta)\) is \(T_1\), \(i = 0, 1, 2\).
3. \((X, \delta)\) is \(T_i\).
4. \((X, \delta)\) is \(PreT_i\).
5. \((X, \delta)\) is \(T_2\).
6. \((X, \delta)\) is \(T'_2\).
7. \((X, \delta)\) is \(ST_i\), \(i = 2, 3\).
8. \((X, \delta)\) is \(\Delta T_i\), \(i = 2, 3\).
9. For any distinct pair of points \(a\) and \(b\) in \(X\), \((\{a\}, \{b\})\) \(\notin \delta\).

(iv) By Theorems 3.10, 3.13 and 3.15, if \((X, \delta)\) is \(ST_i\) or \(\Delta T_i\), \(i = 2, 3\), then \((X, \delta)\) is \(T'_0\) or \(PreT_2\). But the reverse of implication is not true, in general. For example, let \(X =\)
\{a, b\} and \(\delta = \{(X, X), (\{a\}, \{a\}), (\{b\}, \{b\}), (X, \{a\}), (X, \{b\}), (\{a\}, X), (\{b\}, X), (\{a\}, \{b\}), (\{b\}, \{a\})\}. Then, \((X, \delta)\) is \(T_0\) and \(\text{Pre}T_2\) but it is not \(ST_i\) or \(\Delta T_i\), \(i = 2, 3\), since \((\{a\}, \{b\}) \in \delta\) but \(a \neq b\).

4. Connectedness and compactness

In this section, the characterization of the notion of the \(\partial\)-connected object and (strongly) Compact object in this category are given. We investigate the relationships between \(\partial\)-connected object and (strongly) connected object in this category.

Recall that the notions of each of (strongly) closed morphisms and (strongly) compact objects in a topological category \(\mathcal{E}\) over \(\text{SET}\) are introduced in [7].

**Definition 4.1.** (cf. [7]) Let \(U : \mathcal{E} \to \text{Set}\) be a topological functor, \(X\) and \(Y\) be objects in \(\mathcal{E}\), and \(f : X \to Y\) a morphism in \(\mathcal{E}\). Then,

1. \(f\) is said to be closed iff the image of each closed subobject of \(X\) is a closed subobject of \(Y\).
2. \(f\) is said to be strongly closed iff the image of each strongly closed subobject of \(X\) is a strongly closed subobject of \(Y\).
3. \(X\) is compact iff the projection \(\pi_2 : X \times Y \to Y\) is closed for each object \(Y\) in \(\mathcal{E}\).
4. \(X\) is strongly compact iff the projection \(\pi_2 : X \times Y \to Y\) is strongly closed for each object \(Y\) in \(\mathcal{E}\).

Note that for the category \(\text{Top}\) of topological spaces, the notions of closed morphism and compactness reduce to the usual ones ([16, p.97 and p.103]).

**Lemma 4.2.** (1). Let \(f : (X, \delta) \to (Y, \delta')\) be a p-map in \(\text{Prox}\). If \(D \subset Y\) is (strongly) closed, so also is \(f^{-1}(D)\).

2. Let \((Y, \delta')\) be an (Efremovich) proximity space. If \(N \subset Y\) is (strongly) closed and \(M \subset N\) is (strongly) closed, so also is \(M \subset Y\).

**Proof.** (1) Suppose \(D \subset Y\) is (strongly) closed and \(x \in f^{-1}(D)\). By 3.5 (2) (3.5 (3)), \(y \in D\) whenever \(y\delta'D\) for all \(y \in Y\). We need to show that, \(x \in f^{-1}(D)\) whenever \(\{x\}\delta f^{-1}(D)\) for all \(x \in X\). Note that \(f(x) \in f(f^{-1}(D)) \subset D\) and \(\{f(x)\}\delta'D\) since \(f\) is p-map and \(D \subset B\) is closed. Thus, \(f^{-1}(D)\) is closed.

The proof for strongly closedness is similar.

2. Suppose \(N \subset Y\) and \(M \subset N\) are strongly closed, \(y \in Y\) and there exists \(a \in M\) such that \(y\delta'a\). By 3.5 (3), we need to show that \(y \in M\). Since \(N \subset Y\) is strongly closed and \(M \subset N\), by 3.5 (3), \(y \in N\). It follows that \(y \in M\) since \(M \subset N\) is strongly closed.

The proof for closedness is similar. \(\square\)

**Lemma 4.3.** All objects in \(\text{Prox}\) are (strongly) compact.

**Proof.** Let \((B, \delta)\) be a (Efremovich) proximity space. By Definition 4.1 (3) (4.1 (4)), we need to show that for all proximity spaces \((A, \delta')\), \(\pi_2 : (B, \delta) \times (A, \delta') \to (A, \delta')\) is (strongly) closed. Suppose \(M \subset B \times A\) is (strongly) closed. To show that \(\pi_2 M\) is (strongly) closed, we assume the contrary and apply Theorem 3.5 (2) (3.5 (3)). Thus for some point \(a \in A\) with \(a \notin \pi_2 M\) whenever \(\{a\}\delta'\pi_2 M\). Since \(M \subset B \times A\) is (strongly) closed, \((b, a) \in M\) whenever \(\{(b, a)\}\delta''M\) for all \((b, a) \in B \times A\), where \(\delta''\) is the product proximity structure on \(B \times A\). Hence \(\pi_2\{(b, a)\}\delta''\pi_2 M = \{a\}\delta'\pi_2 M\), by definition of product proximity structure. Since \((b, a) \in M\), \(\pi_2(b, a) = a \in \pi_2 M\). This is a contradiction since \(M\) is (strongly) closed, by Theorem 3.5 (2) (3.5 (3)). Hence, by Theorem 3.5 (2) (3.5 (3)), \(\pi_2 M\) must be (strongly) closed and consequently, by Definition 4.1 (3) (4.1 (4)), \((B, \delta)\) is (strongly) compact. \(\square\)

**Theorem 4.4.** Let \(f : X \to Y\) be a p-map in \(\text{Prox}\). If \((X, \delta)\) is (strongly) compact, then \((f(X), \delta')\) is (strongly) compact.

**Proof.** It follows from Lemma 4.3. \(\square\)
We now give the characterization of $\partial$-connected object in the category of (Efremovich) proximity spaces and investigate the relationships between $\partial$-connected object and (strongly) connected object in this category.

**Definition 4.5.** Let $\mathcal{E}$ be a topological category over $\textbf{Set}$ and $X$ be an object in $\mathcal{E}$.

1. $X$ is connected iff the only subsets of $X$ both strongly open and strongly closed are $X$ and $\emptyset$, [15].
2. $X$ is strongly connected iff the only subsets of $X$ both open and closed are $X$ and $\emptyset$, [15].
3. $X$ is $\partial$-connected iff the boundary of any non-empty proper subsets of $X$ is non-empty set, i.e., $\partial F \setminus \bar{F} \neq \emptyset$, [23].
4. $X$ is $D$-connected iff any morphism from $X$ to any discrete object is constant, (cf. [15, 34]).

Note that for the category $\textbf{Top}$ of topological spaces, the notion of strongly connectedness, $\partial$-connected and $D$-connectedness coincides with the usual notion of connectedness. If a topological space $X$ is $T_1$, then, by 4.5, the notions of connectedness, strong connectedness and $\partial$-connectedness coincide, [15].

**Theorem 4.6.** A (Efremovich) proximity space $(X, \delta)$ is $\partial$-connected iff for any non-empty proper subset $F$ of $X$, either the condition (1) or (2) holds.

1. $x \notin F$ whenever $\{x\} \delta F$ for some $x \in X$.
2. $x \notin F^c$ whenever $\{x\} \delta F^c$ for some $x \in X$.

**Proof.** Suppose that $(X, \delta)$ is $\partial$-connected but conditions (1) and (2) do not hold for some non-empty proper subset $F$ of $X$. Since the condition (1) does not hold, we get $x \in F$ whenever $\{x\} \delta F$ for all $x \in X$ which means that subset $F$ is (strongly) closed by 3.5 (2) or 3.5 (3). Since the condition (2) does not hold, we get $x \in F^c$ for all $x \in X$, whenever $\{x\} \delta F^c$. This means that $F^c$ is (strongly) closed. So $F$ is (strongly) open by 3.7. Hence $F$ is (strongly) open and (strongly) closed, i.e., $\partial F \setminus \bar{F} = F \setminus \bar{F} = \emptyset$. But this is a contradiction since $(X, \delta)$ is $\partial$-connected.

Conversely, suppose that the condition (1) holds. Then $x \notin F$ whenever $\{x\} \delta F$ for some $x \in X$ and $F$ is not (strongly) closed 3.5 (2) or 3.5 (3). Suppose that the condition (2) holds. Then for some $x \in X$, $x \notin F^c$ whenever $\{x\} \delta F^c$. This means that $F^c$ is not (strongly) closed. So $F$ is not (strongly) open by 3.7. Hence the only subsets of $X$ both (strongly) open and (strongly) closed are $X$ and $\emptyset$. Hence $\partial F \setminus \bar{F} \neq \emptyset$. From here $(X, \delta)$ is $\partial$-connected.

**Example 4.7.** Let $X = \{a, b\}$ and $\delta = \{(X, X), (\{a\}, \{a\}), (\{b\}, \{b\}), (X, \{a\}), (\{a\}, X), (X, \{b\}), (\{b\}, X), (\{a\}, \{b\}), (\{b\}, \{a\})\}$. Then $(X, \delta)$ is $\partial$-connected since non-empty proper subset $F = \{a\}$ of $X, b \notin F$ whenever $\{b\} \delta F$ for some $b \in X$. The case $F = \{b\}$ of $X$ can be handled similarly.

**Theorem 4.8.** A (Efremovich) proximity space $(X, \delta)$ is (strongly) connected iff for any non-empty proper subset $F$ of $X$, either the condition (1) or (2) holds.

1. $x \notin F$ whenever $\{x\} \delta F$ for some $x \in X$.
2. $x \notin F^c$ whenever $\{x\} \delta F^c$ for some $x \in X$, [26].

**Remark 4.9.** Let $(X, \delta)$ be in $\textbf{Prox}$. By Theorem 4.6 and Theorem 4.8, $(X, \delta)$ is (strongly) connected iff $(X, \delta)$ is $\partial$-connected.

**Lemma 4.10.** Let $f : (X, \delta) \to (Y, \delta')$ be a $p$-map in $\textbf{Prox}$. If $(X, \delta)$ is (strongly) connected, $\partial$-connected or $D$-connected, then $f(X)$ is (strongly) connected, $\partial$-connected or $D$-connected, respectively.
\textbf{Proof.} Let $(X, \delta)$, $(Y, \delta')$ be in \textbf{Prox} and $M$ is any non-empty proper subset of $f(X)$. Since $f^{-1}(M) \subseteq X$ and $(X, \delta)$ is (strongly) connected, either conditions (I) or (II) in Theorem 4.8 holds. Suppose condition (I) in Theorem 4.8 holds. Then, $x \notin f^{-1}(M)$ whenever $\{x\} \delta f^{-1}(M)$ for some $x \in X$. Hence, $f(x) \notin f(f^{-1}(M)) \subseteq M \Rightarrow f(x) \notin M$ whenever $\{f(x)\} \delta f(f^{-1}(M))$ for some $f(x) \in f(X)$. Similarly, if the condition (II) of Theorem 4.8 holds, $f(X)$ is strongly connected.

The proof for $\partial$-connected and $D$-connectedness is similar. \qed 

5. Tychonoff objects

In this section, the characterization of Tychonoff objects in this category is given. Furthermore, we investigate the relationships between Tychonoff objects and $ST_2$, $\Delta T_2$, $ST_3$, $\Delta T_3$, generalized separation properties and separation properties at a point $p$ in this category.

\textbf{Definition 5.1.} (cf. [7, 8, 14]). Let $\mathcal{U} : \mathcal{E} \rightarrow \textbf{Set}$ be a topological functor and $X$ an object in $\mathcal{E}$ with $\mathcal{U}(X) = B$.

1. $X$ is $C\Delta T_{\frac{3}{2}}$ iff $X$ is a subspace of a compact $\Delta T_2$.
2. $X$ is $CST_{\frac{3}{2}}$ iff $X$ is a subspace of a compact $ST_2$.
3. $X$ is $LT_{\frac{3}{2}}$ iff $X$ is a subspace of a compact $T_2$.
4. $X$ is $S\Delta T_{\frac{3}{2}}$ iff $X$ is a subspace of a strongly compact $\Delta T_2$.
5. $X$ is $SST_{\frac{3}{2}}$ iff $X$ is a subspace of a strongly compact $ST_2$.
6. $X$ is $SLT_{\frac{3}{2}}$ iff $X$ is a subspace of a strongly compact $T_2$.

\textbf{Remark 5.2.} For the category $\textbf{Top}$ of topological spaces, all six of the properties defined in Definition 5.1 are equivalent and reduce to the usual $T_{\frac{3}{2}} = \text{Tychonoff}$, i.e., completely regular $T_1$, spaces ([31, Remark 5.2, and Remark 6.2]).

\textbf{Theorem 5.3.} Let $(X, \delta)$ be a (Efremovich) proximity space. Then the followings are equivalent:

1. $(X, \delta)$ is $C\Delta T_{\frac{3}{2}}$.
2. $(X, \delta)$ is $CST_{\frac{3}{2}}$.
3. $(X, \delta)$ is $LT_{\frac{3}{2}}$.
4. $(X, \delta)$ is $S\Delta T_{\frac{3}{2}}$.
5. $(X, \delta)$ is $SST_{\frac{3}{2}}$.
6. $(X, \delta)$ is $SLT_{\frac{3}{2}}$.
7. $(X, \delta)$ is separated (Hausdorff) (Efremovich) proximity i.e., if $\{a\} \delta \{b\}$, then $a = b$.

\textbf{Proof.} It follows from Theorem 3.10, Lemma 4.3 and Definition 5.1. \qed 

\textbf{Example 5.4.} Let $X = \{a, b\}$, $\delta = \{(X, X), (\{a\}, \{a\}), (\{b\}, \{b\}), (X, \{a\}), (\{a\}, X), (X, \{b\}), (\{b\}, X)\}$ and $\delta_1 = \{(X, X), (\{a\}, \{a\}), (\{b\}, \{b\}), (X, \{a\}), (\{a\}, X), (X, \{b\}), (\{b\}, X), (\{a\}, \{b\}), (\{b\}, \{a\})\}$. Then $(X, \delta)$ is $C\Delta T_{\frac{3}{2}}$, but $(X, \delta_1)$ is not $C\Delta T_{\frac{3}{2}}$, since $(\{a\}, \{b\}) \in \delta$ with $a \neq b$.

By Remark 3.16 and Theorem 5.3, we need only to give explicit relationships among $C\Delta T_{\frac{3}{2}}$, $\widetilde{T_0}$ at $p$, $T_0^0$ at $p$, $\overline{T_0}$, $T_0^0$ and $\text{Pre}\overline{T_2}$ in the topological category of (Efremovich) proximity spaces.

\textbf{Remark 5.5.} Let $(X, \delta)$ be a (Efremovich) proximity space and $p \in A$.

1. By Remark 3.16 and Theorem 5.3, then the followings are equivalent:
   (i) $(X, \delta)$ is $\widetilde{T_0}$ at $p$ for all $p \in A$. 

(ii) \((X, \delta)\) is \(C\Delta T_{3\frac{1}{2}}\).

(iii) For each \(a \neq p\), \(\{a\}, \{p\}\) \(\notin \delta\).

2. By Theorem 3.13 (3) and Theorem 5.3, if \((X, \delta)\) is \(C\Delta T_{3\frac{1}{2}}\), then \((X, \delta)\) is \(T_0\) at \(p\) for all \(p \in A\). But the converse of implication is not true. For example, take \((X, \delta)\) to be the proximity space in Remark 3.16 (ii). Then \((X, \delta)\) is \(T_0\) at \(a\) but it is not \(C\Delta T_{3\frac{1}{2}}\) at \(a\).

3. By Remark 3.16, Theorem 5.3 and Definition 3.14, then the followings are equivalent:

(i) \((X, \delta)\) is \(T_0\).

(ii) \((X, \delta)\) is \(T_0\).

(iii) \((X, \delta)\) is \(C\Delta T_{3\frac{1}{2}}\).

(iv) For each distinct pair of points \(a\) and \(b\) in \(X\), \(\{a\}, \{b\}\) \(\notin \delta\).

4. By Theorem 3.13 (5), Theorem 3.13 (7) and Theorem 5.3, if \((X, \delta)\) is \(C\Delta T_{3\frac{1}{2}}\), then \((X, \delta)\) is \(T_0\) or \(Pre T_2\). But the converse implication is not true. For example, take \((X, \delta)\) to be the proximity space in Remark 3.16 (ii). Then \((X, \delta)\) is \(T_0\) or \(Pre T_2\) but it is not \(C\Delta T_{3\frac{1}{2}}\).

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References


Semiparallel Submanifolds of a Normal Paracontact Metric Manifold

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Abstract
The object of the present paper is to study invariant semiparallel and 2-semiparallel submanifolds of a normal paracontact metric manifold. We see that parallel submanifolds of a normal paracontact metric manifold are totally geodesic.

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1. Introduction
In the modern geometry, the geometry of submanifolds has become a subject of growing interest for its significant applications in applied mathematics and physics. For instance, the notion of invariant submanifold is used to discuss properties of non-linear autonomous system. On the other hand, the notion of geodesics plays an important role in the theory of relativity. For totally geodesic submanifolds, the geodesics of the ambient manifolds remain geodesics in the submanifolds. Therefore, totally geodesic submanifolds are also very much important in physical sciences. The study of geometry of invariant submanifolds was initiated by Bejancu and Papaghiuc [4,5]. Later on the invariant submanifolds have been studied by many geometers to different extent [13]. Invariant submanifolds inherit almost all properties of the ambient manifolds.

Arslan K. and et al. [1,11] defined and studied 2-semiparallel surfaces in space forms. Ishihara I. [7], Yano K. and Kon M. [16] studied anti-invariant submanifolds of a Sasakian space form. In [3–5, 8, 9, 14], authors studied semi-invariant and totally umbilical submanifolds in Sasakian and cosymplectic manifolds. In [2], we discussed the properties of semi-invariant submanifolds of a normal paracontact metric manifold.

Motivated by the above studies, the present paper deals with the study of invariant submanifolds of a normal paracontact metric manifold.

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Let $\mathcal{M}$ be a $(2n+1)$-dimensional manifold and $\phi$, $\xi$ and $\eta$ be a tensor field of type $(1,1)$, a vector field and a 1-form on $\mathcal{M}$, respectively. If $\phi$, $\xi$ and $\eta$ satisfy the conditions
\begin{equation}
\phi^{2}X = -X + \eta(X)\xi, \quad \eta(\xi) = 1,
\end{equation}
for any vector field $X$ on $\mathcal{M}$, then $\mathcal{M}$ is said to be an almost contact manifold. In addition, it is called almost contact metric manifold if $\mathcal{M}$ has a Riemannian metric tensor such that
\begin{equation}
g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi),
\end{equation}
for any $X, Y \in \chi(\mathcal{M})$, where $\chi(\mathcal{M})$ denotes set of the differentiable vector fields on $\mathcal{M}$ [15].

Furthermore, $\mathcal{M}$ is called a normal paracontact metric manifold if we have
\begin{equation}
(\nabla_{X}\phi)Y = -g(X, Y)\xi - \eta(Y)X + 2\eta(X)\eta(Y)\xi
\end{equation}
and
\begin{equation}
\nabla_{X}\xi = -\phi X,
\end{equation}
for any $X, Y \in \chi(\mathcal{M})$, where $\nabla$ denotes the Levi-Civita connection determined by $g$.

The concircular curvature tensor, conformal curvature tensor and quasi-conformal curvature tensor of a normal paracontact metric manifold $\mathcal{M}_{2n+1}$ are, respectively, defined by
\begin{equation}
\tilde{Z}(X, Y)Z = R(X, Y)Z - \frac{\tau}{2n(2n+1)} \{ g(Y, Z)X - g(X, Z)Y \},
\end{equation}
\begin{equation}
C(X, Y)Z = R(X, Y)Z - \frac{1}{2n-1} \{ S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX \\
- g(X, Z)QY \} + \frac{\tau}{2n(2n-1)} \{ g(Y, Z)X - g(X, Z)Y \},
\end{equation}
\begin{equation}
\tilde{C}(X, Y)Z = \lambda R(X, Y)Z + \mu \{ S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY \} \\
- \frac{\tau}{2n+1} \{ \frac{\lambda}{2n} + 2\mu \} \{ g(Y, Z)X - g(X, Z)Y \}
\end{equation}
for any $X, Y, Z \in \chi(\mathcal{M})$, where $R$ denotes the Riemannian curvature tensor of $\mathcal{M}$ and $Q$ is the Ricci operator given by $g(QX, Y) = S(X, Y)$.

Also, on a normal paracontact metric manifold $\mathcal{M}_{2n+1}$, the following relations are satisfied
\begin{equation}
R(\xi, Y)Z = g(Y, Z)\xi - \eta(Z)Y,
\end{equation}
and
\begin{equation}
R(\xi, Y)\xi = \eta(Y)\xi - Y
\end{equation}
for any $X, Y, Z \in \chi(\mathcal{M})$.

Now let $\overline{\mathcal{M}}$ be a submanifold of a normal paracontact metric manifold $\mathcal{M}$ with induced metric tensor $g$. We also denote the induced connections on the tangent bundle $\chi(\overline{\mathcal{M}})$ and the normal bundle $\chi^\bot(\overline{\mathcal{M}})$ by $\nabla$ and $\nabla^\bot$, respectively. Then the Gauss and Weingarten formulas are given by
\begin{equation}
\nabla_{X}Y = \nabla_{X}Y + h(X, Y)
\end{equation}
and
\begin{equation}
\nabla_{X}V = -AV + \nabla_{X^\bot}V,
\end{equation}
for any $X, Y, V \in \chi(\overline{\mathcal{M}})$.
for any $X, Y \in \chi(M)$ and $V \in \chi^+(M)$, where $h$ and $A_V$ are second fundamental form and shape operator, respectively, for the immersion of $M$ into $M$ [12]. $M$ is called totally geodesic submanifold if $h = 0$. $h$ and $A_V$ are related by
\[ g(A_V X, Y) = g(h(X, Y), V), \]
(1.12)

The covariant derivation of $h$ is defined by
\[ (\nabla_X h)(Y, Z) = \nabla_X^h h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z), \]
for any $X, Y, Z \in \chi(M)$. $h$ is said to be parallel if $(\nabla_X h)(Y, Z) = 0$.

For a submanifold $M$ of a normal paracontact metric manifold $M$, if for any $X \in \chi(M)$, then we can write
\[ \phi X = f X + \omega X, \]
(1.14)
where $f X$ and $\omega X$ are the tangent and normal components of $\phi X$, respectively. $M$ is said to be an invariant submanifold if $\omega = 0$ [6]. Throughout this paper, we assume that $M$ is an invariant submanifold of a normal paracontact metric manifold $M$. In this case, we have $\phi(\chi(M)) \subseteq \chi(M)$ and $\phi(\chi^+(M)) \subseteq \chi^+(M)$ [10].

2. Preliminaries

Let $(M, g)$ be a Riemannian manifold and $M$ be a submanifold of $M$. We denote the Levi-Civita connection of $g$ and the second fundamental form of $M$ by $\nabla$ and $h$, respectively. The submanifold $M$ said to be semiparallel if
\[ R(X, Y) \cdot h = 0, \]
(2.1)
for any $X, Y \in \chi(M)$, where $R$ denotes the Riemannian curvature tensor of $M$ and $R(X, Y) \cdot h = 0$ is defined by
\[ (R(X, Y) \cdot h)(Z, U) = R^{\perp}(X, Y)h(Z, U) - h(R(X, Y)Z, U) - h(Z, R(X, Y)U), \]
for any $X, Y, Z, U \in \chi(M)$.

In [1] Arslan et al. defined and studied 2-semiparallel submanifolds. Such submanifolds are defined as, a Riemannian submanifold $M$ is said to be 2-semiparallel if the following relation holds
\[ R(X, Y) \cdot \nabla h = 0, \]
(2.2)
for any $X, Y \in \chi(M)$, where
(2.3)
for any $X, Y, Z, U, W \in \chi(M)$.

Now, let us assume that normal paracontact metric manifold $M^{2n+1}$ is conformal flat. Then from (1.6) we have
\[ R(X, Y)\xi = \frac{1}{2n-1} \{ S(Y, \xi)X - S(X, \xi)Y + \eta(Y)QX - \eta(X)QY \} - \frac{\tau}{2n(2n-1)} \{ \eta(Y)X - \eta(X)Y \}, \]
(2.4)
which implies that
\[ \eta(Y)X - \eta(X)Y = \frac{2n}{2n-1} \{ \eta(Y)X - \eta(X)Y \} + \frac{1}{2n-1} \{ \eta(Y)QX - \eta(X)QY \} \]
\[ - \frac{\tau}{2n(2n-1)} \{ \eta(Y)X - \eta(X)Y \}. \quad (2.5) \]

This is equivalent to
\[ \eta(Y)QX - \eta(X)QY = \{ \eta(Y)X - \eta(X)Y \} \{ \frac{\tau}{2n} - 1 \}. \quad (2.6) \]

For \( Y = \xi \), we obtain
\[ QX = \left( \frac{\tau}{2n} - 1 \right) X + \left( 2n + 1 - \frac{\tau}{2n} \right) \eta(X)\xi, \quad (2.7) \]
that is, conformally flat normal paracontact metric manifold is an Einstein manifold and the Ricci tensor is given by
\[ S(X, Y) = \left( \frac{\tau}{2n} - 1 \right) g(X, Y) + \left( 2n + 1 - \frac{\tau}{2n} \right) \eta(X)\eta(Y). \quad (2.8) \]

The scalar curvature \( \tau \) of \( M^{2n+1} \) is obtained by
\[ \tau = \left( \frac{\tau}{2n} - 1 \right) (2n + 1) + \left( 2n + 1 - \frac{\tau}{2n} \right). \quad (2.9) \]

Thus we have the following theorem for later use.

**Theorem 2.1.** Conformally flat a normal paracontact metric manifold is always an \( \eta \)-Einstein manifold.

Now, let us suppose that normal paracontact metric manifold be Quasi-Conformally flat. Then from (1.7), we have
\[ R(X, Y)Z = -\frac{\mu}{\lambda} \{ S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY \} \]
\[ + \frac{\tau}{n\lambda} \{ \frac{\lambda}{n-1} + 2\mu \} \{ g(Y, Z)X - g(X, Z)Y \}, \]
for any \( X, Y, Z \in \chi(M) \). By the direct calculations, we obtain
\[ \tau = \frac{2n(n - 1) \{ \lambda + \mu(2n - 1) \}}{2\lambda + 3\mu(n - 1)} \]
provided that \( 2\lambda + 3\mu(n - 1) \neq 0 \).

3. **Invariant submanifolds of a normal paracontact metric manifold**

In this section, we study of invariant submanifolds of a normal paracontact metric manifold satisfying the \( \tilde{Z}(X, Y) \cdot h = 0 \) and \( \tilde{Z}(X, Y) \cdot \nabla h = 0 \). Finally we see that these conditions are satisfied if and only if invariant submanifold is totally geodesic.

**Proposition 3.1.** Let \( \overline{M} \) be an invariant submanifold of a normal paracontact metric manifold \( M \). Then the following relations holds:
1) \( \nabla_X \xi = -fX, \quad h(X, \xi) = 0 \)
2) \( \phi h(X, Y) = h(X, fY) \)
3) \( (\nabla_X f)Y = -g(X, Y)\xi - \eta(Y)X + 2\eta(Y)\eta(X)\xi, \)
   for any \( X, Y \in \chi(\overline{M}). \)
\textbf{Proof.} By using (1.4) and taking into account of $\mathcal{M}$ being invariant submanifold, 1) statement is obvious. On the other hand, making use of (1.3) and (1.10), we have

$$((\nabla_X \phi)Y = \nabla_X \phi Y - \phi \nabla_X Y = h(X, fY) + \nabla_X fY - \phi h(X, Y) - f \nabla_X Y = -g(X, Y)\xi - \eta(Y)X + 2\eta(X)\eta(Y)\xi,$$

for any $X, Y \in \chi(\mathcal{M})$, which proves 2) and 3) statements.

Thus we have the following conclusion.

\textbf{Corollary 3.2.} Every invariant submanifold of a normal paracontact metric manifold has a normal paracontact metric structure.

\textbf{Theorem 3.3.} Let $\mathcal{M}$ be an invariant submanifold of a normal paracontact metric manifold $M$. Then the second fundamental form of $\mathcal{M}$ is parallel if and only if $\mathcal{M}$ is a totally geodesic submanifold.

\textbf{Proof.} If the second fundamental form $h$ of $\mathcal{M}$ is parallel, then we have

$$\nabla^2_{\xi, \eta} h(Y, Z) - h(\nabla_Y Z, Y) - h(Y, \nabla_X Z) = 0,$$  \hspace{1cm} (3.1)

for any $X, Y, Z \in \chi(\mathcal{M})$. Setting $Z = \xi$ in (3.1) and taking into account that Proposition 3.1, we get $h(Y, \nabla_X \xi) = -h(X, fY) = 0$, which implies that $\mathcal{M}$ is a totally geodesic submanifold. The converse statement is obvious.

\textbf{Theorem 3.4.} Let $\mathcal{M}$ be an invariant submanifold of a normal paracontact metric manifold $M$. Then $\mathcal{M}$ is semiparallel if and only if $\mathcal{M}$ is a totally geodesic submanifold.

\textbf{Proof.} If $\mathcal{M}$ is semiparallel, then $\nabla h = 0$. This implies that

$$(R(X, Y) \cdot h)(Z, U) = R^\perp(X, Y)h(Z, U) - h(R(X, Y)Z, U) - h(Z, R(X, Y)U),$$  \hspace{1cm} (3.2)

for any $X, Y, Z, U \in \chi(\mathcal{M})$. Putting $X = U = \xi$ in (3.2), we obtain

$$R^\perp(\xi, Y)h(Z, \xi) - h(R(\xi, Y)Z, \xi) - h(Z, R(\xi, Y)\xi) = 0.$$  \hspace{1cm} (3.3)

Here taking into account of (1.8) and (1.9), we reach at

$$\eta(Z)h(Y, \xi) - g(Y, Z)h(\xi, \xi) + h(Y, Z) - \eta(Y)h(Z, \xi) = 0.$$  \hspace{1cm} (3.4)

Here, from (3.4) we conclude $h(Y, Z) = 0$, that is, the submanifold is a totally geodesic. Conversely, if $h = 0$, then $M$ is semiparallel.

\textbf{Theorem 3.5.} Let $\mathcal{M}$ be an invariant submanifold of a normal paracontact metric manifold $M$. Then $\mathcal{M}$ is 2-semiparallel if and only if $\mathcal{M}$ is a totally geodesic submanifold.

\textbf{Proof.} Let us suppose $\mathcal{M}$ be 2-semiparallel. This implies that

$$(R(X, Y) \cdot \nabla h)(Z, U, W) = R^\perp(X, Y)(\nabla_Z h)(U, W) - (\nabla_{R(X, Y)Z} h)(U, W) - (\nabla_Z h)(R(X, Y)U, W) - (\nabla_Z h)(U, R(X, Y)W),$$  \hspace{1cm} (3.5)

for all $X, Y, Z, U, W \in \chi(\mathcal{M})$. Here taking $X = U = \xi$ and we calculate each expression as follows

$$R^\perp(\xi, Y)(\nabla_Z h)(\xi, W) = R^\perp(\xi, Y)\{\nabla^{\perp}_Z h(\xi, W) - h(\nabla_Z \xi, W) - h(\xi, \nabla_Z W)\} = R^\perp(\xi, Y)h(fZ, W),$$  \hspace{1cm} (3.6)
\[
(\nabla_{R(\xi, Y)Z} h)(\xi, W) = \nabla_{R(\xi, Y)Z}^1 h(\xi, W) - h(\nabla_{R(\xi, Y)Z}^{\perp} \xi, W) - h(\nabla_{R(\xi, Y)Z} \xi, W) \\
= -h(\nabla_{Y^g(Y, Z)\xi} Z, W) \\
= \eta(Z) h(\nabla_Y \xi, W) - g(Y, Z) h(\nabla \xi, W) \\
= -\eta(Z) h(fY, W), \quad (3.7)
\]

\[
(\nabla_Z h)(R(\xi, Y)Z, W) = \nabla_Z^1 h(R(\xi, Y)\xi, W) - h(\nabla_Z R(\xi, Y)\xi, W) - h(R(\xi, Y)\xi, \nabla_Z W) \\
= -\nabla_Z^1 h(Y, W) + \nabla_Z^1 h(\eta(Y)\xi, W) + h(\nabla_Z Y, W) \\
= h(\eta(Y)\xi, W) + h(Y, \nabla_Z W) - h(\nabla_Z W, \eta(Y)\xi) \\
= - (\nabla_Z h)(Y, W), \quad (3.8)
\]

and

\[
(\nabla_Z h)(\xi, R(\xi, Y)W) = \nabla_Z^1 h(\xi, R(\xi, Y)W) - h(\nabla_Z \xi, R(\xi, Y)W) - h(\nabla_Z \xi, \nabla_Z W, \eta(Y)\xi) \\
= - h(\nabla_Z \xi, -\eta(W)Y + g(Y, W)\xi) \\
= -\eta(W) h(fZ, Y). \quad (3.9)
\]

Thus, by combining (3.6),(3.7),(3.8) and (3.9), we derive

\[
(R(\xi, Y) \cdot \nabla h)(Z, \xi, W) = R^1(\xi, Y) h(fZ, W) + \eta(Z) h(fY, W) \\
+ (\nabla_Z h)(Y, W) + \eta(W) h(fZ, Y). \quad (3.10)
\]

Since $\mathbb{M}$ is 2-semiparallel and for $W = \xi$, we obtain $h(fY, W) = 0$. This proves our assertion. The converse is obvious. \hfill \Box

**Theorem 3.6.** Let $\mathbb{M}$ be an invariant submanifold of a paracontact metric manifold $M$ with $\tau \neq 2n(2n+1)$. Then $Z(X, Y) \cdot h = 0$ if and only if $\mathbb{M}$ is totally geodesic submanifold.

**Proof.** $Z(X, Y) \cdot h = 0$ implies that

\[
(Z(X, Y) \cdot h)(Z, U) = R^1(X, Y) h(Z, U) - h(Z, Z(X, Y)U) \\
- h(Z, Z(X, Y)U), \quad (3.11)
\]

for any $X, Y, Z, U \in \chi(M)$. By using (1), we have

\[
Z(\xi, \eta(Y)Z) = \left(1 - \frac{\tau}{2n(2n+1)}\right) (g(Y, Z)\xi - \eta(Z)Y). \quad (3.12)
\]

Thus

\[
0 = R^1(\xi, Y) h(Z, \xi) - h(Z, Z(X, Y)Z, \xi) - h(Z, Z(X, Y)\xi) \\
= \left(1 - \frac{\tau}{2n(2n+1)}\right) (h(-\eta(Z)Y + g(Y, Z)\xi, \xi) - h(Z, -Y + \eta(Y)\xi)) \\
= \left(1 - \frac{\tau}{2n(2n+1)}\right) h(Y, Z). \quad (3.13)
\]

This proves our assertion. \hfill \Box

**Theorem 3.7.** Let $\mathbb{M}$ be an invariant submanifold of a paracontact metric manifold $M$ with $\tau \neq 2n(2n+1)$. Then $Z(X, Y) \cdot \nabla h = 0$ if and only if $\mathbb{M}$ is totally geodesic submanifold.
Proof. \( \bar{Z}(X, Y) \cdot \nabla h = 0 \) means that
\[
R^\perp(X, Y)(\nabla_Z h)(U, W) = (\nabla_{\bar{Z}(X, Y)} Z h)(U, W) - (\nabla_Z h)(\bar{Z}(X, Y) U, W)
- (\nabla_Z h)(U, \bar{Z}(X, Y) W) = 0,
\]
for any \( X, Y, Z, U, W \in \chi(M) \). Here,
\[
R^\perp(\xi, Y)(\nabla_Z h)(\xi, W) = R^\perp(\xi, Y)\{\nabla_{\nabla Z} h(\xi, W) - h(\nabla_Z \xi, W) - h(\nabla_Z W, \xi)\}
= R^\perp(\xi, Y) h(f Z, W), \tag{3.15}
\]
\[
(\nabla_{\bar{Z}(\xi, Y)} Z h)(\xi, W) = \nabla_{\nabla Z} h(\bar{Z}(\xi, Y) \xi, W) - h(\nabla_Z \bar{Z}(\xi, Y) \xi, W)
- h(\nabla_Z W, \bar{Z}(\xi, Y) \xi)
= \left(1 - \frac{\tau}{2n(2n + 1)}\right)\{\nabla_{\nabla Z} h(-Y + \eta(Y) \xi, W)
- h(\nabla_Z - Y + \eta(Y) \xi, W) - h(\nabla_Z W, -Y + \eta(Y) \xi)\}
= \left(1 - \frac{\tau}{2n(2n + 1)}\right)\{-\nabla_{\nabla Z} h(Y, W) + \nabla_{\nabla Z} h(Y, \eta(Y) \xi, W)
+ h(\nabla_Z Y, W) - \eta(Y) h(\xi, W) + h(\nabla_Z W, Y)
- \eta(Y) h(\nabla_Z W, \xi)\}
= -\left(1 - \frac{\tau}{2n(2n + 1)}\right)(\nabla_Z h)(Y, W) \tag{3.16}
\]
and
\[
(\nabla_Z h)(\xi, \bar{Z}(\xi, Y) W) = \nabla_{\nabla Z} h(\xi, \bar{Z}(\xi, Y) W) - h(\nabla_Z \bar{Z}(\xi, Y) W, \xi)
- h(\nabla_Z \bar{Z}(\xi, Y) W, \xi)
= \left(1 - \frac{\tau}{2n(2n + 1)}\right)\{\eta(W) h(\nabla_Z \xi, Y)
- g(Y, W) h(\nabla_Z \xi, \xi)\}
= -\left(1 - \frac{\tau}{2n(2n + 1)}\right)\eta(W) h(f Z, Y). \tag{3.17}
\]
Thus we obtain
\[
R^\perp(\xi, Y) h(f Z, W) + \left(1 - \frac{\tau}{2n(2n + 1)}\right)\{\eta(Z) h(f Y, W)
+ (\nabla_Z h)(Y, W) + \eta(W) h(f Z, Y)\} = 0. \tag{3.19}
\]
Here choosing \( W = \xi \), we conclude
\[
\left(1 + \frac{\tau}{2n(2n + 1)}\right)\{h(f Z, Y) - (\nabla_Z h)(Y, \xi)\} = \left(1 + \frac{\tau}{2n(2n + 1)}\right) h(f Z, W).
\]
The converse is obvious. This proves our assertion.
Example 3.8. Let $\overline{M}$ be a submanifold of $\mathbb{R}^7$ is given by the equation
\[
\phi(x_1, y_1, s) = (\cos x_1 \sinh y_1, \sin y_1 \sinh x_1, \cos x_1 \sinh y_1, \\
\sin x_1 \cosh y_1, \cos y_1 \cosh x_1, \sin x_1 \cosh y_1, s).
\]
Then tangent space of $\overline{M}$ is spanned by the vectors
\[
e_1 = -\sin x_1 \sinh y_1 \frac{\partial}{\partial x_1} + \sin y_1 \cosh x_1 \frac{\partial}{\partial x_2} - \sin x_1 \sinh y_1 \frac{\partial}{\partial x_3},
\]
\[
e_2 = \cos x_1 \cosh y_1 \frac{\partial}{\partial x_1} + \cos y_1 \sinh x_1 \frac{\partial}{\partial x_2} + \cos x_1 \cosh y_1 \frac{\partial}{\partial x_3},
\]
\[
e_3 = \cos x_1 \cosh y_1 \frac{\partial}{\partial y_1} - \sin y_1 \cosh x_1 \frac{\partial}{\partial y_2} + \sin x_1 \sinh y_1 \frac{\partial}{\partial y_3},
\]
We define the almost paracontact structure of $\mathbb{R}^7$ by
\[
\phi(x_1, x_2, x_3, y_1, y_2, y_3, s) = (-y_1, -y_2, -y_3, x_1, x_2, x_3, 0),
\]
then we have $\phi^2 X = -X + \eta(X)\xi$ for any $X \in \chi(\mathbb{R}^7)$. By direct calculations,
\[
\phi e_1 = (\cos x_1 \cosh y_1, \sin y_1 \sinh x_1, \cos x_1 \sinh y_1, \\
-\sin x_1 \sinh y_1, \sin y_1 \cosh x_1, -\sin x_1 \sinh y_1,0)
= -e_2,
\]
\[
\phi e_2 = (\sin x_1 \sinh y_1, \sin y_1 \cosh x_1, -\sin x_1 \sinh y_1, \\
\cos x_1 \cosh y_1, \cos y_1 \cosh x_1, \cos x_1 \cosh y_1, s)
= e_1.
\]
Thus $\overline{M}$ is a 3-dimensional invariant submanifold of $\mathbb{R}^7$. On the other hand, Lie-bracket the vector fields of $e_1$ and $e_2$ is
\[
[e_1, e_2] = \sinh(2y_1) \frac{\partial}{\partial x_1} + \sin(2x_1) \frac{\partial}{\partial y_1} \\
- (2\sin x_1 \cos y_1 \sinh y_1 \cosh x_1 + 2\cos x_1 \sin y_1 \sinh x_1 \cosh y_1) \frac{\partial}{\partial x_2} \\
+ (2\sin x_1 \sin y_1 \sinh x_1 \cosh y_1 - 2\cos x_1 \cos y_1 \cosh x_1 \cosh y_1) \frac{\partial}{\partial y_2} \\
+ \sinh(2y_1) \frac{\partial}{\partial x_3} + + \sin(2x_1) \frac{\partial}{\partial y_3}.
\]
By using Koszul-formulae, we obtain
\[
\nabla_{e_1} e_2 = \left[ -\cos x_1 \sinh(2x_1) \cosh y_1 - \sin(2y_1) \sin x_1 \sinh y_1 \\
- \sin x_1 \sinh y_1 \sinh(2y_1) + \cos x_1 \cosh y_1 \sin 2x_1 \\
+ \frac{1}{2} \cos x_1 \cosh y_1 \sinh(2x_1) + \frac{1}{2} \sin x_1 \sinh y_1 \sin(2y_1) \right] e_1 \\
+ \left[ \sin x_1 \sinh y_1 \sin(2x_1) - \frac{1}{2} \sin x_1 \sinh y_1 \sin(2x_1) \\
+ \cos x_1 \cosh y_1 \sinh(2y_1) + \frac{1}{2} \cos x_1 \cosh y_1 \sin(2y_1) \right] e_2
\]
Since $\mathcal{M}$ is a totally-geodesic submanifold, $\mathcal{M}$ is a semiparallel and 2-semiparallel submanifold of $\mathbb{R}^7$. This verifies the statements of Theorem 3.4 and Theorem 3.5.

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References

Interval type-2 fuzzy c-Control charts using ranking methods

Hatice Ercan Teken* and Ahmet Sermet Anagün†

Abstract

Control charts are important for process or product because they provide information about the control situation of process and product. Because of this feature, control charts are used in many fields. Information about the product and/or process, which is under control or not, can be provided by looking the control charts. Fuzzy numbers are used to reduce information losses in operations with crisp numbers. In control charts applications, especially for qualitative control charts, the fuzzy set theory reduces the information losses and provide more flexible decision-making process. In the literature, there are some fuzzy control charts with type-1 fuzzy sets but there are few studies about fuzzy control charts regarding the cases where the data are expressed by type-2 fuzzy sets. The purpose of the study is to create an innovation using the ranking methods, which has not used for control charts in accessible literature, for the fuzzy control charts with interval type-2 fuzzy sets. The fuzzy results are compared with the crisp results. This study introduces ranking methods as new approach to generate interval type-2 fuzzy control charts, which is a different field.

Keywords: Interval type-2 trapezoidal fuzzy sets, Fuzzy control charts, c-Control charts, Nonconformities, Ranking methods

Mathematics Subject Classification (2010): 03E72, 62A86

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1. Introduction

Control charts are one of the statistical process control methods that inform the process or product according to the upper control limit (UCL) and lower control limit (LCL) determined by the data. It allows to take precautions by recognizing abnormal conditions of the product or process. Similarly, the product or process indicates that the system is suitable when normal conditions occur. Beside these, the control charts are easy to understand because of their visual representation.

Control charts applied in various areas were first used in Bell Laboratories [19]. The control charts vary according to the fault type they are interested in. In general, it is divided qualitatively and quantitatively in the general sense. Since quantitative data are measurable, which are used for variable control charts, the data collection is much simpler than quantification. On the other hand, qualitative data are more relevant, which are used for attribute control charts, the data collection is more difficult and subjective.

In many areas, the fuzzy set theory, which was developed by Zadeh was first used in 1965, reduces the subjectivity and data loss of data collection. Since fuzzy numbers are more flexible than crisp numbers, are used in many areas and thought to be beneficial because they can transform linguistic expressions into numbers. There are some studies about control charts which are considered as one of these areas [8, 6, 17, 1, 5, 7, 23, 15, 10].

It is the work of Wang and Raz, and Raz and Wang [23, 15], who first mentioned linguistic definitions for quality character. Then Kanagawa et al. Wang and Raz’s work and talked about fuzzy probability and fuzzy membership approaches [10].

In his work, Asai mentioned that the control charts generated by categorical data and the fuzzy logic for them can be used [2]. Other works mention that fuzzy control charts can be created with the categorical data, Laviolette et al., and Woodall et al. [12, 24].

In the literature, Gülbay et al., Gülbay and Kahraman, Şentürk and Erginel have constructed control graphs for type-1 fuzzy numbers using the $\alpha$-cut method [9, 7, 17].

Some studies in the literature are related to $\bar{X}$ fuzzy control charts. Faraz and Shapiro have studied $\bar{X} - S$ fuzzy control charts using LR fuzzy numbers [6]. Shu and Wu use triangular fuzzy numbers. With these numbers, the fuzzy $\bar{X} - R$ graphics are created separately and investigated whether the process is under control or not [20]. Similarly, Alaeddini et al. generate $\bar{X}$ fuzzy control charts using triangular fuzzy numbers [1].

Cheng has drawn control charts with distance to possibility and control charts with distance to necessity [5].

On the other hand, Gülbay and Kahraman defined fuzzy control limits and calculated fuzzy control limits for type-1 fuzzy numbers. This study allows for more flexible decision making such as “rather in control” and “rather out of control” for fuzzy control charts [8].

Finally, in the accessible literature, it can be mentioned that type-1 fuzzy numbers are used for process capability analysis. While Kaya and Kahraman use trapezoidal and triangular fuzzy numbers for process capability analysis, Senvar and Kahraman intend to provide flexibility for the process capability indices and thus use type-1 fuzzy sets [11, 18].

After talking about the type-1 fuzzy control charts, the type-2 control charts have gradually begun to enter the literature. Şentürk and Antucheviciene draw interval type-2 fuzzy control charts with using defuzzification method [16]. Our previous work has used defuzzification method for interval type-2 fuzzy control charts. In addition to defuzzification method, the likelihood method for interval type-2 fuzzy sets is used to create c-control charts [21]. Similarly, we use different ranking methods for interval type-2 fuzzy sets to generate fuzzy control charts [22].

In this study, ranking methods are used to draw interval type-2 fuzzy control charts based on interval type-2 trapezoidal fuzzy sets. Regarding the accessible literature, this is the first study to use the ranking methods for fuzzy control charts.

The study has been prepared in the following draft. The operations required to calculate the interval type-2 fuzzy control limits are described in Section 2. In Section 3, interval type-2 ranking
methods to compare data are mentioned. In Section 4, the ranking methods which are mentioned will be adapted for interval type-2 fuzzy control charts. A numerical example will be given in Section 5. Section 6 is about conclusion and future research.

2. Interval Type-2 Fuzzy Sets

Zadeh has generated type-2 fuzzy numbers which means fuzzifying membership degrees. Therefore, he has provided more realistic data [25]. Mendel, in his work, has mentioned that type-2 fuzzy numbers are more useful in defining some linguistic expressions [13].

The general representation of type-2 fuzzy numbers is \( \tilde{A} = \{(x, \mu(x, a)) | \forall x \in X, \forall a \in J_1 \subseteq [0, 1], 0 \leq \mu_A(x, a) \leq 1 \} \) where \( J_1 \) is in interval [0, 1]. When all \( \mu_A(x, a) = 1 \), \( \tilde{A} \) is called an interval type-2 fuzzy set [3].

In this study, we use interval type-2 trapezoidal fuzzy sets as given below:

\[
\tilde{A}_i = \left( \left[ a_{iU}^m, a_{iL}^m, a_{iU}^m, a_{iL}^m \right], H_1(\tilde{A}_i^U), H_2(\tilde{A}_i^L), H_3(\tilde{A}_i^U), H_4(\tilde{A}_i^L) \right)
\]

where \( a_{iU}^m \) is the reference point of the interval type-2 fuzzy set \( \tilde{A}_i \), \( k = 1, 2, 3, 4, m = U, L \) (U for upper membership function and L for lower membership function) and \( 1 \leq i \leq n \). \( H_1(\tilde{A}_i^U) \in [0, 1] \) denotes the membership value of the element \( a_{iU}^m \); \( j = 1, 2, 3, 4, m = U, L \) and \( 1 \leq i \leq n \). Figure 1 shows the illustration of trapezoidal interval type-2 fuzzy sets.

Addition, subtraction, multiplication, and multiplication with a scaler number required for control charts are given below. Eqs. 2.1-2.4 show these operations.

\[
\tilde{A}_1 + \tilde{A}_2 = (a_{1U}^m + a_{2U}^m, a_{1L}^m + a_{2L}^m, a_{1L}^m + a_{2U}^m, a_{1U}^m + a_{2U}^m; \min(H_1(\tilde{A}_1^U); H_1(\tilde{A}_2^U)),
\min(H_2(\tilde{A}_1^L); H_2(\tilde{A}_2^L)));
\]

\[
(2.1)
\]

\[
\tilde{A}_1 - \tilde{A}_2 = (a_{1L}^m - a_{2U}^m, a_{1L}^m - a_{2L}^m, a_{1L}^m - a_{2U}^m, a_{1U}^m - a_{2U}^m; \min(H_1(\tilde{A}_1^U); H_1(\tilde{A}_2^U)),
\min(H_2(\tilde{A}_1^L); H_2(\tilde{A}_2^L)));
\]

\[
(2.2)
\]

\[
\tilde{A}_1 \ast \tilde{A}_2 = (a_{1L}^m \ast a_{2U}^m, a_{1L}^m \ast a_{2L}^m, a_{1L}^m \ast a_{2U}^m, a_{1U}^m \ast a_{2U}^m; \min(H_1(\tilde{A}_1^U); H_1(\tilde{A}_2^U)),
\min(H_2(\tilde{A}_1^L); H_2(\tilde{A}_2^L)));
\]

\[
(2.3)
\]

\[
\tilde{A}_1 \ast \tilde{A}_2 = (a_{1L}^m \ast a_{2U}^m, a_{1L}^m \ast a_{2L}^m, a_{1L}^m \ast a_{2U}^m, a_{1U}^m \ast a_{2U}^m; \min(H_1(\tilde{A}_1^U); H_1(\tilde{A}_2^U)),
\min(H_2(\tilde{A}_1^L); H_2(\tilde{A}_2^L)));
\]

\[
(2.4)
\]
3. Ranking Methods for Interval Type-2 Fuzzy Sets

Since type-2 fuzzy sets are somehow difficult to calculate, usually interval type-2 fuzzy sets are preferred. Different methods are being developed for comparing fuzzy sets. Various comparison methods for interval type-2 fuzzy sets are available in the literature. One of these methods is ranking method.

In this study, two ranking methods are used; Chen et al.’s ranking method and Qin and Liu’s ranking method, respectively.

3.1. Chen et al.’s ranking method. Chen et al. proposed ranking method for interval type-2 trapezoidal fuzzy sets. Ranking of A is shown in Eq.(3.1) [4].

\[
RV(\tilde{A}) = \frac{1}{2} H_1(\tilde{A}^U) + H_2(\tilde{A}^L) + H_1(\tilde{A}^U) + H_2(\tilde{A}^L)
\]

\[
(3.1)
\]

\[
K_i \text{ value is the value that makes the numbers positive. The } K_i \text{ value will be evaluated as 0, since there will be no negative data for the control charts, in this study.}
\]

3.2. Qin and Liu’s ranking method. Qin and Liu proposed ranking method for type-2 fuzzy sets. Ranking of A is shown in Eq.(3.2) [14].

\[
\text{Rank}(A) = \sum_{i=1}^{3} (M_i(A^U) + M_i(A^L)) - \frac{1}{3} \sum_{i=1}^{3} (S_i(A^U) + S_i(A^L))
\]

\[
(3.2)
\]

\[
\text{where } M_i(A) = \left( a_{i, p} + a_{i, p+1} \right) / 2 \text{ and } S_i(A) = \sqrt{\frac{1}{2} \sum_{t=1}^{4} (a_{i, t} - \frac{1}{2} \sum_{t=1}^{4} a_{i, t})^2}, i=1,2,3.
\]

4. C-Control Charts with Interval Type-2 Fuzzy Sets

Linguistic data can be represented by fuzzy sets. For this reason, there are lots of applications in many areas. Control charts that can be regarded as one of these fields. It is suitable for control charts, especially attribute control charts, because of the data are linguistic and categorical.

The attribute control charts are separated by the fraction rejected as nonconforming to the specifications, number of nonconforming items, number of nonconformities and number of nonconformities per unit. In this study, we have been working on the fuzzifying of control charts dealing with the number of nonconformities referred to as c control charts. For classical c control charts, control limit are calculated as given below (see Eqs. (4.1)-(4.3)).
\( CL = \bar{c} \)
\( LCL = \bar{c} - 3 \sqrt{\bar{e}} \)
\( UCL = \bar{c} + 3 \sqrt{\bar{e}} \)

where \( \bar{c} \) is the mean of the nonconformities.

In this study, each sample point is expressed as an interval type-2 trapezoidal fuzzy numbers
\[
\left( [a_{1i}, a_{2i}, a_{3i}, a_{4i}; H_1(A_i^U), H_2(A_i^L)] , [a_{1i}, a_{2i}, a_{3i}, a_{4i}; H_1(A_i^L), H_2(A_i^U)] \right)
\]

The fuzzy control limits are then calculated using the operations of interval type-2 trapezoidal fuzzy sets. These equations are shown in Eqs. (4.4)-(4.6).

\[
\bar{CL} = \left( \left( \sum_{i=1}^{m} \frac{a_{1i}}{m}, \sum_{i=1}^{m} \frac{a_{2i}}{m}, \sum_{i=1}^{m} \frac{a_{3i}}{m}, \sum_{i=1}^{m} \frac{a_{4i}}{m} ; \min(H_1(A_i^U)), \min(H_2(A_i^U)) \right), \left( \sum_{i=1}^{m} \frac{a_{1i}}{m}, \sum_{i=1}^{m} \frac{a_{2i}}{m}, \sum_{i=1}^{m} \frac{a_{3i}}{m}, \sum_{i=1}^{m} \frac{a_{4i}}{m} ; \min(H_1(A_i^L)), \min(H_2(A_i^L)) \right) \right)
\]

\[
\bar{UCL} = \left( \left( \sum_{i=1}^{m} \frac{a_{1i}}{m}, \sum_{i=1}^{m} \frac{a_{2i}}{m}, \sum_{i=1}^{m} \frac{a_{3i}}{m}, \sum_{i=1}^{m} \frac{a_{4i}}{m} ; \min(H_1(A_i^U)), \min(H_2(A_i^U)) \right), \left( \sum_{i=1}^{m} \frac{a_{1i}}{m}, \sum_{i=1}^{m} \frac{a_{2i}}{m}, \sum_{i=1}^{m} \frac{a_{3i}}{m}, \sum_{i=1}^{m} \frac{a_{4i}}{m} ; \min(H_1(A_i^L)), \min(H_2(A_i^L)) \right) \right)
\]

\[
\bar{LCL} = \left( \left( \sum_{i=1}^{m} \frac{a_{1i}}{m} + 3 \sqrt{\frac{a_{1i}}{m}}, \sum_{i=1}^{m} \frac{a_{2i}}{m} + 3 \sqrt{\frac{a_{2i}}{m}}, \sum_{i=1}^{m} \frac{a_{3i}}{m} + 3 \sqrt{\frac{a_{3i}}{m}}, \sum_{i=1}^{m} \frac{a_{4i}}{m} + 3 \sqrt{\frac{a_{4i}}{m}} ; \min(H_1(A_i^U)), \min(H_2(A_i^U)) \right), \left( \sum_{i=1}^{m} \frac{a_{1i}}{m} + 3 \sqrt{\frac{a_{1i}}{m}}, \sum_{i=1}^{m} \frac{a_{2i}}{m} + 3 \sqrt{\frac{a_{2i}}{m}}, \sum_{i=1}^{m} \frac{a_{3i}}{m} + 3 \sqrt{\frac{a_{3i}}{m}}, \sum_{i=1}^{m} \frac{a_{4i}}{m} + 3 \sqrt{\frac{a_{4i}}{m}} ; \min(H_1(A_i^L)), \min(H_2(A_i^L)) \right) \right)
\]

After calculating interval type-2 control limits, the ranking methods mentioned in the previous section are used to compare limits based on the data.

5. **Numerical Example**

In this section, numerical example is given so that the methods can be better understood. Data for nonconformities are shown in Table 1, which shows crisp value of data, and Table 2, which shows linguistic values of data. Interval type-2 trapezoidal fuzzy sets are transformed from linguistic data and control limits are obtained as interval type-2 trapezoidal numbers using Eqs. (4.4)-(4.6).

\( \bar{CL}, LCL \) and \( \bar{UCL} \) are calculated as interval type-2 trapezoidal fuzzy sets and these are given below.

\[
\bar{CL} = ((18.13, 22.67, 26.93, 32.07; 0.63, 0.59), (19.37, 23.67, 26.00, 30.30; 0.48, 0.45))
\]
\[
LCL = ((1.14, 7.10, 12.65, 19.29; 0.63, 0.59), (32.57, 38.26, 41.30, 46.81; 0.48, 0.45))
\]
\[
\bar{UCL} = ((30.91, 36.95, 42.50, 49.05; 0.63, 0.59), (32.57, 38.26, 41.30, 46.81; 0.48, 0.45))
\]
Table 1. Crisp values for numerical example

<table>
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<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
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<td>Crisp Value</td>
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Table 2. Linguistic values for numerical example

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</table>

5.1. Solving with Chen et al.’s ranking method. In this study, ranking methods are used to generate control charts. One of these ranking methods is proposed by Chen et al. [4]. We refer to this method as shown in Eq. (3.1) in Section 3.1.

Table 3 shows ranking values using Chen et al.’s ranking method for numerical example.

The ranking values are calculated for the control limits regarding with Chen et. al’s method. These values are obtained as 638.16, 1612.44, and 107.44 for CL, UCL, and LCL, respectively. Based on the calculations, the control chart is drawn using Chen et al.’s method in Figure 2.

Referring to Figure 2 and Table 3, it can be said that sample points of 3, 4, 7, 11, 14, 17 and 30 are out of control, remainings are in control.

5.2. Solving with Qin and Liu’s ranking method. The other ranking method, used in this study, is proposed by Qin and Liu [14]. We refer to this method as shown in Eq. (3.2) in Section 3.2.
Table 3. Ranking values using Chen et al.’s method

<table>
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<td>13</td>
<td>176.84</td>
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<td>956.04</td>
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<td>14</td>
<td>2357.84</td>
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<td>539.74</td>
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<tr>
<td>15</td>
<td>1400.68</td>
<td>30</td>
<td>100.69</td>
</tr>
</tbody>
</table>

Figure 2. Control charts with Chen et al.’s method

Table 4 shows ranking values using Qin and Liu’s ranking method for numerical example.

The ranking values are calculated for the control limits using Qin and Liu’s method. These values are obtained as 148.24, 236.78, and 57.82 for CL, UCL, and LCL, respectively. The control chart is depicted using Qin and Liu’s method in Figure 3.
Table 4. Ranking values using Qin and Liu’s method

<table>
<thead>
<tr>
<th>Sample No</th>
<th>Data</th>
<th>Sample No</th>
<th>Data</th>
</tr>
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<tbody>
<tr>
<td>1</td>
<td>182.07</td>
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<td>147.64</td>
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<tr>
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<td>128.53</td>
<td>21</td>
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<tr>
<td>7</td>
<td>40.82</td>
<td>22</td>
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<tr>
<td>8</td>
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<td>183.62</td>
</tr>
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<td>14</td>
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</tr>
<tr>
<td>15</td>
<td>227.54</td>
<td>30</td>
<td>55.54</td>
</tr>
</tbody>
</table>

Figure 3. Control charts with Qin and Liu’s method

The sample points of 3, 4, 7, 11, 14, 16, 17 and 30 are out of control, the remaining ones are in control when looking at the Table 4 and Figure 3.

Finally, the control charts generated with the crisp data are compared with the ranking methods’ results. Classical c-control chart is drawn with using Minitab and shown in Figure 4.
When looking Fig. 4, the sample points of 3, 4, 7, 8, 11, 14-17 and 30 are out of control, while the others are in control.

The last stage of study is the comparison of methods with each other. Table 5 is about comparison of all methods. Chen et al.’s ranking methods’ results are 90% similar to the results obtained from classical control chart while Qin and Liu’s ranking methods’ results are 93.9% similar to the results obtained from classical control chart.

Table 5. Comparisons of classical control chart with ranking methods

<table>
<thead>
<tr>
<th>Sample No</th>
<th>Classical Control Chart</th>
<th>Chen et al.’s Ranking Method</th>
<th>Qin&amp;Liu’s Ranking Method</th>
<th>Sample No</th>
<th>Classical Control Chart</th>
<th>Chen et al.’s Ranking Method</th>
<th>Qin&amp;Liu’s Ranking Method</th>
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6. Conclusions

This article differs from the studies appeared in accessible literature in regarding with the fuzzy control charts considering the ranking methods. Also, there are very limited studies on interval type-2 fuzzy sets in the accessible literature. From these two perspectives, this study, for the first time, seeks to obtain c-control charts using ranking methods for interval type-2 fuzzy sets.

In the study, firstly, interval type-2 fuzzy control limits are set. After that, control charts are created with two different ranking methods. Then, the results obtained from ranking methods’ are compared with the classical c-control chart, and the consistency of the results are investigated.

An important contribution of this study is that not only the control limits are calculated as interval type-2 fuzzy sets but also the ranking values. The other important point of this study is that it is the first study that tests ranking methods for fuzzy control charts.

For further studies, this research can be extended to include effects of different ranking methods for interval type-2 fuzzy sets.

References


Robust estimation in canonical correlation analysis for multivariate functional data

Miroslaw Krzyśko∗ and Łukasz Smaga†‡

Abstract
In this paper, the canonical correlation analysis for multivariate functional data is considered. The analysis is based on the basis functions representation of the data. The use of non-orthogonal bases is available in contrast to the approach given in the literature. The robust estimation methods of the covariance matrix are also studied in the multivariate functional canonical correlation analysis. Simulation studies and breakdown analysis suggest that the proposed methods may perform better than the classical estimator under non-normal models and in the presence of outlying observations.

Keywords: Basis functions representation, Canonical correlation, Functional data analysis, Multivariate analysis, Robust covariance estimation.

Mathematics Subject Classification (2010): 62H20, 62F35

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1. Introduction
Nowadays, quick and accurate measurement procedures are developed caused by the technological progress. This results in obtaining new types of (usually large) data. In many areas, the observations of random variables are taken over a continuous interval or in larger discretizations of such interval. In functional data analysis (FDA), such data observed longitudinally are expressed as smooth functions or curves, and then the information is drawn from the collection of functions or curves, called functional data. FDA has received considerable attention in such fields of applications as chemometrics, economics, environmental studies, image recognition, spectroscopy, and many others [30, 12, 2, 26].

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†Faculty of Mathematics and Computer Science, Adam Mickiewicz University, Umultowska 87, 61-614 Poznań, Poland, Email: ls@amu.edu.pl
‡Corresponding Author.
Comprehensive surveys about functional data analysis can be found in [31, 13, 19, 45]. Some new developments in FDA can be found, for instance, in [1, 4] and in recent special issue on Econometrics and Statistics [23]. In the literature, the following problems of analysis of univariate and multivariate functional data are of particular interest: analysis of variance [16, 17], canonical correlation analysis [15], classification [21, 11], cluster analysis [44], outlier detection [12], principal component analysis [3, 14], regression [6, 18, 34], repeated measurements analysis [28, 41], variable selection [29, 7].

This paper concerns the canonical correlation analysis for multivariate functional data (MFCCA). The aim of this analysis is to identify and quantify the relations between a $p$-dimensional stochastic process $X(t)$ and a $q$-dimensional stochastic process $Y(t)$. The associations between $X(s)$ and $Y(t)$ are measured by the correlations between linear combinations of both sets of processes. For the case $p = q = 1$, the so-called canonical correlation analysis for (univariate) functional data was developed, for example, in the monograph by Ramsay and Silverman [31]. Recently, Górecki et al. [15] extended this analysis to multivariate functional data $(p \geq 1$ and $q \geq 1)$. Their method is based on orthonormal basis functions representation of the data and the sample covariance matrix as estimator of unknown covariance matrix, which however may cause in poorer results of the canonical correlation analysis. Namely, some non-orthogonal basis may be more appropriate for particular type of data, and what is perhaps more important, the classical estimator of covariance matrix may be sensitive to non-normal data and outlying observations. In this paper, we extend the multivariate functional canonical correlation analysis proposed by Górecki et al. [15] to be available for using non-orthogonal bases and consider robust estimation of covariance matrix in this analysis. Simulation results suggest that the proposed modifications of MFCCA perform promisingly and may be alternatives to existing methods in practical applications.

The remainder of the paper is organized as follows. In Section 2, the canonical correlation analysis for multivariate functional data based on the basis functions representation of the data is considered. Section 3 raises the problem of estimation of unknown covariance matrices in canonical correlation analysis. A brief review of robust estimators of multivariate location and scatter is also given there. In Section 4, efficiency and robustness of the covariance matrix estimation methods in MFCCA are investigated by means of a simulation study and breakdown plots. Conclusions are provided in Section 5.

2. Canonical correlation analysis for multivariate functional data

In this Section, we formally present the canonical correlation analysis for multivariate functional data and show how to deal with it by using the basis functions representation of the data. The obtained results are more general than those of [15].

Let $X(s) = (X_1(s), \ldots, X_p(s))^\top$ and $Y(t) = (Y_1(t), \ldots, Y_q(t))^\top$ be random processes belonging to spaces $L^p(I_1)$ and $L^q(I_2)$, where $L^p(I)$ is a Hilbert space of $p$-dimensional vectors of square integrable functions on the set $I$ and $I_1 = [a, b]$ and $I_2 = [c, d]$, $a, b, c, d \in \mathbb{R}$. Moreover, without loss of generality, we can assume that $E(X(s)) = 0_p$ for $s \in I_1$ and $E(Y(t)) = 0_q$ for $t \in I_2$. By Górecki et al. [15], the functional canonical variables $U$ and $V$ for processes $X(s)$ and $Y(t)$ are defined in the following way:

$$U = \langle u, X \rangle = \int_{I_1} u^\top(s)X(s)ds, \quad V = \langle v, Y \rangle = \int_{I_2} v^\top(t)Y(t)dt,$$

where $u \in L^p(I_1)$ and $v \in L^q(I_2)$ are the vector weight functions chosen to maximize the coefficient

$$\rho_{U,V} = \frac{\text{Cov}(U, V)}{\sqrt{\text{Var}(U^{(N)})\text{Var}(V^{(N)})}}.$$
where \( \text{Var}(U^{(N)}) = \text{Var}(U) + \lambda \text{PEN}_2(u) \), \( \text{Var}(V^{(N)}) = \text{Var}(V) + \lambda \text{PEN}_2(v) \), \( \lambda > 0 \) and
\[
\text{PEN}_2(u) = \int_{t_1} \left( \frac{\partial^2 u(s)}{\partial s^2} \right)^T \frac{\partial^2 u(s)}{\partial s^2} ds, \quad \text{PEN}_2(v) = \int_{t_2} \left( \frac{\partial^2 v(t)}{\partial t^2} \right)^T \frac{\partial^2 v(t)}{\partial t^2} dt,
\]
subject to the constraint that
\[
(2.1) \quad \text{Var}(U^{(N)}) = \text{Var}(V^{(N)}) = 1.
\]
Without adding roughness penalty terms in constraints, functional canonical correlation analysis may not produce a meaningful result as described, for instance, in [15] and [31] (Chapter 11). Further, we define the \( k \)th functional canonical correlation \( \rho_k \) and the associated vector weight functions \( u_k(s) \) and \( v_k(t) \) as follows:
\[
\rho_k = \sup_{u \in L^2(I_1), v \in L^2(I_2)} \text{Cov}(\langle u, X \rangle, \langle v, Y \rangle) = \text{Cov}(\langle u_k, X \rangle, \langle v_k, Y \rangle),
\]
subject to the restrictions given in (2.1), and the \( k \)th pair of canonical variables \( (U_k, V_k) = (\langle u_k, X \rangle, \langle v_k, Y \rangle) \) is uncorrelated with the first \( k - 1 \) canonical variables. The above analysis is called (smoothed) canonical correlation analysis for multivariate functional data.

We deal with the canonical correlation analysis for multivariate functional data by assuming that the random processes \( X(s) \) and \( Y(t) \) belong to finite dimensional subspaces \( L^2(I_1), L^2(I_2) \) of \( L^2(I_1), L^2(I_2) \), respectively, and their components can be represented by a finite number of basis functions \( \{\varphi_i\} \) and \( \{\psi_j\} \), respectively, and their components can be represented by the same spaces
\[
X_i(s) = \sum_{i=1}^{K_1} \alpha_{il} \varphi_l(s), \quad Y_j(t) = \sum_{m=1}^{K_2} \beta_{jm} \psi_m(t)
\]
where \( s \in I_1, t \in I_2, i = 1, \ldots, p, j = 1, \ldots, q \) and \( \alpha_{il} \) and \( \beta_{jm} \) are random variables of mean zero and finite variance. Let
\[
\alpha = (\alpha_{11}, \ldots, \alpha_{1K_1}, \ldots, \alpha_{p1}, \ldots, \alpha_{pK_p})^T, \\
\beta = (\beta_{11}, \ldots, \beta_{1L_1}, \ldots, \beta_{q1}, \ldots, \beta_{qL_q})^T,
\]
\[
(2.2) \quad \Phi(s) = \begin{bmatrix} \varphi_1(s) & 0 & \cdots & 0 \\ 0 & \varphi_2(s) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \varphi_p(s) \end{bmatrix},
\]
\[
(2.3) \quad \Psi(t) = \begin{bmatrix} \psi_1(t) & 0 & \cdots & 0 \\ 0 & \psi_2(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \psi_q(t) \end{bmatrix},
\]
where \( \varphi_i(s) = (\varphi_{i1}(s), \ldots, \varphi_{iK_i}(s)) \) and \( \psi_j(t) = (\psi_{j1}(t), \ldots, \psi_{jL_j}(t)) \). Moreover, we assume that \( E(\alpha) = 0, E(\beta) = 0, \text{Cov}(\alpha) = \Sigma_1 > 0, \text{Cov}(\beta) = \Sigma_2 > 0 \) and \( \text{Cov}(\alpha, \beta) = \Sigma_1 \Sigma_2 \), where the matrices \( \Sigma_1, \Sigma_2 \) and \( \Sigma_1 \Sigma_2 \) are unknown parameters. Therefore, in matrix notation, the processes can be expressed as follows:
\[
(2.4) \quad X(s) = \Phi(s)\alpha, \quad Y(t) = \Psi(t)\beta.
\]
We may assume that the vector weight functions \( u(s) \) and \( v(t) \) belong to the same spaces as the processes \( X(s) \) and \( Y(t) \), respectively, i.e., \( u(s) = \Phi(s)\mu \) and \( v(t) = \Psi(t)\nu \), where
\[
\mu = (\mu_{11}, \ldots, \mu_{1K_1}, \ldots, \mu_{p1}, \ldots, \mu_{pK_p})^T, \quad \nu = (\nu_{11}, \ldots, \nu_{1L_1}, \ldots, \nu_{q1}, \ldots, \nu_{qL_q})^T.
\]
Then, we have

\[ U = <u, X> = \int_{I_1} u^T(s)X(s)ds = \int_{I_1} \mu^T \Phi^T(s)\Phi(s)\alpha ds = \mu^T J_\Phi \alpha, \]

where

\[ J_\Phi := \int_{I_1} \Phi^T(s)\Phi(s)ds = \text{diag} \left( J_{\varphi_1}, \ldots, J_{\varphi_p} \right) \]

is the block diagonal matrix of \( K_1 \times K_1 \) cross product matrices.

Analogously, we obtain \( V = v^T J_{\Psi} \beta \), where

\[ J_{\Psi} := \int_{I_2} \Psi^T(t)\Psi(t)dt = \text{diag} \left( J_{\psi_1}, \ldots, J_{\psi_q} \right) \]

and \( J_{\psi_j} = \int_{I_2} \psi_j(t)\psi_j(t)dt, j = 1, \ldots, q \). Therefore,

\[
\begin{align*}
E(U) &= \mu^T J_\Phi E(\alpha) = 0, \\
E(V) &= v^T J_{\Psi} E(\beta) = 0, \\
\text{Var}(U) &= \mu^T J_\Phi \text{Cov}(\alpha) J_\Phi \mu = \mu^T J_\Phi \Sigma_{11} J_\Phi \mu, \\
\text{Var}(V) &= v^T J_{\Psi} \text{Cov}(\beta) J_{\Psi} v = v^T J_{\Psi} \Sigma_{22} J_{\Psi} v, \\
\text{Cov}(U, V) &= \mu^T J_\Phi \text{Cov}(\alpha, \beta) J_{\Psi} v = \mu^T J_\Phi \Sigma_{12} J_{\Psi} v.
\end{align*}
\]

By Górecki et al. [15], we conclude that

\[
\text{Var}(U^{(N)}) = \mu^T (J_\Phi \Sigma_{11} J_\Phi + \lambda R_\Phi) \mu, \quad \text{Var}(V^{(N)}) = v^T (J_{\Psi} \Sigma_{22} J_{\Psi} + \lambda R_\Psi) v,
\]

where

\[
R_\Phi = \begin{bmatrix}
\int_{I_1} \frac{\partial^2 \varphi_1(s)}{\partial s^2} \frac{\partial^2 \varphi_1(s)}{\partial s^2} ds & \ldots & 0 \\
0 & \int_{I_1} \frac{\partial^2 \varphi_2(s)}{\partial s^2} \frac{\partial^2 \varphi_1(s)}{\partial s^2} ds & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \int_{I_1} \frac{\partial^2 \varphi_q(s)}{\partial s^2} \frac{\partial^2 \varphi_1(s)}{\partial s^2} ds
\end{bmatrix}
\]

and

\[
R_\Psi = \begin{bmatrix}
\int_{I_2} \frac{\partial^2 \psi_1(t)}{\partial t^2} \frac{\partial^2 \psi_1(t)}{\partial t^2} dt & \ldots & 0 \\
0 & \int_{I_2} \frac{\partial^2 \psi_2(t)}{\partial t^2} \frac{\partial^2 \psi_1(t)}{\partial t^2} dt & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \int_{I_2} \frac{\partial^2 \psi_q(t)}{\partial t^2} \frac{\partial^2 \psi_1(t)}{\partial t^2} dt
\end{bmatrix}.
\]

We have thus proved the following theorem.

**2.1. Theorem.** Under the above assumptions and notation, we have

\[
U_k = \mu_k^T J_\Phi \alpha, \quad V_k = v_k^T J_{\Psi} \beta, \quad u_k(s) = \Phi(s)\mu_k, \quad v_k(t) = \Psi(t)\nu_k, \quad s \in I_1, \ t \in I_2,
\]

and

\[
\rho_k = \sup_{\mu_k \in \mathbb{R}^K, \nu_k \in \mathbb{R}^L} \mu_k^T J_\Phi \Sigma_{12} J_{\Psi} v = \mu_k^T J_\Phi \Sigma_{12} J_{\Psi} v, \quad k = 1, \ldots, \min\{K, L\},
\]
where $K = K_1 + \cdots + K_p$, $L = L_1 + \cdots + L_q$, subject to the restrictions

$$\mu_k^\top (J_\Phi \Sigma_{11} J_\Phi + \lambda R_\Phi) \mu_k = 1$$

and

$$v_k^\top (J_\Psi \Sigma_{22} J_\Psi + \lambda R_\Psi) v_k = 1.$$

By Theorem 2.1, MFCCA reduces to the canonical correlation analysis for $K$-dimensional random vector $\alpha_*$ and $L$-dimensional random vector $\beta_*$ with $E(\alpha_*) = 0_K$, $E(\beta_*) = 0_L$, $\text{Cov}(\alpha_*) = J_\Phi \Sigma_{11} J_\Phi + \lambda R_\Phi$, $\text{Cov}(\beta_*) = J_\Psi \Sigma_{22} J_\Psi + \lambda R_\Psi$ and $\text{Cov}(\alpha_*, \beta_*) = J_\Phi \Sigma_{12} J_\Psi$. It is well known that the CCA optimization problem has a fairly simple solution. Namely, the vectors $\mu_k$ and $v_k$ are the eigenvectors corresponding to the eigenvalues $\rho_1^2 \geq \cdots \geq \rho_{\min(K,L)}^2 > 0$ of the matrices

$$(J_\Phi \Sigma_{11} J_\Phi + \lambda R_\Phi)^{-1} (J_\Phi \Sigma_{12} J_\Psi)(J_\Psi \Sigma_{22} J_\Psi + \lambda R_\Psi)^{-1} (J_\Psi \Sigma_{12} J_\Phi),$$

$$(J_\Psi \Sigma_{22} J_\Psi + \lambda R_\Psi)^{-1} (J_\Psi \Sigma_{12} J_\Phi)(J_\Phi \Sigma_{11} J_\Phi + \lambda R_\Phi)^{-1} (J_\Phi \Sigma_{12} J_\Psi),$$

respectively. In practice, we have to estimate the unknown parameters, i.e., the matrices $\Sigma_{11}$, $\Sigma_{22}$ and $\Sigma_{12}$. This problem is discussed in the next Section. The smoothing parameter $\lambda$ can be chosen subjectively or by using an automatic procedure as, for example, cross-validation one analogous to that considered by Ramsay and Silverman [31] (Chapter 11) in the canonical correlation analysis for univariate functional data.

3. Robust estimation in MFCCA

In this Section, we consider the estimation problem of the unknown matrices $\Sigma_{11}$, $\Sigma_{22}$ and $\Sigma_{12}$ to make the theory of Section 2 applicable. Robust estimators are discussed as competitors to the classical estimator.

Assume that we have $n$ independent realizations $x_1(s), \ldots, x_n(s)$ and $y_1(t), \ldots, y_n(t)$ of random processes $X(s)$ and $Y(t)$, $s \in I_1$, $t \in I_2$. They are represented as in (2.4), i.e., $x_r(s) = \Phi(s) \alpha_r$ and $y_r(t) = \Psi(t) \beta_r$, $r = 1, \ldots, n$. The vectors $\alpha_r$ and $\beta_r$ can be estimated by the least squares method or by the roughness penalty approach (see [31] Chapter 5). Let $\hat{\alpha}_r$ and $\hat{\beta}_r$ denote the estimates of $\alpha_r$ and $\beta_r$, respectively. The optimum numbers of basis functions $K_j$ and $L_j$ ($i = 1, \ldots, p$, $j = 1, \ldots, q$) can be selected by using Akaiake or Bayesian information criterion (see, for example, [15]). The Bayesian information criterion is preferred, since it measures goodness of fit better than Akaiake information criterion [40].

The unknown matrices $\Sigma_{11}$, $\Sigma_{22}$ and $\Sigma_{12}$ are estimated based on the vectors $\hat{\alpha}_r$ and $\hat{\beta}_r$, $r = 1, \ldots, n$. More precisely, we estimate the joint covariance matrix

$$\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$$

of $(\alpha_r^\top, \beta_r^\top)^\top$ by using the vectors $(\hat{\alpha}_r^\top, \hat{\beta}_r^\top)^\top$, $r = 1, \ldots, n$ as observations. Górecki et al. [15] used the sample covariance matrix to estimate $\Sigma$. However, it is known that this classical estimator is sensitive to non-normal data and outlying observations. The poor behavior of this estimator may result in worse performance of MFCCA based on it. For this reason, we propose to use robust estimators of the matrix $\Sigma$, which seems to result in better performance of MFCCA than using classical one, as we will see in Section 4. In that Section, we compare the sample covariance matrix with commonly used robust estimates of covariance matrix, which are briefly reviewed in the remainder of the present Section. Other estimation methods of $\Sigma$ can also be applied (see, for example, [42, 46]).

Let $\{z_1, \ldots, z_n\}$ be the data set in $\mathbb{R}^d$. Hence $n$ stands for the number of objects and $d$ for the number of variables. First, we consider the minimum covariance determinant
(MCD) estimator, which is very popular in the literature. It was introduced by Rousseeuw [35]. The MCD estimates for multivariate location and scatter are the mean and a multiple of the sample covariance matrix of $h$ observations $z_{i1}, \ldots, z_{in}$, for which the determinant of the covariance matrix is minimum, i.e.,

$$L_{MCD} = \frac{1}{h} \sum_{j=1}^{h} z_{ij}, \quad S_{MCD} = \frac{c_1 c_2}{h-1} \sum_{j=1}^{h} (z_{ij} - L_{MCD})(z_{ij} - L_{MCD})^\top.$$ 

The consistency correction factor $c_1$ and the small sample correction factor $c_2$ are selected for consistency at the multivariate normal model and unbiasedness at small samples of $S_{MCD}$. With $h = [(n + d + 1)/2]$, the breakdown point of the MCD estimator is high, and hence such a choice of $h$ is recommended. The computation of the MCD estimate by the naive algorithm is possible in sensible time only for very small data sets. However, fact computing algorithms of the estimators are available as that by Rousseeuw and Van Driessen [36], which is usually used in practice. Let $(L_1, S_1)$ be an approximation of the MCD estimator and let $d_1, \ldots, d_n$ denote the distances of the observations $z_1, \ldots, z_n$ related to this approximation, i.e.,

$$d_r = \sqrt{(z_r - L_1)^\top S_1^{-1}(z_r - L_1)}, \quad r = 1, \ldots, n.$$ 

Then, the C-step of this algorithm moves from $(L_1, S_1)$ to the next approximation $(L_2, S_2)$ by computing it for those $h$ data points, which have smallest distances. By this step, lower determinant $\det(S_2)$ may be obtained. Here, “C” stands for “concentration”, because we look for the $h$ observations with smallest distances, and $S_2$ is more concentrated and has possibly lower determinant than $S_1$. The estimators $L_{MCD}$ and $S_{MCD}$ are not very efficient at normal models. This is especially evident, when $h$ is chosen to achieve maximal breakdown point. This low efficiency drawback may be overcome by using reweighted MCD estimators (see, for instance, [8]). Weight $w_r$ assigned to observation $z_r$ is defined as

$$w_r = \frac{1}{w} I_{\left(-\infty, \lambda^2_{d,0.975}\right)} \left( (z_r - L_{MCD})^\top S_{MCD}^{-1}(z_r - L_{MCD}) \right)$$

related to the raw estimators $L_{MCD}$ and $S_{MCD}$, where $I_A$ stands for the usual indicator function on the set $A$. Then, the reweighted MCD estimators are as follows:

$$L_{MCD}^R = \frac{1}{w} \sum_{r=1}^{n} w_r z_r, \quad S_{MCD}^R = \frac{c_1 c_2}{w-1} \sum_{r=1}^{n} w_r (z_r - L_{MCD}^R)(z_r - L_{MCD}^R)^\top,$$

where $w = w_1 + \cdots + w_n$.

Now, we describe the S-estimators introduced by Davies [9] and further investigated by Lopuhaä [24]. In fact, Davies [9] extended an idea of S-estimators by Rousseeuw and Yohai [37] in regression. The S-estimators of location and scatter are the vector $L$ and the positive definite symmetric matrix $S$ minimizing the determinant $\det(S)$, subject to

$$\frac{1}{n} \sum_{r=1}^{n} \rho(d_r) \leq b_0,$$

where the function $\rho$ is non-negative, symmetric and continuously differentiable and strictly increasing on $[0, c_0]$ with $\rho(0) = 0$ and constant on $[c_0, \infty)$ for some $c_0 > 0$, and

$$d_r = \sqrt{(z_r - L)^\top S^{-1}(z_r - L)}, \quad r = 1, \ldots, n$$

and $b_0 < \sup \rho$. The S-estimators are asymptotically normal and $\sqrt{n}$-consistent. Moreover, these estimates can have a very high breakdown point $[(n - d + 1)/2]/n$ and be highly efficient at normal models. However, a high breakdown point and a high efficiency at the normal models may be not simultaneously attained by the S-estimators. Lopuhaä
[25] and Davies [10] proposed some modifications of the S-estimators, which can overcome this drawback. The S-estimates may be computed, for example, by a fast algorithm of [38] and that similar to the one proposed by Salibian-Barrera and Yohai [39] for the regression setting (see [42]).

Finally, we consider multivariate M-estimators defined by Maronna [27], who extended the idea of the univariate M-estimators by Huber [20]. The M-estimators are defined as the vector \( \mathbf{L} \) and the positive definite symmetric matrix \( \mathbf{S} \), which are solutions of the following equations:

\[
\frac{1}{n} \sum_{r=1}^{n} u_1(d_r)(\mathbf{z}_r - \mathbf{L}) = \mathbf{0}, \quad \frac{1}{n} \sum_{r=1}^{n} u_2(d_r^2)(\mathbf{z}_r - \mathbf{L})(\mathbf{z}_r - \mathbf{L})^\top = \mathbf{S},
\]

where \( u_1 \) and \( u_2 \) are weight functions satisfying certain conditions and \( d_r \) are given in (3.1). Unfortunately, the M-estimators may have relatively low breakdown points. However, they can be quite efficient at normal and other models. Modified M-estimators, which perform better than the standards ones, are proposed by Tyler [43] and Kent and Tyler [22]. Lopuhaä [24] shows that M-estimators have a close connection to S-estimators.

In the next Section, the performance of considered estimators of the covariance matrix in the canonical correlation analysis for multivariate functional data is studied under finite samples.

4. Simulation study

In this Section, the methods of estimation of covariance matrix \( \mathbf{\Sigma} \) for MFCCA considered in Section 3 are compared by means of a simulation study.

4.1. Experimental setup. The numbers of observations are \( n = 100, 500, 1000 \) and the number of simulations within each setup was \( nr = 500 \). The dimensions of the processes \( \mathbf{X}(s) \) and \( \mathbf{Y}(t) \) are \( p = 2 \) and \( q = 2, 4 \). The functional observations corresponding to these processes are represented by their values in an equally spaced grid of 50 points \( s_1 = t_1, \ldots, s_{50} = t_{50} \) in \( I_1 = I_2 = [0, 1] \), which are generated in the following way:

\[
\begin{bmatrix}
\mathbf{x}_r(s_u) \\
\mathbf{y}_r(t_u)
\end{bmatrix}
= \begin{bmatrix}
\mathbf{\Phi}(s_u) & 0 \\
0 & \mathbf{\Psi}(t_u)
\end{bmatrix}
\begin{bmatrix}
\boldsymbol{\alpha}_r \\
\boldsymbol{\beta}_r
\end{bmatrix}
+ \varepsilon_{ru},
\]

where \( r = 1, \ldots, n, u = 1, \ldots, 50 \), the matrices \( \mathbf{\Phi}(s) \) and \( \mathbf{\Psi}(t) \) are given in (2.2)-(2.3) and contain the Fourier basis functions only and \( K_1 = 5, i = 1, \ldots, p, L_j = 5, j = 1, \ldots, q \). \( (\alpha_r^\top, \beta_r^\top) \) are \( 5(p + q) \)-dimensional random vectors, and \( \varepsilon_{ru} = (\varepsilon_{u1}, \ldots, \varepsilon_{u(p+q)}) \) are the measurement errors such that \( \varepsilon_{ru} \sim N(0,0.025^2) \) and \( a_{ru} \) is the range of the \( u \)-th row of the following matrix:

\[
\begin{bmatrix}
\mathbf{\Phi}(s_1)\alpha_r & \ldots & \mathbf{\Phi}(s_{50})\alpha_r \\
\mathbf{\Psi}(t_1)\beta_r & \ldots & \mathbf{\Psi}(t_{50})\beta_r
\end{bmatrix}.
\]

We consider the covariance matrices \( \mathbf{\Sigma} \) with \( \mathbf{\Sigma}_{11} = \mathbf{I}_{5p}, \mathbf{\Sigma}_{22} = \mathbf{I}_{5q} \), and

\[
\mathbf{\Sigma}_{12} = \begin{cases}
\text{diag}(0.9, 0.8, 0.7, 0.6, 0.55, 0.5, 0.4, 0.3, 0.2, 0.1), & \text{for } q = 2,
\text{diag}(0.9, 0.8, 0.7, 0.6, 0.55, 0.5, 0.4, 0.3, 0.2, 0.1), & \text{for } q = 4.
\end{cases}
\]

Similarly to Branco et al. [5], we consider the following four cases of generating random vectors \( (\alpha_r^\top, \beta_r^\top) \), \( r = 1, \ldots, n \):

- **NOR** — normal distribution: \( (\alpha_r^\top, \beta_r^\top) \sim N_{5(p+q)}(0_{5(p+q)}, \mathbf{\Sigma}) \),
- **T3** — multivariate \( t_3 \)-distribution with scatter parameter \( \mathbf{\Sigma} \): \( (\alpha_r^\top, \beta_r^\top) \sim T_r/\sqrt{C_r/3} \), where \( T_r \sim N_{5(p+q)}(0_{5(p+q)}, \mathbf{\Sigma}) \) and \( C_r \sim \chi^2_3 \),
- **SCN** — symmetric contamination: \( (\alpha_r^\top, \beta_r^\top) \sim N_{5(p+q)}(0_{5(p+q)}, \mathbf{\Sigma}) \) for \( r = 1, \ldots, 0.95n \), and \( (\alpha_r^\top, \beta_r^\top) \sim N_{5(p+q)}(0_{5(p+q)}, 9\mathbf{\Sigma}) \) for \( r = 0.96n, \ldots, n \).
ACN – asymmetric contamination: $(\alpha^T_r, \beta^T_r) \overset{\sim}{\rightarrow} N_5(p+q)(0_{5(p+q)}, \Sigma)$ for $r = 1, \ldots, 0.95n$, while for $r = 0.96n, \ldots, n$, $(\alpha^T_r, \beta^T_r) \overset{\sim}{\rightarrow} 515(p+q), 7.515(p+q)$ when $q = 2, 4$, respectively. Therefore, we consider the classical (NOR) and non-normal (T3) models as well as the models with different outlying observations (SCN, ACN).

We set $\lambda = 0$. Then, Theorem 2.1 shows that the canonical correlations for the random processes $X(s)$ and $Y(t)$ generated as above are determined by the matrix $\Sigma$, i.e., they are equal to $0.9, 0.8, 0.7, 0.6, 0.55, 0.5, 0.4, 0.3, 0.2, 0.1$. Since the Fourier basis is used to generate the data, the MFCCA is performed with the B-spline basis with $K_i = 5$, $i = 1, \ldots, p$, $L_j = 5$, $j = 1, \ldots, q$ (Other basis does not change the canonical correlations.). To compare the true canonical correlations with their estimators $\hat{\rho}_{kl}$, $k = 1, \ldots, 5p$, $l = 1, \ldots, nr$, obtained in simulation replications, we compute the following measure of mean squared error (MSE) as in [5]:

\begin{equation}
\text{MSE}(\hat{\rho}_k) = \frac{1}{nr} \sum_{l=1}^{nr} (\phi(\hat{\rho}_{kl}) - \phi(\rho_k))^2,
\end{equation}

where $\phi(x) = \tanh^{-1}(x)$ is the Fisher transformation.

To conduct simulation experiments, the R program was used [33]. The cross product matrices in $J_\Phi$ and $J_\Psi$ and the roughness penalty matrices in $R_\Phi$ and $R_\Psi$ can be approximated by using the functions `inprod()` and `getbasispenalty()` from the `fda` package [32]. The functions `create.fourier.basis()` and `eval. basis()` (resp. `create.bspline.basis()` and `Data2fd()`) available in this package were used to compute the values of the Fourier basis functions (resp. estimate the coefficients of the basis functions representation of the data). To estimate the covariance matrix $\Sigma$ by the MCD, S- and M-estimators, the functions `CovMcd()`, `CovSest()` and `CovMest()` from the `rrcov` package [42] were applied. The default values of these functions were used.
Figure 2. Mean squared error for canonical correlations in T3 case \((p = 2, q = 4, nr = 500, SCM – sample covariance matrix)\).

Figure 3. Mean squared error for canonical correlations in SCN case \((p = 2, q = 4, nr = 500, SCM – sample covariance matrix)\).

4.2. Simulation results. The simulation results are depicted in Figures 1-4. They show the mean squared error for canonical correlations (4.1) obtained by using the sample
covariance matrix (SCM) and the MCD, M-, S-estimators for estimating the covariance matrix, when \( p = 2 \) and \( q = 4 \). For the dimensions \( p = q = 2 \), similar results were obtained, and therefore they are not presented. We only mention that the mean squared errors for \( p = q = 2 \) are usually smaller than those for \( p = 2 \) and \( q = 4 \).

In general, the mean squared errors obtained in ACN case are greater than those for T3 case, which are greater than the MSEs for SCN case. As expected, the smallest
Figure 5. Breakdown plot: Mean squared error for canonical correlations and for the percentage of contamination, ranging from 1 to 10%, in ACN case ($p = q = 2$, $n = 500$, $nr = 500$, SCM – sample covariance matrix).
Figure 6. Breakdown plot: Mean squared error for canonical correlations and for the percentage of contamination, ranging from 11 to 20%, in ACN case ($p = q = 2$, $n = 500$, $nr = 500$, SCM - sample covariance matrix).
mean squared errors are obtained in the normal model (NOR). We also observe that the MSEs for all estimation methods decrease as the number of observations increases. In cases T3 and ACN, the estimates of lower order canonical correlations lead usually to the largest MSEs. In the other cases, the largest MSE’s may be also obtained for higher order canonical correlations.

In the normal model (NOR), the sample covariance matrix is the most precise. However, the S-estimator gives only slightly worse results than the SCM. The other robust estimators perform very similarly. When deviating from the normal model (the cases T3, SCN, ACN), the sample covariance matrix is overcome by at least one of the robust methods. For \( n = 100 \), the smallest mean squared errors are obtained by using the S-estimator, and the classical method works similarly to or better than the MCD and M-estimates. For greater number of observations (\( n = 500,1000 \)), all robust estimators significantly outperform the sample covariance matrix. The robust methods give similar results, but the S-estimator seems to perform slightly better than the other ones.

Finally, we study the robustness of the covariance matrix estimators in MFCCA by breakdown analysis. Namely, we investigate the sensitivity of considered estimators to increasing amounts of contamination (proportion of atypical points) in the data. For this purpose, we carried out additional simulation study in ACN case, when \( p = q = 2 \), \( n = 500 \) and the percentage of contamination ranges from 1 to 20%. The resulting mean squared errors for all methods are presented in Figures 5-6. We observe that the MSEs of the sample covariance matrix rapidly increase in presence of contamination. The classical method performs poorly even when the percentage of contamination is very small. The S- and MCD estimators are more stable. Their MSEs remain small up to 6% and 7% of contamination, respectively, but then they also go up, especially for the first canonical correlation. In general, the M-estimator performs best and seems to be very stable up to about 16% of contamination.

5. Conclusions

We have considered the canonical correlation analysis for multivariate functional data based on the basis functions representation of the data. In contrast to Górecki et al. [15], we have developed this analysis in such a way to be available also for using non-orthogonal bases. Moreover, the robust estimation methods of the covariance structure were investigated to increase the performance of the multivariate functional canonical correlation analysis under non-normal models and in the presence of outlying observations. Their results indicate that the new estimation methods perform usually better in the presence of outliers and are more robust to deviations from the normal model than the sample covariance matrix, which is the classical estimator.

References


Exact distribution of Hadi’s ($H^2$) influence measure and identification of potential outliers

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Abstract

This paper proposed an exact distribution of Hadi’s influence measure that can be used to evaluate the potential outliers in a linear multiple regression analysis. The authors explored a relationship between the measure in terms of two independent F-ratios and they derived density function of the measure in a complicated series expression form with Gauss hyper-geometric function. Moreover, the first two moments of the distribution are derived in terms of Beta function and the authors computed the critical points of Hadi’s measures at 5% and 1% significance level for different sample sizes and varying no. of predictors. Finally, the numerical example shows the identification of the potential outliers and the results extracted from the proposed approaches are more scientific, systematic and its exactness outperforms the Hadi’s traditional approach.

Keywords: Hadi’s measure, Potential outliers, Series expression form, Gauss hyper-geometric function, Moments, Beta function, Critical points.

Mathematics Subject Classification (2010): 62H10

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1. Introduction and Related work

While fitting a regression model it is well understood that not all observations in a dataset play an equal role. Some observations have more impact than the others. Those observations which significantly influence the results of a regression analysis are called influential observations. Andrews and Pregibon [2] highlighted the need to find the outliers that matter. This means not all outliers need to be harmful in that they have an undue influence on the estimation of the parameters in the regression model. Hence, examining the residuals alone might not lead us to the detection of aberrant or unusual observations. Thus, other ways for finding influential observations are needed. Hoaglin and Welsch [9] discussed the importance of the projection matrix in linear regression, where the projection matrix is the matrix that projects onto the regression space. They argued that the diagonal elements of the projection matrix are important ingredients in influence analysis. The diagonal elements are referred to as leverages since they can be thought of as the amount of leverage concerning the response value on the corresponding predicted response value. Perhaps the most well-known influence measure was proposed by Cook [6], referred as Cook’s distance. It is an influence measure used for assessing the influence of the observations on the estimated parameter vector in the linear regression model. It is widely used by practitioners for detecting influential observations. There are other influence measures to use in the linear regression analysis for assessing the influence of the observations on various results of the regression analysis. Such as, Andrews and Pregibon [2] derived a measure of the influence of an observation on the estimated parameters. This measure the AP statistic is based on the change in volume of confidence ellipsoids with and without a particular observation. Moreover, Belsley et al. [3] suggested an influence measure for assessing the influence of an observation on the variance of the estimated parameters in the linear regression model, known as COVRATIO. Besides the influence measures mentioned here there exist much more, see e.g. Chatterjee and Hadi [5] and Hadi [8] for excellent overviews of influence measures. Graphical investigation of data is a powerful tool in exploratory analysis. It can be used to examine relationships between variables and discover observations deviating from other. Hence, influential observations can also be detected using graphical tools. Mosteller and Tukey [11] introduced the added variable plot, which is used for graphically detecting observations that have a large influence on the parameter estimates. For details concerning the added variable plot, such as construction and properties, see, Belsley et al. [3], where the plot is referred to as the partial regression leverage plot, and Cook and Weisberg [7]. Other results on graphical tools in influence analysis are provided by Johnson and McCulloch [10]. It is important to note that the graphical tools used in influence analysis are not conclusive, but rather suggestive. From the previous discussions, we can see that the 1970’s and the 1980’s were the decades when most research results on influence analysis in linear regression came to see the light. However, influence analysis in linear regression is still an active research area. Nurunnabi et al. [12] proposed a modification of Cook’s distance. This modification enables the identification of multiple influential observations. Furthermore, Beyaztas and Alin [4] used a combined Bootstrap and Jackknife algorithm to detect influential observations. In applied data analysis, there is an increasing availability of data sets containing a large number of variables. When such data is in the hands of the researcher sparse regression can be implemented, which is another field of research active today. In sparse regression, a penalty term on the regression parameters is added which shrinks the number of parameters. Common approaches to estimate the parameter in the sparse regression are, however, sensitive to influential observations and new methods are needed. Alfons et al. [1] and Park et al. [13] proposed robust
estimation methods, where influential observations are not harmful to the resulting estimates. Considering the above reviews, the authors proposed the exact distribution of Hadi’s influence function \( (H^2) \) which exactly identifies the influential data points and it is discussed in the subsequent sections.

2. Relationship between Hadi’s \( (H^2) \) influence measure and F-ratios

The multiple linear regression models with random error is given by

\[
Y = X\beta + e
\]

where \( Y \) is the matrix of the dependent variable, \( \beta \) is the vector of beta coefficients or partial regression coefficients and \( e \) is the residual followed normal distribution \( N(0, \sigma^2 I_n) \). From (2.1), statisticians concentrate and give importance to the error diagnostics such as outlier detection, identification of leverage points and evaluation of influential observations. Several error diagnostics techniques exist in the literature proposed by statisticians, but Hadi’s \( (H^2) \) influence measure is the interesting technique based on the simple fact that potentially influential observations are outliers in X-space, Y-space or both. The general form of the Hadi’s influence measure of the \( i^{th} \) observation is given by

\[
H^2_i = (p+1) \frac{\hat{c}_{ei}^2}{(1-h_{ii})} + \frac{h_{ii}}{1-h_{ii}}
\]

Where \( \hat{c}_{ei} \) is the vector of squared estimated residuals, \( p \) is the no.of predictors, \( \hat{e} \) is the sum of the squared estimated residuals and \( h_{ii} \) is the hat values of \( i^{th} \) observation or diagonal elements of the hat matrix \( H = X (X'X)^{-1} X' \). This diagnostic measure is the sum of two components each of which has an interpretation. A large value for the first term indicates that the model has a poor fit (a large prediction error) and a large value for the second term indicates the presence of an outlier in the X-space. Similarly, Hadi pointed this diagnostic measure possess several desirable properties and it is also supplemented by a graphical display which shows the source of influence. He suggested, \( (H^2) \) for observations more than a cut-off of \( E (H^2_i) + c \sqrt{V (H^2_i)} \) which is treated as a potential outlier. Hadi’s influence measure can also be written in an alternative form as

\[
H^2_i = \frac{p+1}{(1-h_{ii})} \left( \frac{\hat{c}^2}{\hat{c}_i} / \hat{c}_i - 1 \right) + \frac{h_{ii}}{1-h_{ii}}
\]

It is known the unbiased estimate of the true error variance is \( s^2 = \frac{\hat{c}^2}{n-p-1} \) and substitute \( \hat{e} = s^2 (n-p-1) \) in (2.3) to get

\[
H^2_i = \frac{p=1}{n-p-1} \left( \frac{\hat{c}^2}{\hat{c}_i} / \hat{c}_i - 1 \right) + \frac{h_{ii}}{1-h_{ii}}
\]

Rewrite (2.4), in terms of the internally studentized residual \( (r_i) \) which is equal to \( \hat{e}_i / s \sqrt{1-h_{ii}} \) and it is given as

\[
H^2_i = \frac{p+1}{((n-p-1)/r_i^2) - (1-h_{ii})} + \frac{h_{ii}}{1-h_{ii}}
\]
Though Hadi’s influence measure is scientific and the yardstick used to detect the influential observation is not scientific and the authors believe it is based on the rule of thumb approach. Because \((H_i^2)\) is non-normally distributed and the usage of mean and variance in the cut-off \(E(H_i^2) + c\sqrt{\text{Var}(H_i^2)}\) is meaningless and illogic. Secondly, when using the cut-off, it is not recommended by the author to use a specific and fixed value for the constant \(c\). Finally, the usage of plots and graphs to identify the potential outliers and sources of influence leads to imprecision and ambiguity. In order to overcome this rule of thumb approach of identifying the influential observations, authors proposed the exact distribution for Hadi’s influence measure and established a scientific yardstick to scrutinize the exact influential observations. For this, authors utilize the relationship among the Hadi’s \((H_i^2)\), internally studentized residual \((r_i)\) and hat elements \((h_{ii})\). The terms \((r_i)\) and \((h_{ii})\) are independent because the computation of \((r_i)\) involves the error term \(e_i \sim N(0, \sigma_i^2)\) and \(h_{ii}\) values involves the set of predictors \((H = X'X)^{-1}X'\). Therefore, from the property of least squares \(E(eX) = 0\), so \(r_i\) and \(h_{ii}\) are also uncorrelated and independent. Using this assumption, authors first determine the distribution of \((r_i)\) based on the relationship given by Weisberg (1980) as

\[
(2.6) \quad t_i = r_i \sqrt{\frac{n-p-2}{(n-p-1)} - r_i^2} \sim t_{(n-p-2)}
\]

From (2.6) it follows student’s t-distribution with \((n-p-2)\) degrees of freedom and it can be written in terms of the F-ratio as

\[
(2.7) \quad r_i^2 = \frac{(n-p-1)t_i^2}{(n-p-2) + t_i^2}
\]

From (2.7), if \(t_i\) follows student’s t-distribution with \((n-p-2)\) degrees of freedom, then \(t_i^2\) follows \(F_{(1,n-1)}\) distribution with \((1,n-p-2)\) degrees of freedom. Similarly, authors identify the distribution of \(h_{ii}\) based on the relationship proposed by Belsey et al [3] and they showed when the set of predictors is multivariate normal with \((\mu_X, \Sigma_X)\), then

\[
(2.8) \quad \frac{(n-p)(h_{ii} - 1/n)}{(p-1)(1-h_{ii})} \sim F_{(p-1,n-p)}
\]

From (2.8) it follows F-distribution with \((p-1, n-p)\) degrees of freedom and it can be written in an alternative form as

\[
(2.9) \quad E(H_i^2) = (p+1)(\phi_1(p,n)) + \frac{n}{n-1}(\phi_2(p,n)) - 1
\]

In order to derive the exact distribution of \((H_i^2)\), substitute (2.7) and (2.9) in (2.5), authors get the Hadi’s \((H_i^2)\) measure in terms of the two independent F-ratios with \((1,n-p-2)\) and \((p-1,n-p)\) degrees of freedom respectively and the relationship is given as

\[
(2.10) \quad H_i^2 = \frac{p+1}{\frac{(n-p-2)+F_{(1,n-p-2)}}{F_{(1,n-p-2)}} - \frac{(n-1)/n}{1 + ((p-1)/(n-p))F_{(p-1,n-p)}} + \frac{((p-1)/(n-p))F_{(p-1,n-p)} + 1/n}{(n-1)/n}}
\]
\begin{align}
(2.11) & \quad H_i^2 = \frac{p + 1}{1 + (n - p - 2) F_{(n-p-2,1)}} - \frac{(n-1)/n}{1 + ((p-1)/(n-p)) F_{(p-1,n-p)}} \\
& \quad + \frac{1 + ((p-1)/(n-p)) F_{(p-1,n-p)}}{(n-1)/n} - (n-1)/n
\end{align}

From \(2.11\), it can be further simplified and \(H_i^2\) is expressed in terms of two independent beta variables namely \(\theta_{1i}\) and \(\theta_{2i}\) of the first kind by using the following facts

\begin{align}
(2.12) & \quad \frac{1}{1 + (n - p - 2) F_{(n-p-2,1)}} = \theta_{1i} \sim \beta_1 \left(\frac{1}{2}, \frac{n - p - 2}{2}\right) \\
(2.13) & \quad \frac{1}{1 + ((p-1)/(n-p)) F_{(p-1,n-p)}} = \theta_{2i} \sim \beta_1 \left(\frac{n - p - 1}{2}, \frac{p - 1}{2}\right)
\end{align}

Then, without loss of generality \(2.11\) can be written as

\begin{align}
(2.14) & \quad H_i^2 = \frac{p + 1}{(1/\theta_{1i}) - ((n-1)/n) \theta_{2i}} + \left(\frac{1/\theta_{1i} - (n-1)/n}{n-1}/n\right)
\end{align}

\begin{align}
(2.15) & \quad H_i^2 = \frac{(p + 1) \theta_{1i}}{1 - ((n-1)/n) \theta_{1i} \theta_{2i}} + (n/n - 1) (1/\theta_{2i}) - 1
\end{align}

From \(2.15\), the authors showed the Hadi’s influence measure in terms of \(\theta_{1i} \sim \beta_1 \left(\frac{1}{2}, \frac{n - p - 2}{2}\right)\) and \(\theta_{2i} \sim \beta_1 \left(\frac{n - p - 1}{2}, \frac{p - 1}{2}\right)\) which followed beta distribution of first kind with two shape parameters \(p\) and \(n\) respectively. To avoid complexity further, the relationship from \(2.15\) modified as

\begin{align}
(2.16) & \quad \frac{n - 1}{n} (1 + H_i^2) = \left(\frac{1 + p ((n-1)/n) \theta_{1i} \theta_{2i}}{1 - ((n-1)/n) \theta_{1i} \theta_{2i}}\right) (1/\theta_{2i}) = \psi_i
\end{align}

Based on the identified relationship from \(2.16\), the authors derived the distribution of the Hadi’s \(H_i^2\) and it is discussed in the next section.

\section*{3. Exact Distribution of Hadi’s \(H_i^2\)}

Using the technique of two-dimensional Jacobian of transformation, the joint probability density function of the two Beta variables of kind-1 namely \(\theta_{1i}, \theta_{2i}\) were transformed into density function of \(\psi_i\) and it is given as

\begin{align}
(3.1) & \quad f(\psi_i, u_i) = f(\theta_{1i}, \theta_{2i}) |J|
\end{align}

From \(3.1\), It is known that \(\theta_{1i}, \theta_{2i}\) are independent then rewrite \(3.1\) as

\begin{align}
(3.2) & \quad f(\psi_i, u_i) = f(\theta_{1i}) f(\theta_{2i}) |J|
\end{align}

Using the change of variable technique, substitute \(\theta_{2i} = u_i\) in \(3.16\) to get

\begin{align}
(3.3) & \quad \theta_{1i} = \frac{\psi_i u_i - 1}{((n-1)/n) u_i (p + \psi_i u_i)}
\end{align}

Then partially differentiate \(3.3\) and compute the Jacobian determinant in \(3.2\) as

\begin{align}
(3.4) & \quad f(\psi_i, u_i) = f(\theta_{1i}) f(\theta_{2i}) \left| \frac{\partial (\theta_{1i}, \theta_{2i})}{\partial (\psi_i, u_i)} \right|
\end{align}
\( f (\psi_i, u_i) = f (\theta_{1i}) f (\theta_{2i}) \left| \begin{array}{cc} \frac{\partial \theta_{1i}}{\partial \psi_i} & \frac{\partial \theta_{1i}}{\partial u_i} \\ \frac{\partial \theta_{2i}}{\partial \psi_i} & \frac{\partial \theta_{2i}}{\partial u_i} \end{array} \right| \)

From (3.5), It is known that the \( \theta_{1i} \) and \( \theta_{2i} \) are independent, then the density function of the joint distribution of \( \theta_{1i} \) and \( \theta_{2i} \) is given as

\[
f (\theta_{1i}, \theta_{2i}) = \frac{1}{B \left( \frac{n}{2}, \frac{n-p-2}{2} \right)} \theta_{1i}^{\frac{1}{2} - 1} (1 - \theta_{1i})^{\frac{n-p-2-1}{2}} \times \frac{1}{B \left( \frac{n}{2}, \frac{n-p-2}{2} \right)} \theta_{2i}^{\frac{n-p-2}{2} - 1} (1 - \theta_{2i})^{\frac{p-1}{2} - 1}
\]

where \( 0 \leq \theta_{1i}, \theta_{2i} \leq 1, n, p > 0 \)
and

\[
\left| \begin{array}{cc} \frac{\partial \psi_i}{\partial \theta_{1j}} & \frac{\partial \psi_i}{\partial \theta_{2j}} \\ \frac{\partial u_i}{\partial \theta_{1j}} & \frac{\partial u_i}{\partial \theta_{2j}} \end{array} \right| = \frac{(p+1)}{(n+1/n)(p+\psi_iu_i)^2} = \frac{p+1}{((n-1)/n)(p+\psi_iu_i)^2}
\]

Then substitute (3.6) and (3.7) in (3.5) in terms of the substitution of \( u_i \), to get the joint distribution of \( \psi_i \) and \( u_i \) as

\[
f (\psi_i, u_i) = \frac{1}{B \left( \frac{n}{2}, \frac{n-p-2}{2} \right)} \left( \frac{\psi_i u_i - 1}{(n+1/n) u_i (p+\psi_i u_i)} \right)^{\frac{1}{2} - 1} \left( 1 - \frac{\psi_i u_i - 1}{(n+1/n) u_i (p+\psi_i u_i)} \right)^{\frac{n-p-2}{2} - 1} \times \frac{1}{B \left( \frac{n}{2}, \frac{n-p-2}{2} \right)} u_i^{\frac{n-p-2}{2} - 1} (1 - u_i)^{\frac{p+1}{2} - 1} \times |J|
\]

where \( \frac{n-1}{n} \leq \psi_i < \infty, 0 \leq u_i \leq 1 \) and \( |J| = \frac{p+1}{((n-1)/n)(p+\psi_i u_i)^2} \)

Rearrange (3.8) and integrate with respect to \( u_i \), to get the marginal distribution of \( \psi_i \) as

\[
f (\psi_i; \alpha, \beta) = \int_0^1 \frac{1}{B \left( \frac{n}{2}, \frac{n-p-2}{2} \right)} \left( \frac{\psi_i u_i - 1}{(n+1/n) u_i (p+\psi_i u_i)} \right)^{\frac{1}{2} - 1} \left( 1 - \frac{\psi_i u_i - 1}{(n+1/n) u_i (p+\psi_i u_i)} \right)^{\frac{n-p-2}{2} - 1} \times \frac{1}{B \left( \frac{n}{2}, \frac{n-p-2}{2} \right)} u_i^{\frac{n-p-2}{2} - 1} (1 - u_i)^{\frac{p+1}{2} - 1} \times |J|
\]

where \( \frac{n-1}{n} \leq \psi_i < \infty \) and \( \alpha (p, n) = \frac{(n-1)}{n} u_i^{\frac{n-p-2}{2} + \beta - r - 1} (1 - u_i)^{\frac{p-1}{2} - 1} (1 + (\psi_i/p) u_i) \)

It is known, from (3.9)

\[
\int_0^1 \frac{1}{u_i^{\frac{n-p-3}{2} + \beta - r - 1}} (1 - u_i)^{\frac{n-1}{2} - 1} \left( 1 + (\psi_i/p) u_i \right)^{\frac{1}{2} + \frac{r}{2}} du_i = \frac{\Gamma \left( \frac{n-1}{2} \right)}{(\psi_i/p) \Gamma \left( \frac{r+\frac{1}{2}}{2} \right) \Gamma \left( \frac{r+\frac{1}{2}}{2} \right)} \left( 1 + (\psi_i/p) u_i \right)^{\frac{1}{2} + \frac{r}{2}} \left( \Omega_1 (\psi_i; p, n, r, s) + \Omega_2 (\psi_i; p, n, r, s) \right)
\]

where

\[
\Omega_1 (\psi_i; p, n, r, s) = (\psi_i/p) \left( \Gamma \left( \frac{n-p-3}{2} + s - r \right) + \Gamma \left( \frac{n-p-3}{2} + s - r \right) \Gamma \left( 2r + \frac{1}{2} - \frac{n-p-3}{2} + s \right) \right)
\]

\[
\Omega_2 (\psi_i; p, n, r, s) = (\psi_i/p) \left( \Gamma \left( \frac{n-p-3}{2} + s - r \right) + \Gamma \left( \frac{n-p-3}{2} + s - r \right) \Gamma \left( 2r + \frac{1}{2} - \frac{n-p-3}{2} + s \right) \right)
\]

\[
\begin{array}{c}
\end{array}
\]
\begin{align*}
\omega_2(p, n, r, s) &= \omega(p/r)^{-(r+\frac{3}{2})} \Gamma \left( r + \frac{3}{2} \right) \Gamma \left( \frac{n-p}{2} + s - r \right) \\
2F_1 \left( \frac{n-p-3}{2}, s - r; \frac{p+1}{2}; \frac{r+1}{2} \right) \\
&= \left( \frac{n-p-3}{2} + s - r \right) \left( \frac{r+1}{2} \right) \Gamma \left( \frac{n-p-3}{2} + s - r \right) \\
2F_1 \left( \frac{3}{2}, 1 - \left( \frac{n-7}{2} + s - 2r \right); 1 + r + \frac{3}{2} \left( \frac{n-p-3}{2} + s - r \right); -\frac{1}{\psi_i/p} \right) \\
\end{align*}

Then substitute (3.10) in (3.9) and arrange the terms, to get the density function of \( \psi_i \) in the series expression form as

\begin{equation}
(3.11)
\begin{align*}
f \left( \psi_i; p, n \right) &= \lambda(p, n) \left( \sum_{i=0}^{\frac{n-p-2}{2}} \sum_{i=0}^{\frac{r+1}{2}} \left( \frac{n-p-2}{r} - 1 \right) \left( \frac{r+1}{2} - 1 \right) \left( \frac{n-p}{2} \right)^{r+\frac{1}{2} - 1} p^{r+\frac{1}{2}} \\
&\times (-1)^{2r+s+\frac{1}{2}} \psi_i^{1-\frac{1}{2}} (\Omega_1(\psi_i; p, n, r, s) + \Omega_2(\psi_i; p, n, r, s)) \\
\end{align*}
\end{equation}

where, \((n-1)/n \leq \psi_i < \infty\), \( p > 0 \), \( n > p \) and \( \lambda(p, n) = (p+1) \alpha(p, n) \)

\begin{align*}
\Omega_1(\psi_i; p, n, r, s) &= \frac{\Gamma \left( \frac{n-p-3}{2} + s - r \right) \Gamma \left( r + \frac{3}{2} \right) \Gamma \left( \frac{n-p}{2} + 1 \right)}{\Gamma \left( \frac{n-p-3}{2} + s - r \right) \Gamma \left( \frac{n-p}{2} + 1 \right)} \frac{\psi_i^{1-\frac{1}{2}}}{(\psi_i/p)^{\frac{r+\frac{1}{2}}{2}}} \\
\Omega_2(\psi_i; p, n, r, s) &= \frac{\Gamma \left( \frac{n-p-3}{2} + s - r \right) \Gamma \left( r + \frac{3}{2} \right) \Gamma \left( \frac{n-p}{2} + 1 \right)}{\Gamma \left( \frac{n-p-3}{2} + s - r \right) \Gamma \left( \frac{n-p}{2} + 1 \right)} \frac{\psi_i^{1-\frac{1}{2}}}{(\psi_i/p)^{\frac{r+\frac{1}{2}}{2}}}
\end{align*}

(3.14)

From (3.14), it is the density function of Hadi’s \( H_i^2 \) influence measure which involves the following such as \( \Omega_1(\psi_i; p, n, r, s) \) and \( \Omega_2(\psi_i; p, n, r, s) \) are the auxiliary functions, 1 \( F_1 \) is the Gauss hypergeometric function and the normalizing constant \( \lambda(p, n) \) comprised of Beta and Gamma functions \( (B(\frac{p+1}{2}, \frac{n-p-2}{2}), B(\frac{n-p}{2} + 1, \frac{n-p}{2}), \Gamma \left( \frac{n-p+1}{2} \right)) \) with two shape parameters \((p, n), n\) is the sample size and \( p \) is the no. of predictors used in a multiple linear
regression model. In order to know the location and dispersion of Hadi’s \((H^2_i)\), the authors derived the first two moments in terms of mean, variance from and it is shown as follows. Using (2.15), rewrite in terms of series expression form as

\[
H^2_i = (p + 1) \left( \sum_{k=0}^{\infty} \left( \frac{n - 1}{n} \right)^k \theta_{1i}^{k+1} \theta_{2i}^k \right) + \frac{n}{n - 1} \left( \frac{1}{\theta_{2i}} \right) - 1
\]

Now take expectation and substitute the moments of two independent beta variables \(\theta_{1i}\) and \(\theta_{2i}\) of kind-1, to get the first moment of \((H^2_i)\) as

\[
E (H^2_i) = (p + 1) \left( \sum_{k=0}^{\infty} \left( \frac{n - 1}{n} \right)^k E (\theta_{1i}^{k+1}) E (\theta_{2i}^k) \right) + \frac{n}{n - 1} E \left( \frac{1}{\theta_{2i}} \right) - 1
\]

\[
E (H^2_i) = (p + 1) (\phi_1 (p, n)) + \frac{n}{n - 1} (\phi_2 (p, n)) - 1
\]

where \(\phi_1 (p, n) = \sum_{k=0}^{\infty} \left( \frac{n - 1}{n} \right)^k \frac{B(\frac{k}{2}, \frac{n-p-2}{2})}{B(\frac{k}{2}, \frac{n-p-2}{2})} \frac{B(\frac{n-p-k}{2}, \frac{n-p-1}{2})}{B(\frac{n-p-k}{2}, \frac{n-p-1}{2})}\)

\(\phi_2 (p, n) = \sum_{k=0}^{\infty} \left( \frac{n - 1}{n} \right)^k \frac{B(\frac{n-p-k}{2}, \frac{n-p-1}{2})}{B(\frac{n-p-k}{2}, \frac{n-p-1}{2})} \) and \(B\) is the beta function respectively.

From (2.15), rewrite and square both sides, then take expectation, to get the second moment of \((H^2_i)\) as

\[
(H^2_i + 1)^2 = (p + 1)^2 \theta_{1i}^2 \left( \frac{1}{1 - \left( \frac{n-1}{n} \right) \theta_{1i} \theta_{2i}} \right) + \left( \frac{n}{n - 1} \right)^2 \left( \frac{1}{\theta_{2i}} \right)^2
\]

\[
(H^2_i + 1)^2 = (p + 1)^2 \left( \sum_{k=0}^{\infty} (k + 1) \left( \frac{n - 1}{n} \right)^k \theta_{1i}^{k+2} \theta_{2i}^k \right) + \left( \frac{n}{n - 1} \right)^2 \left( \frac{1}{\theta_{2i}} \right)^2
\]

\[
E (H^2_i + 1)^2 = (p + 1)^2 \left( \sum_{k=0}^{\infty} (k + 1) \left( \frac{n - 1}{n} \right)^k E (\theta_{1i}^{k+2}) E (\theta_{2i}^k) \right) + \left( \frac{n}{n - 1} \right)^2 E \left( \frac{1}{\theta_{2i}} \right)^2
\]

\[
E (H^2_i + 1)^2 = (p + 1)^2 \left( \sum_{k=0}^{\infty} (k + 1) \left( \frac{n - 1}{n} \right)^k E (\theta_{1i}^{k+2}) E (\theta_{2i}^k) \right) + \left( \frac{n}{n - 1} \right)^2 E \left( \frac{1}{\theta_{2i}} \right)^2
\]

\[
E (H^2_i) = (p + 1)^2 \left( \sum_{k=0}^{\infty} (k + 1) \left( \frac{n - 1}{n} \right)^k E (\theta_{1i}^{k+2}) E (\theta_{2i}^k) \right) + \left( \frac{n}{n - 1} \right)^2 E \left( \frac{1}{\theta_{2i}} \right)^2
\]

Therefore, It is known

\[
(3.18) \quad V (H^2_i) = E (H^2_i)^2 - (E (H^2_i))^2
\]

Then substitute (3.16) and (3.17) in (3.18), to get the variance of \((H^2_i)\) as

\[
V (H^2_i) = (p + 1)^2 \left( \Phi_1 (p, n) \right) + \frac{n}{n - 1} \left( \Phi_2 (p, n) \right) + \frac{2n}{n - 1} (p + 1) \left( \Phi_3 (p, n) \right)
\]

where

\[
\Phi_1 (p, n) = \left( \sum_{k=0}^{\infty} (k + 1) \left( \frac{n - 1}{n} \right)^k \frac{B(\frac{k}{2}, \frac{n-p-2}{2})}{B(\frac{k}{2}, \frac{n-p-2}{2})} \frac{B(\frac{n-p-k}{2}, \frac{n-p-1}{2})}{B(\frac{n-p-k}{2}, \frac{n-p-1}{2})}\right)
\]

\[- \left( \sum_{k=0}^{\infty} \left( \frac{n - 1}{n} \right)^k \frac{B(\frac{k}{2}, \frac{n-p-2}{2})}{B(\frac{k}{2}, \frac{n-p-2}{2})} \frac{B(\frac{n-p-k}{2}, \frac{n-p-1}{2})}{B(\frac{n-p-k}{2}, \frac{n-p-1}{2})} \right)^2\]
Hadi’s influential observations in a sample. The approach is to derive the critical points of the authors adopted the test of significance approach of evaluating and identifying the observation is said to be influential and it may be a potential outlier. As a second approach, computed cut-off to identify the influential observation in linear multiple regression models. The established the upper control limit of $H_2 (p, n)$.

By using $(3.19)$, as a first approach, the authors utilize the upper control limit as a measure. By using the critical points, it is possible to test the significance of the measure computed from $(2.10)$ is given as

$$
\Phi_2 (p, n) = \frac{B \left( \frac{n-p-4}{2}, \frac{p-1}{2} \right)}{B \left( \frac{n-p}{2}, \frac{p-1}{2} \right)} - \left( \frac{B \left( \frac{n-p-2}{2}, \frac{p-1}{2} \right)}{B \left( \frac{n-p}{2}, \frac{p-1}{2} \right)} \right)^2
$$

$$
\Phi_3 (p, n) = \left( \sum_{k=0}^{\infty} \left( \frac{n-1}{n} \right)^k \frac{B(k+\frac{1}{2}, \frac{n-p-2}{2}, \frac{p-1}{2})}{B(\frac{1}{2}, \frac{n-p-2}{2})} \right) - \left( \frac{B \left( \frac{n-p-2}{2}, \frac{p-1}{2} \right)}{B \left( \frac{n-p}{2}, \frac{p-1}{2} \right)} \right)^2
$$

By using the mean and variance of Hadi’s measure from $(3.16)$ and $(3.18)$, the authors established the upper control limit of $(H_2^2)$ for different combination of $(p, n)$ by using $(3.20)$. Therefore

$$(3.19) \quad UCL (H_2^2) = E \left( H_2^2 \right) + \sqrt{V \left( H_2^2 \right)}$$

$$(3.20) \quad UCL (H_2^2) = (p + 1) \phi_1 (p, n) + \frac{n}{n-1} \phi_2 (p, n) - 1 + \sqrt{(p + 1)^2 \phi_1 (p, n) + \left( \frac{n}{n-1} \right)^2 \phi_2 (p, n) + \frac{2n(p + 1)}{n-1} \phi_3 (p, n)}$$

By using $(3.19)$, as a first approach, the authors utilize the upper control limit as a cut-off to identify the influential observation in linear multiple regression models. The computed $(H_2^2)$ of any observation is greater than upper control limit, then the observation is said to be influential and it may be a potential outlier. As a second approach, the authors adopted the test of significance approach of evaluating and identifying the influential observations in a sample. The approach is to derive the critical points of the Hadi’s $(H_2^2)$ measure by using the following relationship from $(2.10)$ is given as

$$(3.21) \quad H_{i(p,n)}^2 (\alpha) = \frac{p + 1}{(n - p - 2) + F_{(1,n-p-2)}(\alpha)} - \frac{(n-1)/n}{1 + ((p-1)/(n-p)) F_{(p-1,n-p)}(\alpha)} + \frac{(p-1)/(n-p) F_{(p-1,n-p)}(\alpha) + 1/n}{(n-1)/n}$$

From $(3.21)$ for a different combination of values of $(p, n)$ and for the significance probability $p \left( H_{i(p,n)}^2 > H_{i(p,n)}^2 (\alpha) \right) = \alpha$, authors computed the critical points of Hadi’s $(H_2^2)$ measure. By using the critical points, it is possible to test the significance of the influential observation computed from a multiple linear regression model. The following table-1 visualizes the upper control limit of the Hadi’s $(H_2^2)$ measure computed from $(3.20)$ and tables 2.3 exhibits the significant percentage points of the distribution of Hadi’s $(H_2^2)$ measure for varying sample size $(n)$ and no. of predictors $(p)$ at 5% and 1% significance $(\alpha)$. 

\[544\]
545

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Table 1. Upper control limit of Hadi’s Hi2 for combinations of (p, n)
n
5
6
7
8
9
10
11
12
13
14
15
16
17
18
19
20
25
30
40
60
80
100
120

1
3.8793
2.7869
2.0960
1.6375
1.3244
1.1016
0.93798
0.81418
0.71781
0.64117
0.57888
0.52736
0.48413
0.44735
0.41571
0.38821
0.29153
0.23330
0.16665
0.10604
0.077750
0.061377
0.050698

2
4.7824
4.1205
3.0649
2.4222
1.9860
1.6735
1.4407
1.2618
1.1207
1.0069
0.91340
0.83539
0.76937
0.71281
0.66388
0.49344
0.39220
0.27779
0.17523
0.12797
0.10078
0.083074

3
5.8544
5.3089
3.9434
3.1307
2.5131
2.2003
2.8756
1.6800
3.4478
1.3518
3.7858
1.1278
4.6021
0.96602
3.5089
0.55879
0.39184
0.24485
0.17792
0.13973
0.11502

4
6.8240
6.4396
4.7506
3.8002
3.1624
2.7015
2.3529
2.0806
1.8627
1.6843
1.5361
1.4112
1.3042
0.94324
0.73673
0.51107
0.31616
0.22875
0.17908
0.14716

p
5
7.75516
7.5498
5.7432
4.4486
8.2252
3.1868
9.4703
2.4698
10.869
2.0085
16.095
1.6880
18.110
0.92840
0.63665
0.38966
0.28048
0.21892
0.17970

6
8.7226
8.6542
6.3327
5.0854
4.2587
3.6615
3.2076
2.8508
2.5631
2.3262
2.1281
1.4852
1.1359
0.76936
0.46577
0.33352
0.25970
0.21257

7
9.6091
9.7586
9.2888
5.7147
16.541
4.1290
18.092
3.2258
28.296
2.6393
42.721
1.3616
0.91011
0.54459
0.38796
0.30113
0.24596

8
10.708
10.864
7.8927
6.3400
5.3235
4.5913
4.0347
3.5961
3.2414
2.1576
1.6083
1.0597
0.62640
0.44375
0.34346
0.28005

9
13.987
11.973
15.101
6.9626
22.212
5.0498
46.585
3.9631
76.534
1.8790
1.2188
0.71132
0.50114
0.38659
0.31459

10
12.787
13.082
9.4489
7.5833
6.3734
5.5056
4.8466
3.0180
2.1778
1.3888
0.79969
0.56014
0.43076
0.34982

p-no.of predictors n-Sample Size

Table 2. Signiﬁcant two-tail percentage points of Hadi’s
n
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
19
20
25
30
40
60
80
100
120
∞

p
1
6.5000
.3457
.2550
.2031
.1689
.1446
.1264
.1123
.1010
.0918
.0842
.0777
.0721
.0673
.0631
.0594
.0561
.0531
.0420
.0348
.0259
.0171
.0128
.0102
.0085
0

2
15.9119
4.4885
2.5201
1.7131
1.2866
1.0262
.8518
.7273
.6341
.5618
.5042
.4572
.4182
.3852
.3571
.3328
.3115
.2361
.1900
.1366
.0874
.0643
.0508
.0420
0

3
28.1667
7.8663
4.2275
2.7941
2.0582
1.6185
1.3291
1.1252
.9744
.8585
.7669
.6927
.6314
.5799
.5362
.4985
.3685
.2921
.2064
.1300
.0948
.0746
.0615
0

4
39.8398
11.0202
5.8051
3.7841
2.7595
2.1536
1.7579
1.4812
1.2776
1.1220
.9995
.9006
.8192
.7511
.6933
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.3904
.2712
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0

5
51.2090
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2.1664
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1.3714
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.1153
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6
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9
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25.8841
13.1973
8.3947
6.0066
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3.7226
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.3569
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.1919
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10
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14.6399
9.2924
6.6373
5.0947
4.1025
3.4177
2.9198
1.6624
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.7074
.3974
.2759
.2112
.1710
0


Table 3. Significant two-tail percentage points of Hadi’s

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Table 4. Identification of Potential Outliers, Comparative results of Hadi’s approach and proposed approach-I

<table>
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<th>Model</th>
<th>p</th>
<th>Hadi’s traditional approach</th>
<th>Proposed approach-I</th>
</tr>
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<tbody>
<tr>
<td></td>
<td></td>
<td>*Cut-off $\left( H_i^2 \right)$ = $\left( E \left( H_i^2 \right) + c \sqrt{V \left( H_i^2 \right)} \right)$</td>
<td>**Cut-off $\left( H_i^2 \right)$ = $\left( E \left( H_i^2 \right) + \sqrt{V \left( H_i^2 \right)} \right)$</td>
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<tr>
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<td>$A$</td>
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<td>3</td>
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<tr>
<td>4</td>
<td>4</td>
<td>0.3349</td>
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4. Numerical Results and Discussion

To evaluate the potential outliers based on Hadi’s influence measure of the $i$th observation in a regression model in this section the authors showed the results of a numerical study. For this, the authors fitted Step-wise linear regression models with a different set of predictors in a Brand equity study. The study comprised of 18 different attributes about a car brand. The Step-wise regression results reveal 4 nested models were extracted from the regression procedure. For each model, the Hadi’s ($H^2$) were computed, and a comparison of proposed approaches I and II with the Hadi’s traditional approach of identifying the potential outliers are visualized in the following tables.

$p$-no. of predictors $n=275$ *A-Cut-off $\left( H_i^2 \right)$ = $\left( E \left( H_i^2 \right) + c \sqrt{V \left( H_i^2 \right)} \right)$ **B- Cut-off $\left( H_i^2 \right)$ = $\left( E \left( H_i^2 \right) + \sqrt{V \left( H_i^2 \right)} \right)$ -refer (3.20)
Table 5. Identification of Potential Outliers, Comparative results of Hadi’s approach and proposed approach-II

<table>
<thead>
<tr>
<th>Model</th>
<th>p</th>
<th>Hadi’s Traditional approach</th>
<th>Proposed approach-I</th>
<th>Proposed approach-II</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
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<td>*Cut-off $((H^2) = {E[1/H^2] + c\sqrt{V[1/H^2]}}]$</td>
<td>**Critical $(H^2)$ at 5% level</td>
<td>**Critical $(H^2)$ at 1% level</td>
</tr>
<tr>
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<td></td>
<td>$n &gt; A$</td>
<td>$n &gt; B$</td>
<td>$n &gt; B$</td>
</tr>
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<td>29</td>
<td>0.018943</td>
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<tr>
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<td>0.028376</td>
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</tr>
<tr>
<td>4</td>
<td>4</td>
<td>0.014002</td>
<td>31</td>
<td>0.003031</td>
</tr>
</tbody>
</table>

$p$-no.of predictors $n=275$ *A-Cut-off $(H^2)$ **B-Critical $(H^2)$

Table 6. Identification of Potential Outliers, Comparative results of Proposed approach I and II

<table>
<thead>
<tr>
<th>Model</th>
<th>p</th>
<th>Proposed approach-I</th>
<th>Proposed approach-II</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>*Cut-off $((H^2) = {E[1/H^2] + c\sqrt{V[1/H^2]}}]$</td>
<td>**Critical $(H^2)$ at 5% level</td>
</tr>
<tr>
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<td>$n &gt; A$</td>
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<td>31</td>
</tr>
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</table>

$p$-no.of predictors $n=275$ *A-Cut-off $(H^2)$ **B-Critical $(H^2)$

Table 4 and 5 visualizes the comparative results of Hadi’s traditional approach of evaluating the potential outliers with the proposed approached 1 and 2. Under Hadi’s traditional approach, 4 nested multiple regression models are evaluated and the cutoffs’ for different $c$ values are shown in the table. As far as the fitted model-1 is a concern, the computed Hadi’s influential measure for 17, 14, 7 and 5 observations were above the cut-off value and hence these observations are said to be potential outliers. Similarly, model-2 is concern 15,12,7 and 5 observations are finalized as potential outliers, in the same manner, in model-3, the calculated Hadi’s influential measure for 15,12.5 and 4 observations was above the cut-off and hence these observations are said to be the potential outliers. Moreover, in model-4, 13, 12, 5 and 4 observations are treated as potential outliers because these observations exceeding the cut-off. Under Hadi’s approach at what value of $c$, an analyst can identify the potential outliers in the fitted models? For this question, the proposed approach-I has the answer. Under proposed approach-I, the cut-off was scientifically determined and in model-1, the calculated value of Hadi’s influential measure for 31 observations are above the cut-off and in model-2 29 observations, in model-3, 33 observations and in model-4, 31 observations are exceeding the scientifically determined cut-off. Hence these observations are treated as potential outliers. Under the proposed approach-II, the authors utilize the test of significance approach to identify the potential outliers. As far as the model-1 is a concern, the computed values of Hadi’s influential measure for 78 observations are greater than the critical Hadi’s $H2$ value at 5% significance level. Similarly, model-2, model-3, and model-4 are also evaluated and the authors identified 50, 58, 57 observations are potential outliers at 5% significance level. Likewise, 78, 35, 43 and 47 observations are treated as potential outliers at 1 % significance level for model-1, model-2, model-3 and model-4 respectively. Finally, among the three approaches to evaluate the outliers, the proposed approach-II is systematic and scientific when compared to Hadi’s traditional approach and proposed approach-I because the proposed approach II identified more number of outliers at different significance level and the cut-off critical Hadi’s $(H^2)$ value is also scientifically determined from the distribution of Hadi’s influence measure. Hence the authors observed, the proposed approach-2 outperforms the Hadi’s traditional approach.
and it will be the better when you compared it with the proposed approach-I. Finally, the comparative results emphasize the superiority of proposed approaches over the traditional approach and it is visualized through the graphical display from the following control charts.

**Figure 1.** Control chart for fitted Model-1 shows the Identification of potential outliers based on Hadis approach

**Figure 2.** Control chart for fitted Model-2 shows the Identification of potential outliers based on Hadis approach

**Figure 3.** Control chart for fitted Model-3 shows the Identification of potential outliers based on Hadis approach
Figure 4. Control chart for fitted Model-4 shows the Identification of potential outliers based on Hadis approach.

Figure 5. Control chart for each fitted model shows the Identification of potential outliers based on Proposed approach-I.

Figure 6. Control chart for each fitted model shows the Identification of potential outliers at 5% level based on proposed approach-II.
5. Conclusion

From the previous sections, the authors proposed a scientific approach that is based on the test of significance for Hadi’s influence measure to evaluate the potential outliers in a multiple linear regression model. At first, the exact distribution of the Hadi’s ($H^2_i$) was derived and the authors visualized the density function of $H^2$ in terms of complicated series expression form in terms of Gauss hypergeometric function and with two shape parameters namely $p$ and $n$. Moreover, the authors computed the critical percentage points of ($H^2_i$) at 5 %, 1% level of significance and it is utilized to evaluate the potential outliers. Finally, the proposed approach II is more systematic and scientific because it is based on the test of significance and the results were superior when compared it with Hadi’s traditional approach and proposed approach-I. Hence, the authors conclude, the proposed approach II overrides the use of traditional approach, proposed approach-I and also it outperforms the traditional Hadi’s approach in identifying more potential outliers in multiple regression models. Though Hadi’s measure is used in the applied statistics for many years but authors found the absence of this technique in statistical software, limits the application of this efficient technique in the research. So the authors recommend the software developers and computational data analyst to include this valuable and pragmatic method in academic and commercial software in near future. Similarly, the authors believe that the scientific approach introduced in this study made Hadi’s method a more significant tool in outlier detection as well as to the frequent users of linear multiple regression analysis.

References


Modelling the logistic processes using fuzzy decision approach

Saima Mustafa∗†, Ishrat Fatimah‡ and Young Bae Jun§

Abstract

Logistic processes deal with the information about producing and distributing goods and services from one place to another to fulfill the customer requirements. Every kind of business requires the proper functioning of logistic processes for the success of company or organization. In retail businesses, facility location for logistic processes serves as a backbone and the success or a failure of the company depends upon the geographical location of the company. It is a multiple criteria decision making problem involves multiple criteria to analyze the proper location of a company. In this research work, at first, we developed a multi criteria based structural model in retail facility location selection based on logistic processes. Then by using fuzzy modelling technique, the criteria are converted into triangular fuzzy numbers. We have adopted the technique of order preference similarity to ideal solutions for finding the best alternative. We have also applied the sensitivity analysis on the suggested modelling technique and found the same alternative as best with high degree of certainty which confirms the reliability of the proposed modelling technique.

Keywords: Retail logistic processes, Facility location, Decision making, Triangular fuzzy numbers, Best alternative.

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1. Introduction

Logistic processes deal with the movement of goods, material and other resources like energy and people between the point of origin and point of consumption in order to provide reliable customer service Tseng et al., [22]. It covers the affiliation between the production and the movement of products. It includes the integration of information, inventory, material management and services. The success or failure of any company or organization depends upon the handling of these processes through construction, consumption, storage and disposal. An effective logistic process also depends upon the proper geographical location of all the resources within the organization Mudambi [16].

The main function of logistic processes is to make decisions about manufacturing and distributing goods and services to customers or end users Rushton [18]. As part of these processes, locating the right product at right place is the main function of any kind of these processes. It involves the integration of supplies, transportation, warehousing and customer service station. Among all the processes, the purpose of logistics processes is to make decisions about the movement of products to deliver at right time to the right customer Christopher [5].

For any logistic process, site location has a key role which involves multiple factors to analyze for any company or organization Dornier [6]. Facility location selection plays a vital role in the strategic decision for any kind of logistic process. It is not very easy to change the location very often. Choosing the suitable facility among a particular set of alternatives is a challenging work requiring both qualitative and quantitative factors surrounded by the other activities of an organization or a company. It plays an important role in a strategic design of a company. It is an extensive and persistent subject, affecting several operational and logistical decisions, and the locational projects generally involve long term investments. Hence a successful facility location for operating the logistic processes would enable a leading edge to the company. On the other hand, a bad facility location is burden and it may damage the company. Once the mistake is made for the location of facility, it becomes tremendously difficult and costly to change it especially in large organizations. Thus, the decision makers mandatory select not only the well-performing facility location for operating the logistic processes, but also a profitable facility for the lifetime of a company Mintzberg et al., [13]. For operating the logistic processes in retail, facility location is a key element Mentzer et al., [12]. It has a strategic importance in making decisions. Selection of site is a strategic decision which is difficult to return. It is considered as a backbone of the whole logistic processes as the end product is finally sent in the retail market. The success or failure of the company or firm depends upon the geographical location of the organization. Due to complicated decision making, the selection of location is essential. A proper site location depends upon the decision-making process where the strategies are planned. There involve many factors that affect the facility location. Thus, it is a multi-criteria decision making process. The main factors that influence the location are terms of lease, cost of development, accessibility, lead time, parking area facility, population coverage and security risk. It is a long-term management process involves the strategic decision making analysis. Business decision makers always work with inadequate and uncertain data. Modelling of logistic processes for facility location is a multiple criteria decision making problem involves several factors. It examines both quantitative and qualitative factors, which are mostly based on uncertain data. Therefore, the conditions under uncertainty are handled by using multiple criteria decision making. While some decisions are quite easy to make, other decisions can be very hard to make, and generally causes a loss of energy and strength. Alike, the information or facts fluctuates very much that are needed for adequate decision making process. Decision making in retail depends upon the judgment of decision makers who
proposed their opinion as a strategic partner of the company or organization Abdi et al., [1]. There are different approaches for making decisions.

1.1. Decision analysis. The objective of decision analysis is to develop techniques and help decision makers but not substitute the decision makers. Therefore, decision analysis can be recognized as the process and policy of modelling, weighting and choosing an appropriate action for the given decision problem Kikeret et al., [10]. Decision analysis procedure generally precedes a comprehensive range of tools and a simple methodology in which decision maker breakdowns the problem into controllable parts in order to make it simple. During this development, the decision maker got a good awareness to the problem, analysis complex conditions and regulates an action which is companionable with their values and knowledge.

1.2. Multiple criteria decision making. Decision making in the occurrence of multiple and non-commensurable criteria is called multi-criteria decision making Monghasemi et al., [15]. Logically, in most of the circumstances the criteria are qualitative as well as quantitative that offers uncertainty in decision making process. There are severe practical restrictions in real world decisionmaking scenario because of the existence of vagueness and integral inaccuracy in the formation of criteria. To tackle such kind of problems, fuzzy decision approach is used.

To solve composite decision making problem in an organized way, fuzzy decision approach supports decision makers in reliable and productive way. For the modelling of vagueness on criteria, fuzzy set theory makes it possible to mathematically describe an accurate way. Fuzzy set theory is marked as the birth of new mathematical discipline. It helps to get more realistic mathematical models that can handle the real-world problems. Therefore, this theory is considered as a new way of modelling the decision problems as it offers organized tools to deal with the imprecision present in human judgment. To deal with the qualitative data, mostly in the form of linguistic variable, fuzzy set theory makes it possible to transform them into numbers by using the concept of membership functions and helps decision makers to deal with ambiguity. This theory is basically designed to mathematically present the vagueness and uncertainty inherent to decision to decision making problems. It provides organized tools to handle ambiguity. Fuzzy set theory deals very precisely for this kind of problems Chen et al., [4]. Therefore, the primary objective of this research is to select the best facility location for logistic processes by using the modelling technique of fuzzy decision approach, to tackle the problem of uncertainty and ambiguity. The research methodology is based on fuzzy decision approach for the modelling of logistic processes. In this research work, we applied the concepts of fuzzy set theory to express the opinion of decision makers in linguistic terms to overcome uncertainty for the estimation of qualitative factors. The linguistic judgment is then transformed into fuzzy number. Finally, multiple criteria decision making model based on fuzzy sets theory and fuzzy Topsis has been used to select the best facility location for logistic processes. To deal with conflicting and non-commensurable criteria, the proposed model is applied through triangular fuzzy numbers. Sensitivity analysis is also presented at the end to check out the consistency of the proposed modelling technique.

Under many real-world situations, crisp data is inadequate to deal with real life problems since human judgement are vague and cannot be estimated with exact numeric values. To resolve the ambiguity frequently arising in information from human judgement, fuzzy set theory has been corporate in many multiple criteria decision making problems including Topsis. It is the most classical method for solving multi-criteria decision making problem, was first developed by Hwang and Yoon in 1981. It was further modified by Lai and Liu in 1993. It is based on the principle that the chosen alternative should have the longest distance from the negative ideal solution that maximize the cost criteria and
minimize the benefit criteria; and the shortest distance from the positive ideal solution that is the solution that maximize the benefit criteria and minimize the cost criteria. In fuzzy Topsis, all the ratings and weights are defined by means of linguistic variables. A triangular fuzzy number can be denoted as \((e, f, g)\). A fuzzy multi-criteria decision making matrix is constructed as follows, in which possible alternatives are \(X_{11}, X_{12}, \ldots, X_{nm}\) and evaluation criteria are presented as \(K_1, K_2, \ldots, K_m\).

\[
D_S = \begin{bmatrix}
K_1 & K_2 & \cdots & K_m \\
X_{11} & X_{12} & \cdots & X_{1m} \\
X_{21} & X_{22} & \cdots & X_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
X_{n1} & X_{n2} & \cdots & X_{nm}
\end{bmatrix}
\]

where \(K_1, K_2, \ldots, K_m\) denotes the different choices for alternative and \(X_{11}, \ldots, X_{nm}\) are the different combinations of criteria and alternatives ratings. The best non-fuzzy number denoted by BNF and crisp values for triangular fuzzy numbers are calculated as

\[
(1.1) \quad C_v = \frac{e + (4 \times f) + g}{6}
\]

where \(e, f\) and \(g\) are the corresponding fuzzy numbers and \(C_v\) represents the crisp value.

After normalization, fuzzy decision matrix is constructed as

\[
R = [n_{ij}]_{m \times n} \quad i = 1, 2, \ldots, m; j = 1, 2, \ldots, n.
\]

where,

\[
n_{ij} = \frac{e_{ij}}{g_j}, \quad f_{ij}, g_{ij}, g_j^* \neq 0 \text{ and } g_j^* = \max(g_{ij}) \quad \text{(profit criteria)}
\]

And

\[
n_{ij} = \frac{e_j}{e_{ij}}, \quad f_{ij}, g_{ij}, e_{ij} \neq 0, \quad g_{ij} \neq 0, \quad e_j = \min(e_{ij}) \quad \text{(price criteria)}
\]

\[
(1.2) \quad N_j = \frac{X_{ij}}{\sqrt{\sum X_{ij}^2}}, \quad X_{ij} \neq 0
\]

Where \(X_{ij}\) possible alternatives. Fuzzy positive ideal solution and fuzzy negative ideal solution of the alternatives are calculated as:

\[
M^+ = \max(N_w) \quad \text{and} \quad M^- = \min(N_w)
\]

Where,

\[
N_w = V_{ij} \times \frac{X_{ij}}{\sqrt{\sum X_{ij}^2}}, \quad X_{ij}^2 \neq 0, \quad i = 1, 2, \ldots, m; \quad j = 1, 2, \ldots, n.
\]

From the fuzzy positive ideal solution and fuzzy negative ideal solution, Euclidean distance for each alternative is given by:

\[
D^+(X_j) = \sqrt{\sum_{j=1}^{m} \left( W_{ij} \times \frac{X_{ij}}{\sqrt{\sum X_{ij}^2}} - \max \left( \frac{W_{ij} \times X_{ij}}{\sqrt{\sum X_{ij}^2}} \right) \right)^2}, \quad X_{ij}^2 \neq 0
\]

and

\[
D^-(X_j) = \sqrt{\sum_{j=1}^{m} \left( W_{ij} \times \frac{X_{ij}}{\sqrt{\sum X_{ij}^2}} - \min \left( \frac{W_{ij} \times X_{ij}}{\sqrt{\sum X_{ij}^2}} \right) \right)^2}, \quad X_{ij}^2 \neq 0
\]
Closeness coefficient $CC^+$ for each alternative from the distance of (FPIS) and (FNIS) is computed as follows:

\[
CC^+ = \frac{D^+(X_j)}{D^+(X_j) + D^-(X_j)}, \quad D^+(X_j) \neq 0, \quad D^-(X_j) \neq 0.
\]

After the identification of location based on closeness coefficient, ranks are provided for each alternative and on the bases of these ranking positions, best alternative is selected.

2. Numerical computation

Multi-criteria decision making method has been used under fuzzy surrounding for the selection of facility location as best alternative in retail logistics. The present research work is based on primary data collection. As the research is based on the decision analysis for logistics processes under fuzzy decision approach, data was collected from the decision makers who operate the logistic processes management in the company. For the application of fuzzy decision approach, fuzzy Topsis model has been used to evaluate and select the most suitable location for the general medical and cash and carry D.Watson store in Islamabad. Different criteria have been developed for the selection of best facility location in retail logistic processes for three alternative places as G-13 markaz Islamabad, G-15 markaz Islamabad and I-14 markaz Islamabad.

2.1. Selection criteria. Modelling of logistic processes in retail for facility location is a complex multi criteria problem. At first stage, we have developed multiple criteria for the alternative places on the basis of strategic planning made by the decision makers for the selection of facility location in retail logistic processes. For evaluating the best possible location in retail logistics, the necessary criteria are computed. Criterion 1 represented terms of lease, criterion 2 represented cost of development, criterion 3 represented the cost of improvement, criterion 4 represented the equipment cost, criterion 5 represented by accessibility, visibility of location represented as criterion 6, criterion 7 shown population coverage, criterion 8 presented the future growth and development, criterion 9 illustrated the distance from competitors, criterion 10 represented the number of located facilities, criterion 11 shown quality of competitors, criterion 12 shown the availability of staff, criterion 3 presented the lead time, criterion 14 presented parking area facility and criterion 15 shown the security risk. A group of 4 decision makers is made to express their personal judgement in linguistic term, for estimating importance ratings for the alternatives.

<table>
<thead>
<tr>
<th>Linguistic term for rating</th>
<th>Fuzzy numbers</th>
</tr>
</thead>
<tbody>
<tr>
<td>Very less suitable</td>
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</tr>
<tr>
<td>Less suitable</td>
<td>(0.0,0.25,0.5)</td>
</tr>
<tr>
<td>Medium suitable</td>
<td>(0.25,0.5,0.75)</td>
</tr>
<tr>
<td>Highly suitable</td>
<td>(0.5,0.75,1)</td>
</tr>
<tr>
<td>Very highly suitable</td>
<td>(0.75,0.75,1)</td>
</tr>
</tbody>
</table>

To provide weightage among different alternatives, four decision makers provide their personal judgment. The following five-member linguistic scale for weighting among each criterion are given where Very Low (VL), Low (L), Medium (M), High (H) and Very High (VH) has been developed.
**Table 2.** Linguistic terms for priority weights for each criterion

<table>
<thead>
<tr>
<th>Linguistic variables</th>
<th>Fuzzy numbers</th>
</tr>
</thead>
<tbody>
<tr>
<td>Very Low</td>
<td>(0.00,0.00,0.25)</td>
</tr>
<tr>
<td>Low</td>
<td>(0.00,0.25,0.50)</td>
</tr>
<tr>
<td>Medium</td>
<td>(0.25,0.50,0.75)</td>
</tr>
<tr>
<td>High</td>
<td>(0.50,0.75,1)</td>
</tr>
<tr>
<td>Very high</td>
<td>(0.75,1.00,1.00)</td>
</tr>
</tbody>
</table>

Four decision makers or experts allocated weights to each criterion according to their own preferences.

**Table 3.** Fuzzy importance weights of indices allocated by DMs in linguistic scale

<table>
<thead>
<tr>
<th>Criteria</th>
<th>DM1</th>
<th>DM2</th>
<th>DM3</th>
<th>DM4</th>
</tr>
</thead>
<tbody>
<tr>
<td>K_1</td>
<td>VH</td>
<td>H</td>
<td>VH</td>
<td>VH</td>
</tr>
<tr>
<td>K_2</td>
<td>H</td>
<td>M</td>
<td>L</td>
<td>H</td>
</tr>
<tr>
<td>K_3</td>
<td>L</td>
<td>L</td>
<td>L</td>
<td>M</td>
</tr>
<tr>
<td>K_4</td>
<td>L</td>
<td>L</td>
<td>L</td>
<td>M</td>
</tr>
<tr>
<td>K_5</td>
<td>H</td>
<td>VH</td>
<td>H</td>
<td>VH</td>
</tr>
<tr>
<td>K_6</td>
<td>H</td>
<td>VH</td>
<td>H</td>
<td>VH</td>
</tr>
<tr>
<td>K_7</td>
<td>M</td>
<td>H</td>
<td>M</td>
<td>M</td>
</tr>
<tr>
<td>K_8</td>
<td>M</td>
<td>M</td>
<td>M</td>
<td>H</td>
</tr>
<tr>
<td>K_9</td>
<td>L</td>
<td>VL</td>
<td>L</td>
<td>L</td>
</tr>
<tr>
<td>K_10</td>
<td>VL</td>
<td>L</td>
<td>L</td>
<td>M</td>
</tr>
<tr>
<td>K_11</td>
<td>L</td>
<td>M</td>
<td>M</td>
<td>M</td>
</tr>
<tr>
<td>K_12</td>
<td>H</td>
<td>VH</td>
<td>VH</td>
<td>H</td>
</tr>
<tr>
<td>K_13</td>
<td>VL</td>
<td>M</td>
<td>VL</td>
<td>M</td>
</tr>
<tr>
<td>K_14</td>
<td>M</td>
<td>H</td>
<td>H</td>
<td>H</td>
</tr>
<tr>
<td>K_15</td>
<td>H</td>
<td>VH</td>
<td>H</td>
<td>H</td>
</tr>
</tbody>
</table>

Here three different locations are represented as alternatives. Alternative 1 shown by 1 A which represents G-13 Markaz Islamabad, Alternative 2 shown by 2 A which represents G-15 Markaz Islamabad and Alternative 3 shown by 3 A which represents I-14 Markaz Islamabad. Decision makers allocated their preferences in linguistic terms as NS, LS, MS, HS, and VHS, for alternative 1 which are shown in table 4.

**Table 4.** Rating for alternative 1 in terms of linguistic scale

<table>
<thead>
<tr>
<th>Criteria</th>
<th>DM1</th>
<th>DM2</th>
<th>DM3</th>
<th>DM4</th>
</tr>
</thead>
<tbody>
<tr>
<td>K_1</td>
<td>VHS</td>
<td>VHS</td>
<td>VHS</td>
<td>VHS</td>
</tr>
<tr>
<td>K_2</td>
<td>HS</td>
<td>HS</td>
<td>MS</td>
<td>HS</td>
</tr>
<tr>
<td>K_3</td>
<td>LS</td>
<td>HS</td>
<td>HS</td>
<td>MS</td>
</tr>
<tr>
<td>K_4</td>
<td>MS</td>
<td>HS</td>
<td>HS</td>
<td>MS</td>
</tr>
<tr>
<td>K_5</td>
<td>HS</td>
<td>VHS</td>
<td>VHS</td>
<td>VHS</td>
</tr>
<tr>
<td>K_6</td>
<td>HS</td>
<td>HS</td>
<td>HS</td>
<td>VHS</td>
</tr>
<tr>
<td>K_7</td>
<td>MS</td>
<td>MS</td>
<td>MS</td>
<td>MS</td>
</tr>
<tr>
<td>K_8</td>
<td>MS</td>
<td>HS</td>
<td>HS</td>
<td>HS</td>
</tr>
<tr>
<td>K_9</td>
<td>LS</td>
<td>MS</td>
<td>LS</td>
<td>LS</td>
</tr>
<tr>
<td>K_10</td>
<td>VLS</td>
<td>MS</td>
<td>LS</td>
<td>LS</td>
</tr>
<tr>
<td>K_11</td>
<td>LS</td>
<td>MS</td>
<td>MS</td>
<td>MS</td>
</tr>
<tr>
<td>K_12</td>
<td>HS</td>
<td>HS</td>
<td>HS</td>
<td>HS</td>
</tr>
<tr>
<td>K_13</td>
<td>VLS</td>
<td>MS</td>
<td>HS</td>
<td>MS</td>
</tr>
<tr>
<td>K_14</td>
<td>HS</td>
<td>HS</td>
<td>VHS</td>
<td>HS</td>
</tr>
<tr>
<td>K_15</td>
<td>HS</td>
<td>MS</td>
<td>MS</td>
<td>HS</td>
</tr>
</tbody>
</table>
Using the fuzzy functioning rules in equation (1.1) and (1.2), combined fuzzy weights and aggregated fuzzy rating for every selected criterion are estimated. Linguistic variables are then converted into fuzzy numbers. The calculated values of aggregated fuzzy ratings for each alternative are shown in the given table.

**Table 5. Aggregated importance ratings for the alternatives**

<table>
<thead>
<tr>
<th>Criteria</th>
<th>Alternatives</th>
<th>A1</th>
<th>A2</th>
<th>A3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_1$</td>
<td>(0.75,1,1)</td>
<td>(0.25,0.5,0.75)</td>
<td>(0.5,0.75,0.9375)</td>
<td></td>
</tr>
<tr>
<td>$K_2$</td>
<td>(0.4375,0.6875,0.9375)</td>
<td>(0.1875,0.4375,0.6875)</td>
<td>(0.3125,0.5625,0.8125)</td>
<td></td>
</tr>
<tr>
<td>$K_3$</td>
<td>(0.375,0.625,0.875)</td>
<td>(0.0625,0.25,0.5)</td>
<td>(0.375,0.625,0.875)</td>
<td></td>
</tr>
<tr>
<td>$K_4$</td>
<td>(0.375,0.625,0.875)</td>
<td>(0.1875,0.4375,0.6875)</td>
<td>(0.3125,0.5625,0.8125)</td>
<td></td>
</tr>
<tr>
<td>$K_5$</td>
<td>(0.6875,0.9375,1)</td>
<td>(0.4375,0.6875,0.9375)</td>
<td>(0.1875,0.375,0.625)</td>
<td></td>
</tr>
<tr>
<td>$K_6$</td>
<td>(0.5625,0.8125,1)</td>
<td>(0.25,0.5,0.6875)</td>
<td>(0.5,0.75,1)</td>
<td></td>
</tr>
<tr>
<td>$K_7$</td>
<td>(0.25,0.5,0.75)</td>
<td>(0.125,0.375,0.625)</td>
<td>(0.1875,0.4375,0.6875)</td>
<td></td>
</tr>
<tr>
<td>$K_8$</td>
<td>(0.4375,0.6875,0.9375)</td>
<td>(0.3125,0.5625,0.8125)</td>
<td>(0.4375,0.6875,0.9375)</td>
<td></td>
</tr>
<tr>
<td>$K_9$</td>
<td>(0.0625,0.3125,0.5625)</td>
<td>(0.125,0.3125,0.5625)</td>
<td>(0.1875,0.4375,0.6875)</td>
<td></td>
</tr>
<tr>
<td>$K_{10}$</td>
<td>(0.1875,0.375,0.625)</td>
<td>(0.0,0.1875,0.4375)</td>
<td>(0.125,0.25,0.5)</td>
<td></td>
</tr>
<tr>
<td>$K_{11}$</td>
<td>(0.1875,0.4375,0.6875)</td>
<td>(0.125,0.3125,0.5625)</td>
<td>(0.0,0.25,0.5)</td>
<td></td>
</tr>
<tr>
<td>$K_{12}$</td>
<td>(0.5,0.75,1)</td>
<td>(0.3125,0.5625,0.8125)</td>
<td>(0.3125,0.5625,0.8125)</td>
<td></td>
</tr>
<tr>
<td>$K_{13}$</td>
<td>(0.25,0.4375,0.6875)</td>
<td>(0.375,0.625,0.875)</td>
<td>(0.375,0.625,0.875)</td>
<td></td>
</tr>
<tr>
<td>$K_{14}$</td>
<td>(0.5,0.75,1)</td>
<td>(0.25,0.5,0.75)</td>
<td>(0.4375,0.6875,0.9375)</td>
<td></td>
</tr>
<tr>
<td>$K_{15}$</td>
<td>(0.375,0.625,0.875)</td>
<td>(0.125,0.3125,0.5625)</td>
<td>(0.3125,0.5625,0.8125)</td>
<td></td>
</tr>
</tbody>
</table>

The estimated importance ratings and priority weights are transformed the indices into crisp values. Crisp values for the 3 alternatives based on 15 criteria are given in the following table.

**Table 6. Fuzzy multi-criteria decision matrix**

<table>
<thead>
<tr>
<th>Criteria</th>
<th>Alternatives</th>
<th>A1</th>
<th>A2</th>
<th>A3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_1$</td>
<td>0.916667</td>
<td>0.5</td>
<td>0.729167</td>
<td></td>
</tr>
<tr>
<td>$K_2$</td>
<td>0.6875</td>
<td>0.4375</td>
<td>0.5625</td>
<td></td>
</tr>
<tr>
<td>$K_3$</td>
<td>0.625</td>
<td>0.270833</td>
<td>0.625</td>
<td></td>
</tr>
<tr>
<td>$K_4$</td>
<td>0.625</td>
<td>0.4375</td>
<td>0.5625</td>
<td></td>
</tr>
<tr>
<td>$K_5$</td>
<td>0.875</td>
<td>0.6875</td>
<td>0.3955833</td>
<td></td>
</tr>
<tr>
<td>$K_6$</td>
<td>0.791667</td>
<td>0.479167</td>
<td>0.75</td>
<td></td>
</tr>
<tr>
<td>$K_7$</td>
<td>0.5</td>
<td>0.375</td>
<td>0.4375</td>
<td></td>
</tr>
<tr>
<td>$K_8$</td>
<td>0.6875</td>
<td>0.5625</td>
<td>0.6875</td>
<td></td>
</tr>
<tr>
<td>$K_9$</td>
<td>0.3125</td>
<td>0.333333</td>
<td>0.4375</td>
<td></td>
</tr>
<tr>
<td>$K_{10}$</td>
<td>0.395833</td>
<td>0.208333</td>
<td>0.291667</td>
<td></td>
</tr>
<tr>
<td>$K_{11}$</td>
<td>0.4375</td>
<td>0.333333</td>
<td>0.25</td>
<td></td>
</tr>
<tr>
<td>$K_{12}$</td>
<td>0.75</td>
<td>0.5625</td>
<td>0.5625</td>
<td></td>
</tr>
<tr>
<td>$K_{13}$</td>
<td>0.458333</td>
<td>0.625</td>
<td>0.625</td>
<td></td>
</tr>
<tr>
<td>$K_{14}$</td>
<td>0.75</td>
<td>0.5</td>
<td>0.6875</td>
<td></td>
</tr>
<tr>
<td>$K_{15}$</td>
<td>0.625</td>
<td>0.333333</td>
<td>0.5625</td>
<td></td>
</tr>
</tbody>
</table>
Table 7. Positive and negative ideal solutions

<table>
<thead>
<tr>
<th>Criteria</th>
<th>PIS</th>
<th>NIS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_1$</td>
<td>0.652288</td>
<td>0.355793</td>
</tr>
<tr>
<td>$K_2$</td>
<td>0.390552</td>
<td>0.248533</td>
</tr>
<tr>
<td>$K_3$</td>
<td>0.2112757</td>
<td>0.091552</td>
</tr>
<tr>
<td>$K_4$</td>
<td>0.206056</td>
<td>0.144239</td>
</tr>
<tr>
<td>$K_5$</td>
<td>0.632804</td>
<td>0.286268</td>
</tr>
<tr>
<td>$K_6$</td>
<td>0.5677</td>
<td>0.343608</td>
</tr>
<tr>
<td>$K_7$</td>
<td>0.409616</td>
<td>0.307212</td>
</tr>
<tr>
<td>$K_8$</td>
<td>0.344282</td>
<td>0.281685</td>
</tr>
<tr>
<td>$K_9$</td>
<td>0.136879</td>
<td>0.097771</td>
</tr>
<tr>
<td>$K_{10}$</td>
<td>0.193037</td>
<td>0.101598</td>
</tr>
<tr>
<td>$K_{11}$</td>
<td>0.309267</td>
<td>0.176724</td>
</tr>
<tr>
<td>$K_{12}$</td>
<td>0.585953</td>
<td>0.439465</td>
</tr>
<tr>
<td>$K_{13}$</td>
<td>0.17001</td>
<td>0.124674</td>
</tr>
<tr>
<td>$K_{14}$</td>
<td>0.427271</td>
<td>0.284848</td>
</tr>
<tr>
<td>$K_{15}$</td>
<td>0.554224</td>
<td>0.295586</td>
</tr>
</tbody>
</table>

We have computed fuzzy positive ideal solution and fuzzy negative ideal solution from the weighted fuzzy multi-criteria normalized decision matrix where positive ideal solution represented the maximum values and negative ideal solution represents the minimum values shown in table.

From the fuzzy positive ideal solution and fuzzy negative ideal solution, which are agreeing to the technique of Topsis, we calculated the Euclidean distance for each alternative which has shown in table 8 below.

Table 8. Calculated Euclidean distance

<table>
<thead>
<tr>
<th>Alternatives</th>
<th>Positive distance</th>
<th>Negative distance</th>
</tr>
</thead>
<tbody>
<tr>
<td>A1</td>
<td>0.059873</td>
<td>0.667464</td>
</tr>
<tr>
<td>A2</td>
<td>0.576823</td>
<td>0.223744</td>
</tr>
<tr>
<td>A3</td>
<td>0.439108</td>
<td>0.387128</td>
</tr>
</tbody>
</table>

According to the closeness coefficient, ranks are given below:

Table 9. Calculated Euclidean distance

<table>
<thead>
<tr>
<th>Alternatives</th>
<th>Closeness coefficient</th>
<th>Ranking</th>
</tr>
</thead>
<tbody>
<tr>
<td>A1</td>
<td>0.917681</td>
<td>1</td>
</tr>
<tr>
<td>A2</td>
<td>0.279482</td>
<td>3</td>
</tr>
<tr>
<td>A3</td>
<td>0.468544</td>
<td>2</td>
</tr>
</tbody>
</table>
3. Application of sensitivity analysis on the proposed modelling technique

Sensitivity analysis is a systematic review which involves the sequence of decisions applied on the pre-specified study to analyze the results of the whole process under consideration and to make final decision. Sensitivity analysis referred that the overall results are not affected by the other decisions, which are conducted for reviewing the process, the results of the appraisal can be regarded with the higher degree of assurance. On the other hand, conflicting results shows the uncertainty and imprecision in decision making analysis. In this case, sensitivity analysis examines the particular decisions which are greatly influence on the findings of review.

In our case, to check out the accuracy of the proposed fuzzy modelling technique on logistic processes for facility location, sensitivity analysis is conducted. Sensitivity analysis is then applied on the same alternatives for facility location. After developing the best facility location for logistic processes using fuzzy modelling technique for decision analysis with the help of four decision makers, the decision makers are then again requested to assign weights for the three alternatives for reviewing the overall process and to make final decision on the proposed work. As described before that facility location plays the key role in logistic processes. Within the logistic processes, the most important research area is strategic decision planning and thus there are multiple factors involved in the planning of facility location for logistic processes. Once the decision is implemented on the company or on an organization, it is not easy to change the facility location again because it impacts on the significant level of company cost and it effects the revenue generation capabilities of an organization. Therefore, we conducted sensitivity analysis in which decision makers give their personal judgment once again for assigning the weights to perform sensitivity analysis.

3.1. Weight assigned by decision makers. The four decision makers once again requested to assign weights in linguistic terms for multiple criteria made for facility location selection in logistic processes. The decision makers have shown their decisions about the given criteria in linguistic terms which are shown in table 9. Fifteen criteria
are made for the selection of facility location among logistic processes. $K_1$ shows terms of lease, $K_2$ shows cost of development, $K_3$ shows cost of improvement, $K_4$ shows equipment cost, $K_5$ shows accessibility, $K_6$ shows visibility of location, $K_7$ gives population coverage, $K_8$ gives future growth and development, $K_9$ illustrates the distance from competitors, $K_{10}$ presents number of located facilities, $K_{11}$ shows quality of competitors, $K_{12}$ shows availability of staff, $K_{13}$ shows lead time, $K_{14}$ shows parking area facility and $K_{15}$ shows security risk.

**Table 10. Weight assigned by decision makers**

<table>
<thead>
<tr>
<th>Criteria</th>
<th>DM1</th>
<th>DM2</th>
<th>DM3</th>
<th>DM4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_1$</td>
<td>H</td>
<td>VH</td>
<td>H</td>
<td>H</td>
</tr>
<tr>
<td>$K_2$</td>
<td>L</td>
<td>M</td>
<td>H</td>
<td>L</td>
</tr>
<tr>
<td>$K_3$</td>
<td>L</td>
<td>L</td>
<td>M</td>
<td>VL</td>
</tr>
<tr>
<td>$K_4$</td>
<td>VL</td>
<td>L</td>
<td>L</td>
<td>VL</td>
</tr>
<tr>
<td>$K_5$</td>
<td>VH</td>
<td>H</td>
<td>VH</td>
<td>VH</td>
</tr>
<tr>
<td>$K_6$</td>
<td>H</td>
<td>H</td>
<td>H</td>
<td>VH</td>
</tr>
<tr>
<td>$K_7$</td>
<td>H</td>
<td>M</td>
<td>VH</td>
<td>H</td>
</tr>
<tr>
<td>$K_8$</td>
<td>M</td>
<td>H</td>
<td>H</td>
<td>H</td>
</tr>
<tr>
<td>$K_9$</td>
<td>L</td>
<td>L</td>
<td>M</td>
<td>L</td>
</tr>
<tr>
<td>$K_{10}$</td>
<td>M</td>
<td>M</td>
<td>H</td>
<td>VL</td>
</tr>
<tr>
<td>$K_{11}$</td>
<td>L</td>
<td>L</td>
<td>VH</td>
<td>L</td>
</tr>
<tr>
<td>$K_{12}$</td>
<td>H</td>
<td>H</td>
<td>H</td>
<td>H</td>
</tr>
<tr>
<td>$K_{13}$</td>
<td>L</td>
<td>M</td>
<td>M</td>
<td>L</td>
</tr>
<tr>
<td>$K_{14}$</td>
<td>H</td>
<td>VH</td>
<td>VH</td>
<td>VH</td>
</tr>
<tr>
<td>$K_{15}$</td>
<td>H</td>
<td>H</td>
<td>L</td>
<td>H</td>
</tr>
</tbody>
</table>

We then estimated the aggregated priority weights by using equation (1.2) and the results are presented in the following table.

**Table 11. Aggregated priority weight**

<table>
<thead>
<tr>
<th>Criteria</th>
<th>Aggregated weights</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_1$</td>
<td>(0.5625, 0.8125, 1)</td>
</tr>
<tr>
<td>$K_2$</td>
<td>(0.1875, 0.4375, 0.6875)</td>
</tr>
<tr>
<td>$K_3$</td>
<td>(0.0625, 0.25, 0.5)</td>
</tr>
<tr>
<td>$K_4$</td>
<td>(0.0, 0.1875, 0.4375)</td>
</tr>
<tr>
<td>$K_5$</td>
<td>(0.6875, 0.9375, 1)</td>
</tr>
<tr>
<td>$K_6$</td>
<td>(0.5625, 0.8125, 1)</td>
</tr>
<tr>
<td>$K_7$</td>
<td>(0.5625, 0.8125, 1)</td>
</tr>
<tr>
<td>$K_8$</td>
<td>(0.375, 0.625, 0.875)</td>
</tr>
<tr>
<td>$K_9$</td>
<td>(0.1875, 0.4375, 0.6875)</td>
</tr>
<tr>
<td>$K_{10}$</td>
<td>(0.1875, 0.4375, 0.6875)</td>
</tr>
<tr>
<td>$K_{11}$</td>
<td>(0.1875, 0.4375, 0.625)</td>
</tr>
<tr>
<td>$K_{12}$</td>
<td>(0.5, 0.75, 1)</td>
</tr>
<tr>
<td>$K_{13}$</td>
<td>(0.125, 0.3125, 0.5625)</td>
</tr>
<tr>
<td>$K_{14}$</td>
<td>(0.6875, 0.9375, 1)</td>
</tr>
<tr>
<td>$K_{15}$</td>
<td>(0.375, 0.625, 0.875)</td>
</tr>
</tbody>
</table>
By applying the same steps using equations (1.1), (1.2) and (1.3), we have been calculated the fuzzy normalized matrices, positive and negative ideal solution and found the closeness coefficient. The result has presented in the following figure.

![Closeness Coefficient](figure_2.png)

**Figure 2.** Closeness coefficient

4. Conclusion

In any business environment, decision making for logistic processes is essential. The operations on logistics processes are based on the multi criteria decision making analysis and criteria are conflicted based on the problem which is under consideration as the decisions. In this research work, the fuzzy modelling of logistic processes is conducted in retail facility location among the alternatives and selected the most suitable location facility and the results are discussed in detail. In section 3, by using fuzzy Topsis technique with triangular fuzzy numbers best facility location is estimated. Different fuzzy rules are used for making multiple criteria decision analysis. Decision matrix is normalized for comparing the criteria and through closeness coefficient, alternatives are ranked and selected the alternative 1 as most suitable place. In section 4, sensitivity analysis is conducted by using the same modelling technique and reviewing the whole process to check out the degree of assurance for the selection of location. For both analysis, the same alternative is chosen as a most suitable place and associates with higher degree of closeness coefficient which represents that the applied modelling technique of fuzzy Topsis for selecting the best location is the most suitable one because same alternative is chosen which is G-13 Markaz Islamabad and the result shown ranking as $A_1 > A_3 > A_2$. It is analyzed that proposed fuzzy modelling technique can be used for locating the facility location in any company or organization for operating the logistic processes.

References


Comparing Bhattacharyya and Kshirsagar bounds with bootstrap method

M. Khorashadizadeh*†, S. Nayeban‡, A.H. Rezaei Roknabadi§ and G.R. Mohtashami Borzadaran¶

Abstract

In the class of unbiased estimators for the parameter functions, the variance of estimator is one of the basic criteria to compare and evaluate the accuracy of the estimators. In many cases the variance has complicated form and we can not compute it, so, by lower bounds, we can approximate it. Many studies have been done on the lower bounds for the variance of an unbiased estimator of the parameter. Another common and popular method that is used in many statistical problems such as variance estimation, is bootstrap method. This method has some advantages and disadvantages that must be careful when using them. In this paper, first we briefly introduce the two famous lower bounds named "Kshirsagar" (one parameter case) and "Bhattacharyya" (one and multi parameter case) bounds and then we extend the Kshirsagar bound in multi parameter case. Also, by giving some examples in different distributions, we compare one and multi parameter Bhattacharyya and Kshirsagar lower bounds with bootstrap method for approximating the variance of the unbiased estimators and show that the mentioned bounds have a better performance than bootstrap method.

Keywords: Bhattacharyya bound, Bootstrap method, Cramer-Rao bound, Hammersley-Chapman-Robbins bound, Fisher information, Kshirsagar bound.


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1. Introduction

One of the main branches of statistical inference is estimation theory which introduce different estimators of unknown parameters and verify various properties of estimators such as unbiasedness, consistency and efficiency. In estimation theory, one of the fundamental things about accuracy of an estimator is finding a good lower bound for the variance of the estimator. In many cases the variance has complicated form and we can not compute it, and researchers are looking for ways to approximate its value. The variance of unbiased estimators of the parameter functions, is one of the basic criteria to compare and evaluate the accuracy of the estimators. In addition, the variance of unbiased estimators is one of the main components of statistical inference such as hypothesis testing and confidence intervals.

One of the most popular methods for approximating the variance of estimators which is introduced by Efron (1979) is bootstrap method. This method has some advantages and disadvantages that must be careful when using them. The other common method for estimating the variance of the estimator, is using the lower bounds. The important point of these bounds is that in many cases, they are too close to the actual value of the variance of estimator and therefore they are considered as a good approximation for the variance of the estimators. Some well-known and applicable lower bounds are Cramer-Rao (by Rao, 1945 and Cramer, 1946), Bhattacharyya (by Bhattacharyya, 1946, 1947), Hammersley-Chapman-Robbins (by Hammersley, 1950 and Chapman and Robbins, 1951) and Kshirsagar (by Kshirsagar, 2000).

In this paper the extended version of Kshirsagar bound in multi parameter case is proposed and it is proven that the new bound is increasing with respect to its order. Also, by focusing on Bhattacharyya and Kshirsagar bounds, we compare these bounds in one and multi parameter cases, with bootstrap method for approximating the variance of unbiased estimators via some examples. It should be noted that our comparisons show that the Bhattacharyya and Kshirsagar bounds are much more better methods with respect to bootstrap method for approximating the variance of the unbiased estimators.

2. One parameter Bhattacharyya and Kshirsagar bounds

In this section, we briefly introduce the structure of one parameter Bhattacharyya and Kshirsagar lower bounds.

2.1. Bhattacharyya lower bound.

One of the most famous lower bounds for the variance of estimators which has many applications in various fields, is Cramer-Rao bound. But this inequality states that under certain conditions, the variance of estimators can not be less than a certain value and the fact that how much the variance is greater than the quantity, is not considered. In fact the Cramer-Rao lower bound is an insufficient amount for the actual amount of the variance of estimator even for large samples. Thus we need a better Bound than Cramer-Rao bound.

Bhattacharyya (1946, 1947) under some regularity conditions, obtained a series of lower bounds for any unbiased estimator of parameter functions. If $X$ has a probability density function $f(X|\theta)$ and $T(X)$ be an unbiased estimator of $g(\theta)$ then the Bhattacharyya bounds are defined as follows,

\begin{equation}
Var_\theta(T(X)) \geq J_\theta^T W^{-1} J_\theta := B_\theta(\theta),
\end{equation}
where $t$ refers to the transpose, $J_\theta = (g^{(1)}(\theta), g^{(2)}(\theta), \ldots, g^{(k)}(\theta))^T$, $g^{(j)}(\theta) = \frac{\partial f(\theta)}{\partial \theta}$ for $j = 1, 2, \ldots, k$ and $W^{-1}$ is the inverse of the Bhattacharyya matrix, where

$$W = (W_{rs}) = \left( Cov_\theta \left\{ \frac{(r)(X|\theta)}{f(X|\theta)}, \frac{(s)(X|\theta)}{f(X|\theta)} \right\} \right),$$

such that $E_\theta \left\{ \frac{(r)(X|\theta)}{f(X|\theta)} \right\} = 0$ for $r, s = 1, 2, \ldots, k$.

In $k \times k$ Bhattacharyya matrix $(W)$, $k$ is the order of it. It is clear that $(1,1)^{th}$ element of the Bhattacharyya matrix is the Fisher information and if we substitute $k = 1$ in (2.1), then it indeed reduces to the Cramer-Rao inequality. By using the properties of the multiple correlation coefficient, it is easy to show that as the order of the Bhattacharyya matrix $(k)$ increases, the Bhattacharyya bound becomes sharper. For details and properties of Bhattacharyya bound one can see Shanbhag (1972,1979), Blight and Rao (1974), Tanaka and Akahira (2003), Tanaka (2003, 2006), Mohtashami Borzadaran (2001, 2006), Khorashadizadeh and Mohtashami Borzadaran (2007), Mohtashami Borzadaran et al. (2010).

### 2.2. Kshirsagar lower bound

Kshirsagar (2000) extended the Hammersley-Chapman-Robbins lower bound which was introduced by Hammersley (1950) and Chapman and Robbins (1951) in the same manner of the Bhattacharyya inequality. This bound does not need the regularity assumptions and states that for any unbiased estimator $T(X)$ of $g(\theta)$,

$$Var_\theta(T(X)) \geq \sup_{\phi} \lambda_0 \Sigma^{-1} \lambda_0 := K_k(\theta),$$

where $t$ refers to the transpose, $\lambda_0 = (g(\phi_1) - g(\theta), g(\phi_2) - g(\theta), \ldots, g(\phi_k) - g(\theta))^T$ and $\Sigma^{-1}$ is the inverse of matrix with elements as follow,

$$\Sigma_{rs} = Cov_\theta(\psi_r, \psi_s), \quad r, s = 1, 2, \ldots, k,$$

where, $\psi_i = \frac{f(\phi_i|\theta) - f(X|\theta)}{f(X|\theta)}$ and the supremum is taken over the set of all $\phi_i \in \Theta, (i = 1, 2, \ldots, k)$, satisfying

$$S(\phi_k) \subset S(\phi_{k-1}) \subset \ldots \subset S(\phi_1) \subset S(\theta).$$

In case $k = 1$ the lower bound (2.3) is reduced to Hammersley-Chapman-Robbins lower bound. Kshirsagar (2000) showed that his bound is sharper than the Bhattacharyya bound with corresponding order. Although, computing the Kshirsagar bound and taking the supremums are difficult, but, nowadays, using computers makes it a little easier. Tsuda and Matsumoto (2005) by improving the Kshirsagar bound, expressed its applications in quantum theory. Qin and Nayak (2008) evaluated the Kshirsagar bound for the mean square error of predictor variable and showed that these bounds are sharper than their Bhattacharyya bounds. Nayeban et al. (2013, 2014) computed and compared Kshirsagar bounds with Bhattacharyya bounds in some applicable distributions.

### 3. Comparing the Bhattacharyya and Kshirsagar bounds with bootstrap method

In statistics literature, bootstrapping is a method for assigning measures of accuracy to sample estimates which was first introduced by Efron (1979). This technique allows estimation of the sampling distribution of almost any statistic using only very simple methods. Generally, it falls in the broader class of resampling methods. Bootstrapping is the practice of estimating properties of an estimator (such as its variance) by measuring those properties when sampling from an approximating distribution.
In the next three examples, we compute and compare the Bhattacharyya and Kshirsagar lower bounds with bootstrap method for approximating the variance of the unbiased estimators.

3.1. Example. Suppose a random variable $X$ has a negative binomial distribution with unknown parameter $p$ and known parameter $r$ and the probability mass function as follows,

$$P(X = x) = \binom{x-1}{r-1} p^r q^{x-r}, \quad x = r, r+1, \ldots, r \geq 1.$$ 

It can be easily shown that the UMVUE of parameter $p$ is,

$$\hat{p} = \frac{r-1}{X-1}, \quad X > 1,$$

but a simple expression for the exact variance of this estimator is not proposed yet. Blight and Rao (1974) and Haldan (1945) introduced the following expression which is made by elements of Bhattacharyya matrix,

$$Var(T(X)) = p^2 \sum_{i=1}^{\infty} \binom{r+i-1}{i}^{-1} q^i.$$ (3.1)

We estimate the variance of $T(X)$ by bootstrap method with 2000, 10000 and 100000 replications. Also we compute the Eq. (3.1) by the third, fourth and tenth Bhattacharyya bounds and present the results in the Table 1.

<table>
<thead>
<tr>
<th>$r$</th>
<th>$p$</th>
<th>$B_3$</th>
<th>$B_4$</th>
<th>$B_{10}$</th>
<th>Bootstrap Reps. 2000</th>
<th>Bootstrap Reps. $10^4$</th>
<th>Bootstrap Reps. $10^5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0.5</td>
<td>0.013778</td>
<td>0.013800</td>
<td>0.013805</td>
<td>0.13634</td>
<td>0.013836</td>
<td>0.013911</td>
</tr>
<tr>
<td>5</td>
<td>0.25</td>
<td>0.012472</td>
<td>0.012754</td>
<td>0.012973</td>
<td>0.013703</td>
<td>0.013264</td>
<td>0.012753</td>
</tr>
<tr>
<td>4</td>
<td>0.1</td>
<td>0.003424</td>
<td>0.003611</td>
<td>0.003876</td>
<td>0.003967</td>
<td>0.003808</td>
<td>0.004005</td>
</tr>
<tr>
<td>20</td>
<td>0.9</td>
<td>0.004089</td>
<td>0.004089</td>
<td>0.004089</td>
<td>0.004179</td>
<td>0.004103</td>
<td>0.004076</td>
</tr>
</tbody>
</table>

In Figure 1 we compute the variance of the UMVUE of $p$ for $r = 5$ by the 50th order Bhattacharyya bound, which is so sharp. Also the variance estimation by bootstrap with 10000 replications for all values of $p$ is shown in the Figure 1.
According to the simulation studies presented in the Table 1 and Figure 1, we can see that the Bhattacharyya bounds adapt the actual amount of the variance better than the bootstrap method. It should be noted that the Bhattacharyya bounds of order more than ten, all were equal up to fifteen decimals, that indicates the precise and fast convergence of Bhattacharyya bounds.

3.2. Example. Suppose $X_1, \ldots, X_n$ be a sample of exponential distribution with mean $\theta$. One of the important factors in this distribution which is more applicable in reliability theory, is reliability function $g(\theta) = R_\theta(t) = e^{-\theta t}$.

Pugh (1962), Basu (1964) and Patil and Wani (1966) found an unbiased estimator of this function as follows,

$$T(X) = (1 - \frac{t}{\sum_{i=1}^{n} X_i})^{n-1},$$

where $t < \sum_{i=1}^{n} X_i$.

Zacks and Even (1966) calculated the variance of this estimator as below,

$$Var_\theta(T(X)) = \frac{1}{(n-1)!} \int_{\lambda}^{\infty} \left( 1 - \frac{\lambda}{u} \right)^{2n-2} e^{-u} u^{n-1} du - e^{-2\lambda},$$

where $\lambda = \frac{t}{\theta}$. Computing the exact amount of this variance needs numerical methods and is not easily calculated. We present the Bhattacharyya and Kshirsagar bounds and also bootstrap approximations for the variance of the unbiased estimator of $g(\theta)$ for some values of $\lambda$ in Table 2.
Figure 2. Comparing the exact variance (Solid line), Kshirsagar bound (Dot-Dot line), Bhattacharyya bound (Dash-Dot line) and bootstrap method (Dash-Dash line) with 50000 replications in exponential distribution for variance of unbiased estimators of the reliability function ($n = 4$). (Example 3.2)

Table 2. Approximation of the variance of any unbiased estimator of reliability function in exponential distribution with Bhattacharyya and Kshirsagar bounds and bootstrap method for $n = 4$.

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$\text{Var}_\theta(T(X))$</th>
<th>$B_1$</th>
<th>$B_2$</th>
<th>$K_1$</th>
<th>$K_2$</th>
<th>Bootstrap (Reps. 2000)</th>
<th>Bootstrap (Reps. 10^4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.25</td>
<td>0.01974</td>
<td>0.00947</td>
<td>0.01237</td>
<td>0.01582</td>
<td>0.01889</td>
<td>0.01936</td>
<td>0.01903</td>
</tr>
<tr>
<td>0.5</td>
<td>0.03010</td>
<td>0.02299</td>
<td>0.02816</td>
<td>0.02828</td>
<td>0.02999</td>
<td>0.03308</td>
<td>0.03278</td>
</tr>
<tr>
<td>1</td>
<td>0.03854</td>
<td>0.03383</td>
<td>0.037218</td>
<td>0.03685</td>
<td>0.03784</td>
<td>0.03039</td>
<td>0.03055</td>
</tr>
<tr>
<td>1.5</td>
<td>0.03241</td>
<td>0.02800</td>
<td>0.02870</td>
<td>0.02780</td>
<td>0.03010</td>
<td>0.01901</td>
<td>0.01930</td>
</tr>
<tr>
<td>2</td>
<td>0.02030</td>
<td>0.01831</td>
<td>0.01831</td>
<td>0.01901</td>
<td>0.01987</td>
<td>0.01099</td>
<td>0.01094</td>
</tr>
<tr>
<td>2.5</td>
<td>0.01333</td>
<td>0.01052</td>
<td>0.01079</td>
<td>0.01091</td>
<td>0.01218</td>
<td>0.00619</td>
<td>0.00618</td>
</tr>
<tr>
<td>3</td>
<td>0.00685</td>
<td>0.00577</td>
<td>0.00613</td>
<td>0.00621</td>
<td>0.00665</td>
<td>0.00358</td>
<td>0.00351</td>
</tr>
</tbody>
</table>

Figure 2 shows the Bhattacharyya and Kshirsagar bounds and also bootstrap method for approximating the variance of the unbiased estimator of reliability function in exponential distribution.
It is observed that the Bhattacharyya and Kshirsagar bounds offer more accurate approximations than the bootstrap method. So an important point about the Bhattacharyya and Kshirsagar bounds is approximating the variance of unbiased estimator with their help.

3.3. Example. Suppose we have a sample of size \( n \) from Burr XII distribution. Since the cdf of Burr XII has closed forms, it is easy to see that its quantile \( x_q \) of order \( q \) is as follow,

\[
x_q = \left( (1-q)^{-\frac{1}{\theta}} - 1 \right)^{\frac{1}{\alpha}}.
\]

So, the median in Burr XII distribution is obtained for \( q = \frac{1}{2} \) as,

\[
Median = \left[ 2^{\frac{1}{\alpha}} - 1 \right]^{\frac{1}{\theta}}.
\]

<table>
<thead>
<tr>
<th>Table 3. Bhattacharyya and Kshirsagar bounds for the variance of any unbiased estimator of the median in Burr XII.</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \theta )</td>
</tr>
<tr>
<td>0.7</td>
</tr>
<tr>
<td>0.5</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>( \theta )</td>
</tr>
<tr>
<td>0.7</td>
</tr>
<tr>
<td>0.5</td>
</tr>
<tr>
<td>2</td>
</tr>
</tbody>
</table>

In Table 3, we evaluate the first five Bhattacharyya and Kshirsagar bounds for the variance of any unbiased estimator of the median in Burr XII distribution for some values of \( \theta \) and \( \alpha \).

Furthermore, in Figure 3 we compare the first order Bhattacharyya and first order Kshirsagar lower bounds with the bootstrap approximation of the variance of the unbiased estimator of the median in Burr XII, which indicates that, with respect to the bootstrap approximation, the Bhattacharyya and Kshirsagar lower bounds are much more nearer to the exact value of the variance. This comparison shows that the two lower bounds are good approximations for the variance of the unbiased estimators.
4. Multi-parameter Bhattacharyya and Kshirsagar bounds

In this section, we first introduce the structure of multi-parameter Bhattacharyya bound and then define the multi-parameter Kshirsagar lower bound. Also, we compare the Bhattacharyya and Kshirsagar bounds with bootstrap method by two examples.

4.1. Multi-parameter Bhattacharyya lower bound. The multi-parameter version of the Bhattacharyya bound is also defined by Bhattacharyya (1947) and more considered by Pommeret (1997), Bartoszewicz (1980), Alharbi (1994) and Tanaka (2006). Bhattacharyya bounds have always been regarded as good approximations for the variance of unbiased estimators but in multi-parameter case, due to the complex structure of the matrix and the difficulty of calculating the matrix inversion, has received less attention.

One of important properties of multi-parameter Bhattacharyya bound is its convergence which has been studied by Ghosh and Sathe (1987) and Tanaka (2006). They showed that Bhattacharyya bounds converge to the variance of unbiased estimator.

Suppose $X$ has density function $f(x; \theta)$ with an unknown parameter vector $\theta := (\theta_1, \ldots, \theta_r) \in \Theta \subset \mathbb{R}^r$. Let following operators:

$$\partial_z := \left( \frac{\partial^z}{\partial \theta_1^{i_1} \cdots \partial \theta_r^{i_r}} \right)_{0 \leq i_1, \ldots, i_r \sum_{j=1}^r i_j = z}, \quad z \in \mathbb{N},$$

and

$$D_k := (\partial_1, \ldots, \partial_k)^t,$$

where $\partial_z$ is all possible partial derivatives of the form $\frac{\partial^z}{\partial \theta_1^{i_1} \cdots \partial \theta_r^{i_r}}$, which $i_1 + \ldots + i_r = z$ and $i_j$ takes integer values $\{0, 1, 2, \ldots\}$ for $j = 1, \ldots, r$. 

**Figure 3.** Comparing Bhattacharyya and Kshirsagar bounds of orders 1 and Bootstrap method (with 10000 replications) for the variance of any unbiased estimator of median in Burr XII distribution with $\alpha = 1$. (Example 3.3)
Considering the above definitions, the Bhattacharyya matrix of order \( k \) is the covariance matrix of random vector,

\[
\mathbf{D}_k f(X; \theta) \overline{f(X; \theta)}.
\]

Also note that \( \partial_x \) has \( m_x = \binom{z + r - 1}{r - 1} \) members, so the multi-parameter Bhattacharyya matrix of order \( k \), is \( n_k \times n_k \) where \( n_k = \sum_{i=1}^{k} m_i \).

Finally the multi-parameter Bhattacharyya inequality under suitable regularity conditions, is defined as follows:

If \( T(X) \) be a real valued unbiased estimator of \( g(\theta) \in \mathbb{R} \) then,

\[
\text{Var}_g(T(X)) \geq \eta_2 V^{-1} \eta_2 := B_{r,k}(\theta),
\]

where,

1. \( \overline{.} \) denotes the transpose and \( \eta_2 = \mathbf{D}_k g(\theta) \).
2. \( V^{-1} \) is the inverse of multi-parameter Bhattacharyya matrix.

### 4.1. Remark
We denote the multi-parameter Bhattacharyya bound of order \( k \) with \( r \) unknown parameters with \( B_{r,k}(\theta) \), which \( B_{1,k} \) is the one parameter Bhattacharyya bound \( (B_k(\theta)) \).

### 4.2. Multi-parameter Kshirsagar lower bound.
In this section we present our new bound in the next theorem, which is an extension of Kshirsagar bounds.

Let \( \Omega \) be a sample space and \( \mathcal{A} \) be a \( \sigma \)-field of subsets of \( \Omega \), and assume that \( \Theta \) be a parameter space on open set of \( \mathbb{R}^r \) and \( \theta = (\theta_1, \ldots, \theta_r) \). Suppose that \( \{f(x|\theta); \theta \in \Theta \} \) be a class of probability density functions or probability functions according as \( x \) is continuous or discrete.

Let \( \tau(\theta) \) be a real-valued function defined on \( \Theta \) and \( T(X) \) be an unbiased estimator of \( \tau(\theta) \), i.e., \( T(X) \) is a real-valued measurable function defined on \( \Omega \) with property that,

\[
E_\theta(T(X)) = \tau(\theta), \quad \forall \theta \in \Theta.
\]

For presenting lower bound for the variance of \( T(X) \), first we should define some notation and symbols:

Let \( \Phi_i(\theta) \) be a set of subsets of parameter spaces which are defined on \( \Theta \) such that for \( i = 1, 2, \ldots, \)

\[
\Phi_i(\theta) := \left( \Phi_{i_1}(\theta_1), \ldots, \Phi_{i_r}(\theta_r) \mid \sum_{j=1}^{r} i_j = i, \quad i_j \in \{0, 1, \ldots\} \right);
\]

where \( \Phi_{i_j}(\theta_j) \) is a function of \( \theta_j \) such that \( \Phi_0(\theta_j) = \theta_j \) for all \( j = 1, \ldots, r \) and

\[
S(\Phi_k(\theta)) \subset \ldots \subset S(\Phi_1(\theta)) \subset S(\theta),
\]

where \( S(\theta) = \{x|f(x|\theta) > 0\} \). It is known that, for any \( i \), there are \( m_i = \binom{i + r - 1}{r - 1} \) sets of non-negative \( (i_1, \ldots, i_r) \) which satisfies \( \sum_{j=1}^{r} i_j = i \).

For an example of such \( \Phi_i(\theta) \), we can take for \( j = 1, \ldots, r \), \( \Phi_{ij}(\theta_j) = \theta_j + i_j \theta_j \) for some proper values of \( \delta_j \) satisfying (4.3). For instance,

\[
\Phi_1(\theta) = \{ (\theta_1 + \delta_1, \theta_2), (\theta_1 + \delta_2, \theta_2, \theta_3), \ldots, (\theta_1 + \delta_r, \theta_2, \ldots, \theta_r) \};
\]

or

\[
\Phi_2(\theta) = \{ (\theta_1 + 2\delta_1, \theta_2, \ldots, \theta_r), (\theta_1 + 2\delta_2, \theta_2, \ldots, \theta_r), \ldots, (\theta_1 + 2\delta_r, \theta_2, \ldots, \theta_r + 2\delta_r), \}
\]

\[
(\theta_1 + \delta_1, \theta_2 + \delta_2, \theta_3, \ldots, \theta_r), \ldots, (\theta_1, \theta_2, \ldots, \theta_{r-1} + \delta_{r-1}, \theta_r + \delta_r) \}.
\]
Consider the row vector $\lambda_k$ as follow,

$$\lambda_k = (h_1, \ldots, h_k),$$

where $h_i = (\tau(\Phi_i(\theta)) - \tau(\theta))1$ is a row vector of size $m_i$ and $1$ is a row vector of ones and therefore $\lambda_k$ is a $1 \times n_k$ row vector where $n_k = \sum_{i=1}^k m_i = \sum_{i=1}^k \left( \frac{i + r - 1}{r - 1} \right)$.

Let $\Psi_i$, for $i = 1, \ldots, k$ be a vector of form below,

$$\Psi_i = \frac{f(X|\Phi_i(\theta)) - f(X|\theta)}{f(X|\theta)}, \quad i = 1, \ldots, k,$$

and let further, $\Sigma_k$ be the covariance matrix of random vector:

$$(\Psi_1, \ldots, \Psi_k).$$

By defining the following operator,

$$D_k f(x|\theta) := (f(x|\Phi_1(\theta)), \ldots, f(x|\Phi_k(\theta))),$$

$\Sigma_k$ is a $n_k \times n_k$ covariance matrix of random row vector

$$\frac{D_k f(X|\theta)}{f(X|\theta)} - 1,$$

where $1$ is a row vector of ones of convenient size.

Now we present our new bound in the next theorem, which is an extension of Barankin and Kshirsagar bounds.

4.2. Theorem. If $T(X)$ be any unbiased estimator of the parameter function $\tau(\theta)$ then,

$$\text{Var}_\theta(T(X)) \geq \sup \lambda_k \Sigma_k^{-1} \lambda_k' := K_{r,k}(\theta) \text{ say},$$

where, the supremum is with respect to all $\Phi_i(\theta)$ satisfying (4.3).

Proof:

It is easy to see that for any $i$, $E_\theta(\Psi_i) = 0$ and therefore for any $k$,

$$E_\theta \left( \frac{D_k f(X|\theta)}{f(X|\theta)} - 1 \right) = 0,$$

(4.4)

(\text{where } 0 \text{ is a row vector of zeros of size } n_k). We can partition the matrix of $\Sigma_k$ as follow:

$$\Sigma_k = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} & \cdots & \Sigma_{1k} \\ \vdots & \vdots & \ddots & \vdots \\ \Sigma_{k1} & \Sigma_{k2} & \cdots & \Sigma_{kk} \end{pmatrix},$$

(4.5)

where the $m_l \times m_s$ block matrix $\Sigma_{ls}$ is of the form,

$$\Sigma_{ls} = E_\theta(\Psi_l, \Psi_s), \quad l, s = 1, \ldots, k.$$

Hence, the covariance between $T(X)$ and every elements of the set $\Psi_i$ is as follow,

$$\text{Cov}_\theta(T(X), \Psi_i) = E_\theta(T(X), \Psi_i)$$

$$= \int T(x).f(x|\Phi_i(\theta))d\mu(x) - \int T(x).f(x|\theta)d\mu(x)$$

$$= \tau(\Phi_i(\theta)) - \tau(\theta)1$$

$$= h_i.$$

Finally, it is easy to see that the multiple correlation coefficient ($\rho_k$) between $T(X)$ and $(\Psi_1, \ldots, \Psi_k)$ is given by,

$$\rho_k^2 = \frac{\lambda_k \Sigma_k^{-1} \lambda_k'}{\text{Var}_\theta(T(X))},$$

(4.6)

which yields the required result. $\triangle$
4.3. Remark. It is well known that \( K_{r,1}(\theta) \) is identical to the Barankin bound and \( K_{1,k}(\theta) \) is identical to the Kshirsagar lower bound \((K_k(\theta))\) and finally \( K_{1,1}(\theta) \) leads to the Hammersley-Chapman-Robbins lower bound.

In the next theorem we try to show that as the order of the matrix \( \Sigma_k \) increases the bound get sharper and sharper.

4.4. Theorem. The lower bound for the variance of \( T(X) \) in Theorem 4.2, is increasing with respect to \( k \), i.e.

\[
K_{r,k}(\theta) \leq K_{r,k+1}(\theta).
\]

Proof: Using the partition (4.5) it is straightforward that,

\[
\Sigma_{k+1}^{-1} = \begin{pmatrix}
\Sigma_k & A \\
\cdots & \cdots & \cdots \\
A' & B
\end{pmatrix},
\]

where \( A = \begin{pmatrix}
\Sigma_{1(k+1)} \\
\vdots \\
\Sigma_{k(k+1)}
\end{pmatrix} \) is a \( n_k \times m_{k+1} \) matrix and \( B = \Sigma_{(k+1)(k+1)} \) is a \( m_{k+1} \times m_{k+1} \) matrix.

Then, the block inverse of the matrix \( \Sigma_{k+1} \) is,

\[
\Sigma_{k+1}^{-1} = \begin{pmatrix}
C_{11} & C_{12} \\
C_{21} & C_{22}
\end{pmatrix},
\]

where

\[
C_{11} = (\Sigma_k - AB^{-1}A')^{-1},
C_{12} = -\Sigma_k^{-1}A(B - A'\Sigma_k^{-1}A)^{-1},
C_{21} = -B^{-1}A'(\Sigma_k - AB^{-1}A')^{-1},
C_{22} = (B - A'\Sigma_k^{-1}A)^{-1}.
\]

Therefor the lower bound of order \( k + 1 \) is expressed as,

\[
K_{r,k+1}(\theta) = \sup[\lambda_k \ h_{k+1}] \Sigma_{k+1}^{-1} \begin{bmatrix}
\lambda_k \\
h_{k+1}
\end{bmatrix},
\]

where \( h_{k+1} = (\tau(\Phi_{k+1}(\theta)) - \tau(\theta)1) \) is a row vector of size \( m_{k+1} \). Now substituting (4.9) in to (4.10) leads to,

\[
K_{r,k+1}(\theta) = \sup [\lambda_k C_{11} \lambda_k + h_{k+1}C_{21}\lambda_k + \lambda_k C_{12}h_{k+1} + h_{k+1}C_{22}h_{k+1}].
\]

Using the lemma presented by Miller (1981) we can write \( C_{11} \) as follow,

\[
C_{11} = \Sigma_k^{-1} + (I - \Sigma_k^{-1}AB^{-1}A')^{-1}\Sigma_k^{-1}AB^{-1}A'\Sigma_k^{-1},
\]

hence by substitution we have,

\[
K_{r,k+1}(\theta) = K_{r,k}(\theta) + \sup \left[ \lambda_k (I - \Sigma_k^{-1}AB^{-1}A')^{-1}\Sigma_k^{-1}AB^{-1}A'\Sigma_k^{-1}\lambda_k + h_{k+1}C_{21}\lambda_k + \lambda_k C_{12}h_{k+1} + h_{k+1}C_{22}h_{k+1} \right],
\]

where the second term in right hand side is positive and the inequality is satisfied.

4.5. Example. Suppose \( X_1, \ldots, X_n \) and \( Y_1, \ldots, Y_m \) be random samples of exponential distributions with means \( \theta_1 \) and \( \theta_2 \). Tong (1974, 1975) found the UMVUE of \( P(X < Y) = \frac{\theta_2}{\theta_1 + \theta_2} \) for the first time which is as follow,
\[ \hat{R} = \begin{cases} Q_1(n, m, \sum_{i=1}^{n} X_i; \sum_{j=1}^{m} Y_j); & \sum_{j=1}^{m} Y_j \leq \sum_{i=1}^{n} X_i, \\ Q_2(n, m, \sum_{i=1}^{n} X_i; \sum_{j=1}^{m} Y_j); & \sum_{j=1}^{m} Y_j > \sum_{i=1}^{n} X_i, \end{cases} \]

where,

\[ Q_1(a, b, u, v) = \left( \sum_{i=0}^{a-2} (-1)^i \frac{\Gamma(a)\Gamma(b)}{\Gamma(a-i)\Gamma(b+i+1)} \left( \frac{v}{u} \right)^{i+1} \right), \]

\[ Q_2(a, b, u, v) = \left( \sum_{i=0}^{b-1} (-1)^i \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+i)\Gamma(b-i)} \left( \frac{u}{v} \right)^i \right). \]

Computation of the variance of this estimator has been considered by many researchers. Kotz et al. (2003) computed the following expression for the UMVUE of \( \text{Var}(\hat{R}) \),

\[ \text{Var}(\hat{R}) = (\hat{R})^2 - \frac{(n-1)(n-2)(m-1)(m-2)}{n^2m^2n^{-1}X^{-1}} H(n, m, X, Y), \]

where, \( H(n, m, X, Y) \) is obtained by,

\[ H(n, m, X, Y) = \int \int \int_B \left( X - \frac{x_1 + x_2}{n} \right)^{n-3} \left( Y - \frac{y_1 + y_2}{m} \right)^{m-3} dx_1 dx_2 dy_1 dy_2, \]

and space \( B \) is expressed as,

\[ B = \{ (x_1, x_2, y_1, y_2) : x_1 + x_2 < nX, y_1 + y_2 < mY, \]
\[ 0 < x_1 < y_1, 0 < x_2 < y_2 \}. \]

Computing above integrals, is very difficult and should be computed by numerical methods.

In Table 4 we approximate the variance of UMVUE of \( P(X < Y) \) with Bhattacharyya bounds and bootstrap method.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( m )</th>
<th>( \rho = \frac{n_2}{\rho_2} )</th>
<th>( k = 3 )</th>
<th>( k = 4 )</th>
<th>Bootstrap (2000 Replications)</th>
<th>Bootstrap (10000 Replications)</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>5</td>
<td>0.25</td>
<td>0.01213</td>
<td>0.01224</td>
<td>0.009615</td>
<td>0.00954</td>
</tr>
<tr>
<td>5</td>
<td>10</td>
<td>0.75</td>
<td>0.01924</td>
<td>0.01926</td>
<td>0.020078</td>
<td>0.019865</td>
</tr>
<tr>
<td>10</td>
<td>10</td>
<td>1.00</td>
<td>0.01312</td>
<td>0.01312</td>
<td>0.013587</td>
<td>0.013492</td>
</tr>
<tr>
<td>10</td>
<td>10</td>
<td>0.50</td>
<td>0.01049</td>
<td>0.01050</td>
<td>0.007453</td>
<td>0.007357</td>
</tr>
<tr>
<td>5</td>
<td>10</td>
<td>0.25</td>
<td>0.00830</td>
<td>0.00832</td>
<td>0.008243</td>
<td>0.008291</td>
</tr>
</tbody>
</table>

It can be seen in Figure 4 that although the Cramer-Rao bound is weaker than Bhattacharyya bounds, but it still acts better than bootstrap approximations.
Figure 4. Approximating the variance of UMVUE of \( P(X < Y) \) with Cramer-Rao and Bhattacharyya bounds and Bootstrap method in exponential distribution \((n=m=5)\).

4.6. Example. Suppose \( X_1, \ldots, X_n \) be a sample of Pareto distribution with unknown parameters \( \alpha \) and \( \beta \). Asrabadi (1990) showed that the UMVUE of mean function in Pareto distribution is,

\[
T(X) = \frac{(n-1)! \bar{\beta}}{(\ln t - n \ln \bar{\beta})^{1-n}} \left( \frac{t}{\beta^n} - 1 - \sum_{i=1}^{n-2} \frac{(\ln t - n \ln \bar{\beta})^i}{i!} \right),
\]

where, \( \bar{\beta} = \min\{X_i\} \) and \( t = \prod_{i=1}^n X_i \).

Unfortunately, calculating the variance of this estimator has not been discussed yet and here we try to give the best approximation of it by comparing the Kshirsagar bounds with bootstrap method. In Table 5 we present the variance approximation for the UMVUE of the mean function in Pareto distribution by Kshirsagar bounds and bootstrap method.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( \alpha )</th>
<th>( \beta )</th>
<th>( K_{2.1} )</th>
<th>( K_{2.2} )</th>
<th>Bootstrap</th>
<th>Bootstrap</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>3</td>
<td>1</td>
<td>3.58847</td>
<td>4.52821</td>
<td>1.512415</td>
<td>3.79487</td>
</tr>
<tr>
<td>5</td>
<td>10</td>
<td>2</td>
<td>4.00102</td>
<td>10.21191</td>
<td>7.85186</td>
<td>8.36154</td>
</tr>
<tr>
<td>5</td>
<td>20</td>
<td>3</td>
<td>0.00009</td>
<td>0.00089</td>
<td>0.00012</td>
<td>0.00054</td>
</tr>
<tr>
<td>10</td>
<td>25</td>
<td>5</td>
<td>0.00301</td>
<td>0.02891</td>
<td>0.00652</td>
<td>0.00541</td>
</tr>
</tbody>
</table>
Figure 5 shows the importance of Kshirsagar bounds against the bootstrap method in approximating the variance of unbiased estimator of the mean in Pareto distribution. It can be seen that the bootstrap approximations are usually less than the actual amount of the variance, because they are less than the second order Kshirsagar bound.

![Graph showing the comparison between Kshirsagar bounds and bootstrap method.](image)

**Figure 5.** Approximating the variance of UMVUE of $P(X < Y)$ with Cramer-Rao and Bhattacharyya bounds and Bootstrap method in exponential distribution ($n=m=5$).

5. Conclusion

In this paper, we define the Bhattacharyya and Kshirsagar bounds in both one and multi parameter cases and compute these bounds for the variance of any unbiased estimator of the parameter function in some applicable distributions. Also, we compare the Bhattacharyya and Kshirsagar lower bounds with bootstrap method in some examples. We see that the Bhattacharyya and Kshirsagar bounds are closer to the exact amount of the variance of unbiased estimators than bootstrap method.

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References


DE- and $\text{EDP}_M$- compound optimality for the information and probability-based criteria

W. A. Hassanein*† and N. M. Kilany‡

Abstract

Several optimality criteria have been considered in the literature as information-based criteria. The probability-based criteria have been recently proposed for maximizing the probability of a desired outcome. However, designs that are optimal for the information-based criteria may be inadequate for probability-based criteria. This paper introduces the DE- and $\text{EDP}_M$ – optimum designs for multi aims of optimality for Generalized Linear Models (GLMs). An equivalence theorem is proved for both compound criteria. Finally, two numerical examples are given to illustrate the potentiality of the proposed compound criteria.

Keywords: Optimum design, E-optimality, D-optimality, P-optimality, Compound criteria, Generalized linear models.

Mathematics Subject Classification (2010): 62Kxx; 62K05.

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1. Introduction

An optimality criterion is a criterion, which summarizes how good the design is, and it is maximized or minimized by an optimal design. Information-based criteria is one of the popular types of optimality criteria that related to the Fisher information matrix of the design. This type included many common optimality criteria such as; D-, G-, I-, A- and E-optimality.

The most important and popular design criterion for parameter estimation is D-optimality. It has been central to work on optimum experimental designs. Several publications on D-optimality can be seen in Atkinson et al. [3]. D-optimal designs are mainly

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intended to obtain efficient parameter estimation by the way of minimizing the generalized variance of the estimated regression coefficients by maximizing the determinant of the Fisher information matrix.

E-optimal design is devoted to minimize the maximum variance of all possible normalized linear combinations of the parameter estimation via maximizing the smallest eigenvalue of the information matrix.

Most of the literature concentrates on D-optimal designs but much less attention has been paid to E-optimal designs in nonlinear regression models (see Dette and Haines [11]; Dette and Wong [13]. Dette et al. [12] established that in the exponential regression models the E-optimal designs are usually more efficient for estimating individual parameters than D-optimal designs. Moreover, E-optimal designs usually behave substantially more reliably with respect to minimize the variances of the parameter estimates than do D-optimal designs. However, the problem of determining E-optimal designs is substantially harder than the D-optimal design problem.

The probability-based optimality criterion was initially introduced by McGree and Eccleston [13] that maximizing a probability of a particular event that assess an importance to the experimenter. Moreover, the DP- compound optimality criterion was proposed and discussed.

Some designs could be adequate for optimality criterion but inadequate for others hence, the motivation of constructing compound criteria is to satisfy multi objective aims of optimality. Many authors have developed optimality criteria which are applicable to the multiple objective problems (for example Clyde and Chaloner [6], McGree et al. [20], Atkinson [2], Denman et al. [8], Kilang[17], Kilang et al.[18] and Mwan et al.[21]).

The main objective of this paper is to construct new compound criteria via E-, D-, and P_M-optimality criterion to achieve the multi optimality problem of efficient parameter estimation, minimizing the maximum variance of all possible normalized linear combinations of the parameter estimation and obtaining the maximum probability of a desired outcome.

The paper is organized as follows; Section 2 is devoted to represent the optimum design background. In Section 3, E-, D-, P_M- and D_P_M-optimum designs are recalled. Section 4 is dedicated to propose the DE-optimality and EDP_M-optimality criteria. The equivalence theorem is stated and proved for both. Finally, Section 5 is devoted to introduce the applications for the offered criteria.

2. Optimum Design Background

Throughout this paper, the generalized linear models (GLMs) are considered. GLMs extend normal theory of regression to encompass non-normal response distributions belonging to the one-parameter exponential family. As well as the normal, this includes gamma, Poisson, and binomial distributions, all of which are important in the analysis of data. GLMs relate the random term (the independent response Y) to the systematic term to the linear predictor \(X\theta\) via a link function \(g(\cdot)\), see Agresti [1].

Consider the generalized linear model GLMs

\[ g(E(y)) = X\theta \]

Three components are involved:

1. **Random component**, which describes the response variable \(y\) and its probability distribution. The observations of \(y = (y_1, \ldots, y_n)^T\) are independent.
2. **A link function** \(g(\cdot)\) that is applied to each component of \(E(y)\).
3. **Linear Predictor** is \(X\theta\) for the parameter vector \(\theta = (\theta_1, \ldots, \theta_p)^T\) and a \(n \times p\) model matrix \(X\) involved \(p\) explanatory variables for \(n\) observations.
GLMs are commonly used to model binary or count data. Some common link functions are used such that the identity, logit, log and probit link to induce the traditional linear regression, logistic regression, Poisson regression models.

An approximate (continuous) design is represented by the probability measure $\xi$ over $\delta$. If the design has trials at $n$ distinct points in $\delta$, it can be written as

$$\xi = \begin{pmatrix} x_1 & x_2 & \ldots & x_n \\ w_1 & w_2 & \ldots & w_n \end{pmatrix}$$

A design $\xi$ defines, for $i = 1, \ldots, n$, the vector of experimental conditions $x_i \in \chi$ related to $y_i$, where $\chi$ is a compact experimental domain and the experimental weights $w_i$ corresponding to each $x_i$, where $\sum_{i=1}^{n} w_i = 1$. The design space can be then expressed as

$$\delta = \{ \xi_i \in X^n \times [0,1] : \sum_{i=1}^{n} w_i = 1 \}$$

The cornerstone in optimal design is the Fisher information matrix. The Fisher information matrix $M(\theta, \xi)$ is defined as

$$M(\theta, \xi) = -E[\frac{\partial^2 l(\theta; y)}{\partial \theta \partial \theta^T}]$$

where $l(\theta; y)$ is the log-likelihood function. The inverse of $M(\theta, \xi)$ is the variance-covariance matrix of the unbiased parameter $\theta$. From this point, $M(\theta, \xi)$ is used to measure the amount of information that $y$ carries about the parameter $\theta$.

Due to Atkinson et al. [3], for the continuous design $\xi$, the information matrix is

$$M(\theta, \xi) = \int_{\chi} f(x) f^T(x) \xi(x) \, dx = \sum_{i=1}^{n} w_i f(x_i) f^T(x_i)$$

where $f^T(x_i)$ is the $i^{th}$ row of $X$.

Consider a Bernoulli random variable $b_i$. The likelihood for it is $L(\theta; b_i) = \pi_{b_i} (1 - \pi_{b_i})^{1-b_i}$, $b_i = 0, 1$. For logistic link function, $\log(\frac{\pi_i}{1-\pi_i}) = x_i \theta$ where $E(\pi_i) = b_i$. In the case of logistic model, the Fisher information matrix $M(\theta, \xi)$ is defined as

$$M(\theta, \xi) = X^T WX$$

where $W$ is diag of $(w_1 \pi_1 (1 - \pi_1) \ldots \ldots w_n \pi_n (1 - \pi_n))$.

3. E-, D-, P$_M$- and DP$_M$-Optimality

3.1. E-optimality. E-optimality was firstly introduced by Ehrenfeld [14]. Heiligers [15] derived the E-optimal polynomial regression designs and presented several numerical examples for some efficiency functions. Pakelsheim and Studden [22] determined the E-optimal design for the polynomial regression model on the interval [-1,1] where the variances of different observations are assumed to be constant and also investigated the relationship between E- and c-optimality. Dette [9] generalized the results of Pakelsheim and Studden [22] for polynomial regression models with non-constant variances proportional to specific functions. Dette and Studden [10] studied the geometry of E-optimality. E-optimal designs for polynomial regression without intercept was introduced by Chang and Heiligers [5], also E-optimal designs for polynomial spline regression were presented by Heiligers [16].

The E-optimality criterion determines the design such that the minimal eigenvalue, say $\lambda_{min}(M(\theta, \xi))$, of the information matrix $M(\theta, \xi)$ is maximal. This corresponds
to the minimization of the worst variance of the least squares estimator for the linear combination of parameter estimation. It can be expressed as the following form
\[
\Phi_E(\xi) = \max_{\lambda_{\text{min}}(M(\theta, \xi))} \lambda_{\text{min}}(M(\theta, \xi))
\]

Following Pakelsheim and Studden [22], the equivalence theorem for E-optimal design is stated that \(\xi_E^*\) is the E-optimum design if and only if there exists a nonnegative definite matrix \(A^*\) such that \(\text{tr} A^* = 1\) and,
\[
\max_{\lambda_{\text{min}}(M(\theta, \xi))} f^T(x) A^* f(x) \leq \lambda_{\text{min}}(M(\theta, \xi))
\]
The matrix \(A^*\) can be represented as \(A^* = \sum_{i=1}^s k_i p_i p_i^T\), where \(s\) is the multiplicity of the minimal eigenvalue, \(k_i \geq 0\), \(\sum_{i=1}^s k_i = 1\), \(\{p_i\}_{i=1,2,\ldots,s}\) is a system of orthonormal eigenvectors corresponding to the minimal eigenvalue. The E-efficiency of a design \(\xi\) relative to the optimum design \(\xi_E^*\) is given by

\[(3.1) \quad E_{f E}(\xi) = \frac{\lambda_{\text{min}}(M(\theta, \xi))}{\lambda_{\text{min}}(M(\theta, \xi_E^*))}\]

3.2. D-optimality. D-optimality is the vital design criterion, introduced by Wald [23], which interested of the efficient parameter estimates. The idea of D-optimality depends on maximization of logarithm the determinant of the information matrix \(M(\theta, \xi) \log |M(\theta, \xi)|\), or equivalently, minimizes logarithm determinant of the inverse of information matrix, \(\log |M^{-1}(\theta, \xi)|\). Hence minimizes the generalized variance of \(\hat{\theta}\), the BLUE of \(\theta\) is obtained.

A design \(\xi_D\) is a D-optimum design iff \(d(x, \xi_D) = q, x \in \chi\), where
\[
d(x, \xi_D) = f^T(x) M^{-1}(\theta, \xi_D^*) f(x)
\]
and \(q\) is the number of parameters for each model. The D-efficiency of any design \(\xi\) is given by

\[(3.2) \quad E_{f D}(\xi) = \left( \frac{|M(\theta, \xi)|}{|M(\theta, \xi_D)|} \right)^{1/q}\]

3.3. P\(_M\)-optimality. McGree and Eccleston [19] proposed two types of probability-based optimality criteria that applied for GLMs. One of the forms of P-optimality criteria is \(P_M\)-optimality criterion that defined as a maximization of the minimum probability of success. The form of this criterion is as follows:
\[
\Phi_{P_M}(\xi) = \min \{\pi_i(\theta, \xi)\}, \quad i = 1, 2, \ldots, n
\]
where, \(\pi_i(\theta, \xi)\) is the \(i\)-th probability of success given by \(\xi\).

Such a criterion seems useful in situations in which relatively high-expected number of successes are desired across all observations. This means, avoiding design points with a low to moderate probability of success.

A design \(\xi_{P_M}^*\) is a \(P_M\)-optimum design for high probability of success iff
\[
\psi_{P_M}(x, \xi_{P_M}^*) \leq 0, \quad x \in \chi,
\]
where
\[
\psi_{P_M}(x, \xi_{P_M}^*) = \frac{\Phi_{P_M}(x) - \Phi_{P_M}(\xi_{P_M}^*)}{\Phi_{P_M}(\xi_{P_M}^*)}
\]
is the directional derivative of \(\Phi_{P_M}(\xi)\). The \(P_M\)-efficiency of design \(\xi\) relative to the optimum design \(\xi_{P_M}^*\) is

\[(3.3) \quad E_{f P_M}(\xi) = \min \{\pi_i(\theta, \xi)\} \quad i = 1, 2, \ldots, n\]
3.4. DP_M-optimality. For the aim of obtaining efficient parameter estimation and maximizing the minimum probability of success, McGree and Eccleston [19] have proposed DP_M-optimality criterion to combine D- and P_M-optimality criteria. In order to obtain design for both D- and P_M-optimality, consider a maximization of a weighted product of the efficiencies:

\[(3.4) \quad \left( \frac{|M(\theta, \xi)|}{|M(\theta, \xi_D^*)|} \right)^{\alpha/q} \left( \frac{\min\{\pi_i(\theta, \xi_i)\}}{\min\{\pi_i(\theta, \xi_{P_M}^*)\}} \right)^{1-\alpha} \]

where, the coefficients \(0 \leq \alpha \leq 1\). Taking the logarithm of (3.4) yields,

\[(3.5) \quad \Phi_{DP_M}(\xi) = \frac{\alpha}{q} \log |M(\theta, \xi)| + (1-\alpha) \log \min\{\pi_i(\theta, \xi_i)\} \]

The terms containing \(\xi_D^*\) and \(\xi_{P_M}^*\) have been ignored, since they are constants when a maximization is taken over \(\xi\). A DP_M-optimum design, \(\xi_{DP_M}^*\), maximizes \(\Phi_{DP_M}(\xi)\).

The derivative function for \(\Phi_{DP_M}(\xi)\) is given by

\[\psi_{DP_M}(x, \xi_{DP_M}^*) = \frac{\alpha}{q} f^T(x) M^{-1}(\theta, \xi_{DP_M}^*) f(x) + (1-\alpha) \times \]

\[
\frac{\Phi_{P_M}(x) - \Phi_{P_M}(\xi_{DP_M}^*)}{\Phi_{P_M}(\xi_{DP_M}^*)}
\]

4. DE- and EDP_M- Compound Design Criteria

Several competing objectives may be relevant in the experimental design. The compound design criterion, which defined as a geometric weighted mean of efficiencies is contributed to achieve the possible requirement objectives.

In this section, we will introduce two new compound criteria; namely DE- and EDP_M-optimality. DE-optimality criterion aimed to obtain the dual goal of efficient parameter estimation and minimum variance. On the other hand, the EDP_M-optimality criterion can satisfy the triple objectives of DE-optimality criterion in addition to maximum probability. An approach to these design problems is to weight each criterion and find the design that optimizes the weighted average of the criteria.

4.1. DE-optimality. To combine D- and E-optimality, we need a common scale of comparison, as they are different completely in the behavior. In this case, the efficiencies of both criteria can be used. In other words, the weighted product of the efficiencies are maximized as

\[(4.1) \quad \left( \frac{|M(\theta, \xi)|}{|M(\theta, \xi_D^*)|} \right)^{\alpha/q} \left( \frac{\lambda_{\min}(M(\theta, \xi))}{\lambda_{\min}(M(\theta, \xi_D^*))} \right)^{1-\alpha} \]

where, the coefficients \(0 \leq \alpha \leq 1\). Taking logarithm of (4.1),

\[\frac{\alpha}{q} \log |M(\theta, \xi)| - \frac{\alpha}{q} \log |M(\theta, \xi_D^*)| + (1-\alpha) \log \lambda_{\min}(M(\theta, \xi)) - (1-\alpha) \log \lambda_{\min}(M(\theta, \xi_D^*)) \]

which can be reduced to

\[(4.2) \quad \Phi_{DE}(\xi) = \frac{\alpha}{q} \log |M(\theta, \xi)| + (1-\alpha) \log \lambda_{\min}(M(\theta, \xi)) \]

As the terms involving \(\xi_D^*\) and \(\xi_E^*\) are constants when a maximum is taken over \(\xi\). Design maximized \(\Phi_{DE}(\xi)\) are called DE-optimum and denoted by \(\xi_{DE}^*\). The equivalence theorem for DE-criterion can be stated as follows:
4.1. Theorem. A design $\xi_{DE}$ is DE-optimal if and only if it satisfies the following inequality,

$$\psi_{DE}(x, \xi_{DE}^*) \leq 1, x \in \chi$$

where the derivative function

$$\psi_{DE}(x, \xi_{DE}^*) = \frac{\alpha}{q} f^T(x) M^{-1}(\theta, \xi_{DE}) f(x) + (1 - \alpha) \frac{f^T(x) A^* f(x)}{\lambda_{\min}(M(\theta, \xi_{DE}^*)))}$$

Moreover, the upper bound of $\psi_{DE}(x, \xi_{DE}^*)$ is attained at the support points of the DE-optimum design.

Proof. Since $0 \leq \alpha \leq 1$, the criterion in (4.2) is a convex combination of two functions. The first one is D-optimality criterion which is concave optimality criterion. The second term is the logarithm of minimum eigenvalue of information matrix $M$. Since the information matrix $M = X^T X$ is real symmetric matrix, then its minimal eigenvalues can be written as follows:

$$\lambda_{\min}(M) = \min_{\|\nu\|=1} \langle M \nu, \nu \rangle$$

where, $\nu$ is a fixed vector and $\langle M \nu, \nu \rangle$ is a linear function of $M$. From the fact that the minimum of any family of linear functions is concave, thus $\lambda_{\min}(M)$ is concave function. Moreover, since $M$ is symmetric matrix with positive diagonal elements, then $M$ is positive definite matrix and therefore all its eigenvalues are positive. From convex analysis (see Boyd and Vandenberghe [4]), we can conclude that, $\log \lambda_{\min}(M(\theta, \xi))$ is concave function of concave design criterion. Thus, the ED-criterion is a convex combination of two concave functions and therefore satisfies the conditions of convex optimum design theory and the proof is done.

4.2. EDP$M$-Optimum Designs. The formula of EDP$M$-optimality can be derived using the weighted geometric mean of efficiencies design for E-, D- and P$M$-optimum design as follows:

$$\Phi_{EDP_M}(\xi) = \alpha (1 - \alpha) \log \lambda_{\min}(M(\theta, \xi)) + (\alpha - 1)^2 \log |M(\theta, \xi)| + \alpha \log(\min\{\pi_i(\theta, \xi)\})$$

The terms containing $\xi_E$, $\xi_D$ and $\xi_{P_M}$ have been ignored, since they are constants when a maximum is taken over $\xi$. Design maximizing $\Phi_{EDP_M}(\xi)$ is called EDP$M$-optimum design and denoted by $\xi_{EDP_M}^*$. This optimum design satisfies the following general equivalence theorem:

4.2. Theorem. For EDP$M$-optimal design, $\xi_{EDP_M}^*$, the following three statements are equivalent:
(1) A necessary and sufficient condition for a design \( \xi_{\text{EDP}_M} \) to be \( \text{EDP}_M \)-optimum is fulfillment of the inequality, \( \psi_{\text{EDP}_M}(x, \xi_{\text{EDP}_M}) \leq 1 \), \( x \in \chi \), where,

\[
\psi_{\text{EDP}_M}(x, \xi_{\text{EDP}_M}) = \alpha (1 - \alpha) \frac{f^T(x) A^f(x)}{\lambda_{\min}(M(\theta, \xi_{\text{EDP}_M}))} + \frac{(\alpha - 1)^2}{q} f^T(x) M^{-1}(\theta, \xi_{\text{EDP}_M}) f(x) + \alpha \left( \Phi_{P_M}(x) - \Phi_{P_M}(\xi_{\text{EDP}_M}) \right) \phi_{P_M}(\xi_{\text{EDP}_M})
\]

(4.6)

is the directional derivative of the criterion function (4.5).

(2) The upper bound of \( \psi_{\text{EDP}_M}(x, \xi_{\text{EDP}_M}) \) is attained at the points of the optimum design.

(3) For any non-optimum design \( \xi \), that is a design for which \( \Phi_{\text{EDP}_M}(\xi) < \Phi_{\text{EDP}_M}(\xi_{\text{EDP}_M}) \),

\[
\sup_{x \in \chi} \psi_{\text{EDP}_M}(x, \xi_{\text{EDP}_M}) > 1
\]

Proof. Since \( 0 \leq \alpha \leq 1 \) and the sum of coefficients \( \alpha (1 - \alpha) \), \( (\alpha - 1)^2 \) and \( \alpha \) equals one, the criterion in (4.5) is a convex combination of three functions. The first and the second one for E- and D- criterion, respectively, are concave function (see proof of Theorem 1). Since, the third function of the convex combination (4.5) is the logarithm of minimum probability of success and \( \pi_i(\theta, \xi) \geq 0 \), so that, \( \log(\min\{\pi_i(\theta, \xi)\}) \) is concave function. Thus, the \( \text{EDP}_M \)-criterion is a convex combination of three concave functions and therefore satisfies the conditions of convex optimum design theory. In addition, the upper bound of \( \psi_{\text{EDP}_M}(x, \xi_{\text{EDP}_M}) \) over \( x \in \chi \) is one achieved at the points of the optimum design because the terms in (4.6) have been scaled. Thus, the theorem has been proved.

5. Applications

In the following sections two separate illustrative examples are considered for logistic GLMs.

5.1. Application of the DE-Optimum Design. In this section, the DE - optimality criterion is applied to Logistic GLMs for binary data. By using the simulated designs (given in Corana et al. [7]), the DE - compound criterion can achieve the dual goal of obtaining efficient parameter estimation and minimizing the maximum variance of all possible normalized linear combinations of the parameter estimation.

The considering model has two main factor effects besides the interaction with initial parameter estimates \( \theta = [1, -2, 1, -1]^T \) as follows.

Consider the Logistic GLM;

\[
\log \left( \frac{\pi}{1 - \pi} \right) = 1 - 2x_1 + x_2 - x_1x_2
\]

(5.1)

DE-optimal designs and their D- and E-efficiencies for \( \alpha = 0.25, 0.5, 0.75 \), 1 are obtained and presented in Table 1.

Table 1 shows the design that maximize the DE-criterion. It can be noticed that there is little changes in the design points with high variation in design weights. Figure 1 illustrates the E- and D-efficiencies for \( \alpha = 0.25, 0.5, 0.75 \) and 1. The dot-dashed line represents the D-efficiency of the designs, and the solid line shows their E-efficiencies. The E-optimal design has a D-efficiency of 0.6383 and the D-optimal design has E-efficiency of 0.691145. By using the compound DE-criterion and compute \( \Phi_{DE}(\xi) \) corresponding...
Table 1. DE-optimum design and their E-and D- efficiencies for different values of $\alpha$.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$w_i$</th>
<th>$\pi$</th>
<th>$E_{eff}$</th>
<th>$D_{eff}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.25</td>
<td>1.000</td>
<td>-1.000</td>
<td>0.0835</td>
<td>0.2689</td>
<td>1</td>
<td>0.6383</td>
</tr>
<tr>
<td></td>
<td>0.8020</td>
<td>1.000</td>
<td>0.0999</td>
<td>0.3999</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>-1.000</td>
<td>-1.000</td>
<td>0.1983</td>
<td>0.7311</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>-0.3980</td>
<td>1.000</td>
<td>0.6182</td>
<td>0.9596</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>1.000</td>
<td>-1.000</td>
<td>0.1570</td>
<td>0.2689</td>
<td>0.953696</td>
<td>0.9054</td>
</tr>
<tr>
<td></td>
<td>1.000</td>
<td>1.000</td>
<td>0.1600</td>
<td>0.2889</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>-1.000</td>
<td>-1.000</td>
<td>0.2802</td>
<td>0.7311</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>-0.1059</td>
<td>1.000</td>
<td>0.4028</td>
<td>0.9103</td>
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</tr>
<tr>
<td>0.75</td>
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<td>1.000</td>
<td>0.2121</td>
<td>0.2689</td>
<td>0.802473</td>
<td>0.9864</td>
</tr>
<tr>
<td></td>
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<td>-1.000</td>
<td>0.2121</td>
<td>0.2689</td>
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</tr>
<tr>
<td></td>
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<td>-1.000</td>
<td>0.2740</td>
<td>0.7311</td>
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<td></td>
</tr>
<tr>
<td></td>
<td>0.0148</td>
<td>1.000</td>
<td>0.3017</td>
<td>0.8761</td>
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</tr>
<tr>
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<td>-1.000</td>
<td>0.2500</td>
<td>0.2689</td>
<td>0.691145</td>
<td>1</td>
</tr>
<tr>
<td></td>
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<td>0.2500</td>
<td>0.2689</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>-1.000</td>
<td>-1.000</td>
<td>0.2500</td>
<td>0.7311</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.0680</td>
<td>1.000</td>
<td>0.2500</td>
<td>0.8577</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

to those values of $\alpha$, we will prefer the optimality criterion with the largest common efficiency for the dual aim, i.e. choosing $\alpha = 0.5$, the E-efficiency is increased to 0.953696 and achieving a D-efficiency of 0.9054. The DE-optimal design is then

$$\xi^{*}_{DE} = \begin{pmatrix}
1.000 & -1.000 & 0.1570 \\
0.9669 & 1.000 & 0.1600 \\
-1.000 & -1.000 & 0.2802 \\
-0.1059 & 1.000 & 0.4028 \\
\end{pmatrix}$$

Figure 1. E- and D-efficiencies of DE-optimal designs for different values of $\alpha$. 
5.2. Application of EDP\textsubscript{M} - Optimum Design. In this section, the EDP\textsubscript{M} -optimality criterion is applied to Logistic GLMs for binary data. The EDP\textsubscript{M} -compound criterion can provide triple goals of obtaining efficient parameter estimation plus maximizing the minimum eigenvalue of the information matrix and maximizing the minimum probability of a desired outcome. For the GLM which considered in (5.1). Let us consider another simulated designs given in Corana et al. [7], the EDP\textsubscript{M} -optimal designs and their D-, E- and P\textsubscript{M}- efficiencies is obtained for $\alpha = 0.25, 0.5, 0.75, 1$. The results are shown in Table 2.
Table 2. EDP<sub>M</sub>-optimum designs and their E- D- and P<sub>M</sub>-efficiencies for different values of \( \alpha \)

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( w_i )</th>
<th>( \pi )</th>
<th>( E_{\text{eff}} )</th>
<th>( D_{\text{eff}} )</th>
<th>( P_{\text{M eff}} )</th>
</tr>
</thead>
<tbody>
<tr>
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<td>0.335688</td>
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<td>0.7360</td>
</tr>
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<td>0.2500</td>
<td>0.7311</td>
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<td></td>
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</tr>
<tr>
<td>-1.000</td>
<td>-1.000</td>
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<td>1.000</td>
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<td></td>
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</tr>
<tr>
<td>0.5</td>
<td>0.2369</td>
<td>-0.8374</td>
<td>0.2500</td>
<td>0.4718</td>
<td>0.712605</td>
<td>0.7155</td>
<td>0.6507</td>
</tr>
<tr>
<td>0.7043</td>
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<td>0.2500</td>
<td>0.4718</td>
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<tr>
<td>-0.1074</td>
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<td>0.2500</td>
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<tr>
<td>-1.0000</td>
<td>-1.000</td>
<td>0.2500</td>
<td>0.7311</td>
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<tr>
<td>0.75</td>
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<td>1.000</td>
<td>0.2500</td>
<td>0.8638</td>
<td>0.97064</td>
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<td>0.9016</td>
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<td>0.2500</td>
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<tr>
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<td>-1.000</td>
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<td>0.7311</td>
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<td></td>
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</tr>
<tr>
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<tr>
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<td>-1.000</td>
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</tr>
<tr>
<td>0.0680</td>
<td>1.000</td>
<td>0.2500</td>
<td>0.8577</td>
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</tr>
</tbody>
</table>

Searching for the most higher common efficiencies for the three criteria, it is found that at \( \alpha = 0.5 \), where the E-efficiency is 0.7126, D-efficiency is 0.7155 and P<sub>M</sub>-efficiency is 0.6507 as illustrated in Figure 2. Hence, the EDP<sub>M</sub> optimal design is then

\[
\xi_{EDP_M}^* = \begin{bmatrix}
0.2369 & -0.8374 & 0.2500 & 0.4718 \\
0.7043 & 1.000 & 0.2500 & 0.4718 \\
-0.1074 & 1.000 & 0.2500 & 0.9107 \\
-1.000 & -1.000 & 0.2500 & 0.7311 
\end{bmatrix}
\]

Figure 2. E- and D- and P<sub>M</sub> efficiencies of EDP<sub>M</sub>-optimal designs for different values of \( \alpha \)
6. Conclusion

Most experimenters are interested in designing the experiments, which satisfy different goals. This requires developments of the field of constructing the compound optimality criteria. Hence, in this paper, two compound criteria named by DE and EDP_M are proposed. They offered multi-objective optimality properties of having efficient parameter estimation, minimizing the maximum variance of all possible normalized linear combinations of the parameter estimation and obtaining the maximum probability of a desired outcome. By applying these designs on logistic GLM, the largest common efficiency for the multi aim described above is achieved which indicated the benefits of using the proposed compound criteria.

References


Agreement and adjusted degree of distinguishability for square contingency tables

Ayfer Ezgi Yılmaz*† and Tulay Saracbasi‡

Abstract

In square contingency tables, analysis of agreement between the row and column classifications is of interest. In such tables, kappa or weighted kappa coefficients are used to summarize the degree of agreement between two raters. In addition to investigate the agreement between raters for square contingency tables, category distinguishability should be considered. Because the kappa coefficient is insufficient to measure the category distinguishability, the degree of distinguishability is suggested to use. In practice, some problems have occurred with regards to the use of the degree of distinguishability. The aim of this study is to assess the agreement coefficient and degree of distinguishability in square contingency tables together. In this study, the adjusted degree of distinguishability is suggested to solve the problem of calculating the degree of distinguishability falls outside the defined range. A simulation study is performed to compare the proposed adjusted degree of distinguishability and the classical degree of distinguishability. Furthermore, interpretation levels for the degree of distinguishability are determined based on a simulation study. The results are discussed over numerical examples and simulation.

Keywords: Agreement, Degree of distinguishability, Kappa coefficient, Square contingency table.

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1. Introduction

Square contingency tables are often used in medical, sociology, and behavioral sciences. These tables may arise in different ways, such as: When a sample of individuals or subjects is cross-classified according to two essentially similar categorical variables; when samples of pairs of matched individuals or subjects are classified according to some categorical variable of interest; in panel studies where each individual or subject in a sample is classified according to the same criterion at two different points in time; in rating experiments in which a sample of individuals or subjects is rated independently by the same two raters into one of the categories [12].

In square contingency tables, analysis of agreement between the row and column classifications is of interest. Interrater agreement represents the extent to which different judges tend to assign exactly the same rating for each object [18]. The agreement between objects rated independently by two raters or in two different time points by the same rater is investigated with the agreement coefficients. The degree of agreement is assessed using Cohen’s kappa coefficient [4].

Even though the raters rate the items independently, there occurs correlation between their decisions. There are two main components of agreement [6, 19]:

1. Marginal homogeneity which corresponds to the differences in the marginal distributions of raters.
2. The category distinguishability which is the ability for raters to distinguish the categories.

In the agreement studies, it is necessary to determine if the categories of the table are distinguishable from one to another [14]. If the categories are indistinguishable, then there could occur some differences between raters’ perceptions. Different raters may understand the categories differently or the same rater may not distinguish the categories correctly. It is discussed that these two problems can occur because the raters may not be experts in their fields or it may be difficult to distinguish the categories. The measure to calculate the distinguishability level of the categories is called degree of distinguishability [6].

In practice, there occurs some problems to the use of the degree of distinguishability. The value of the measure falls outside the defined range in some tables. Furthermore, there is not any information about how to interpret the degree of distinguishability except the general one. In this article, the adjusted degree of distinguishability is suggested to solve the problem of calculating the degree of distinguishability falls outside the defined range. It is aimed to assess the agreement coefficient and the adjusted degree of distinguishability in square contingency tables together. A simulation study is performed to compare the proposed adjusted degree of distinguishability with the classical one. Furthermore, interpretation levels for the degree of distinguishability are determined based on a simulation study. The results are discussed over three numerical examples and simulation.

Agreement coefficients and degree of distinguishability are reviewed in Section 2. Section 3 presents the suggested adjusted degree of distinguishability. The simulation study results are summarized in Section 4. The illustrative examples are discussed in Section 5, followed by conclusion in Section 6.
2. Agreement Coefficients

There is a large literature on agreement coefficients. There are numerous agreement coefficients for each table structure or number of raters. The well-known agreement coefficient for nominal categories is Cohen’s kappa coefficient \([4]\). When the categories are ordinal, instead of kappa, Cohen’s weighted kappa coefficient is suggested for use \([5]\). Darroch and McCloud \([6]\) recommend the degree of distinguishability to be used in place of kappa.

### 2.1. Cohen’s Kappa and Weighted Kappa Coefficients

Consider two raters classify the objects from a population \(n\) on a \(R\) scale. Let \(n_{ij}\) denote the number of objects \((i, j = 1, 2, \ldots, R)\). The cell probabilities are \(p_{ij}\) and \(p_i\) indicates the \(i\)th row total probability, \(p_j\) indicates the \(j\)th column total probability of an \(R \times R\) contingency table. The kappa coefficient \(\kappa\) is calculated as

\[
\kappa = \frac{\sum_{i=1}^{R} p_{ii} - \sum_{i=1}^{R} p_i p_i}{1 - \sum_{i=1}^{R} p_i p_i}.
\]

For ordinal responses, instead of kappa, weighted kappa coefficient is suggested by Cohen (1968). The coefficient allows each \((i, j)\) cell to be weighted according to the degree of agreement between \(i\)th and \(j\)th categories \([16]\). The weighted kappa coefficient \(\kappa_w\) is calculated as

\[
\kappa_w = \frac{\sum_{i=1}^{R} \sum_{j=1}^{R} w_{ij} p_{ij} - \sum_{i=1}^{R} \sum_{j=1}^{R} w_{ij} p_i p_j}{1 - \sum_{i=1}^{R} \sum_{j=1}^{R} w_{ij} p_i p_j}
\]

where \(w_{ij}\) is the weight ranges \(0 \leq w_{ij} \leq 1\). The popular weights for weighted kappa are the linear and the quadratic weights shown in Equations (2.3) and (2.4), respectively \([3, 7]\).

- Linear weights:
  \[
  w_{ij} = 1 - \frac{|i - j|}{(R - 1)}
  \]
- Quadratic weights:
  \[
  w_{ij} = 1 - \frac{(i - j)^2}{(R - 1)^2}.
  \]

In the literature, there are several interpretations of \(\kappa\) coefficient. Landis and Koch \([10]\) define the agreement levels of kappa coefficient as: “<0.00” poor, “0.00-0.20” slight, “0.21-0.40” fair, “0.41-0.60” moderate, “0.61-0.80” substantial, and “0.81-1.00” almost perfect.

### 2.2. Degree of Distinguishability

Degree of distinguishability (DD) is suggested to investigate the ability of the raters to distinguish between two categories \([6]\). The category distinguishability is defined in terms of the following odds ratio.

\[
\tau_{ij} = \frac{n_{ii}n_{jj}}{n_{ij}n_{ji}}, \quad i < j.
\]

The degree of distinguishability \(\delta_{ij}\) of \(i\)th and \(j\)th categories is

\[
\delta_{ij} = 1 - \tau_{ij}^{-1}.
\]
where $0 \leq \delta_{ij} \leq 1$. When $\delta_{ij} \cong 1$, then there is a perfect distinguishability between these two categories. When $\delta_{ij} \cong 0$, then it is impossible to distinguish between these two categories and this is not a preferred situation in the studies.

3. The Adjusted Degree of Distinguishability

Darroch and McCloud [6] defined the degree of distinguishability between two categories ranges from 0 to 1. In the applications, the degree of distinguishability may be calculated outside the defined range as negative.

Carcinoma in situ of uterine cervix data, one of the most common data in agreement studies, is an illustrative example where this problem observed. The data is discussed by Holmquist et al. [9], Landis and Koch [11], Becker and Agresti [2], and Sarachasi [15]. In order to investigate the variability in the classification of carcinoma in situ of the uterine cervix, seven pathologists classified 118 slides into the 5 categories: (1) Negative, (2) Atypical squamous hyperplasia, (3) Carcinoma in situ, (4) Squamous carcinoma with early stromal invasion, and (5) Invasive carcinoma. Two of the seven pathologists are chosen and given in Table 1.

<table>
<thead>
<tr>
<th>Pathologist 1</th>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
<th>(4)</th>
<th>(5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>26</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(2)</td>
<td>20</td>
<td>6</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(3)</td>
<td>10</td>
<td>19</td>
<td>9</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(4)</td>
<td>5</td>
<td>5</td>
<td>11</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>(5)</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>3</td>
</tr>
</tbody>
</table>

Table 1. Independent classifications by two pathologists of most involved histological lesion

For Table 1, the degree of distinguishabilities are calculated and given in Table 2. The results in Table 2 show that the degree of distinguishability of (3) Carcinoma in situ and (4) Squamous carcinoma with early stromal invasion is $\delta_{34} = -0.21$ and the degree of distinguishability of (4) Squamous carcinoma with early stromal invasion and (5) Invasive carcinoma is $\delta_{45} = -4.11$ which fall outside of the defined range.

<table>
<thead>
<tr>
<th>$\delta_{12}$</th>
<th>$\delta_{13}$</th>
<th>$\delta_{14}$</th>
<th>$\delta_{15}$</th>
<th>$\delta_{23}$</th>
<th>$\delta_{24}$</th>
<th>$\delta_{25}$</th>
<th>$\delta_{34}$</th>
<th>$\delta_{35}$</th>
<th>$\delta_{45}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Estimate</td>
<td>0.94</td>
<td>0.98</td>
<td>0.79</td>
<td>0.99</td>
<td>0.84</td>
<td>0.15</td>
<td>0.97</td>
<td>-0.21</td>
<td>0.99</td>
</tr>
</tbody>
</table>

Table 2. The category distinguishabilities measures of carcinoma in situ of uterine cervix data

This well-known data illustrates the problem of the degree of distinguishability and shows the necessity of a new formulation. In this study, we proposed the adjusted degree of distinguishability (ADD) to calculate the distinguishabilities between adjacent categories.

We proposed the adjusted degree of distinguishability under two arguments. Firstly, when the category distinguishability is discussed, it should be considered that the distinguishability of $i$th and $j$th categories is equal to the distinguishability of $j$th and $i$th
categories. Secondly, if categories (1) and (2) are distinguishable and categories (2) and (3) are distinguishable, then it is reasonable if categories (1) and (3) are distinguishable as well. For this reason, it is sufficient to calculate degree of distinguishability for only adjacent categories instead of all the pairs.

The adjusted degree of distinguishability \((ADD)\) for \(i\) and \(i + 1\) categories is calculated as

\[
ADD_{i,i+1} = \begin{cases} 
1 - \tau_{i,i+1}^{-1} & \text{if } \tau_{i,i+1} \geq 1 \\
1 - \tau_{i,i+1} & \text{if } \tau_{i,i+1} < 1 
\end{cases}
\]

where \(0 \leq ADD_{i,i+1} \leq 1\), \(i = 1, 2, \ldots, (R - 1)\). The odds ratio for square contingency tables is

\[
\tau_{i,i+1} = \frac{n_{ii} n_{i+1,i+1}}{n_{i,i+1} n_{i+1,i}}.
\]

For Table 1, the adjusted degree of distinguishabilities are calculated and given in Table 3. The results in Table 3 show that the adjusted degree of distinguishability of (3) Carcinoma in situ and (4) Squamous carcinoma with early stromal invasion is \(ADD_{34} = 0.17\), (4) Squamous carcinoma with early stromal invasion and (5) Invasive carcinoma is \(ADD_{45} = 0.81\).

| Table 3. The adjusted degree of distinguishabilities of carcinoma in situ of uterine cervix data |
|---------------------------------|-----------|-----------|-----------|-----------|
| Estimate                        | ADD_{12}  | ADD_{23}  | ADD_{34}  | ADD_{45}  |
|                                 | 0.94      | 0.84      | 0.17      | 0.81      |

If the table contains sampling zeros, then the odds ratio is

\[
\tau_{i,i+1} = \frac{(n_{ii} + c)(n_{i+1,i+1} + c)}{(n_{i,i+1} + c)(n_{i+1,i} + c)}
\]

where \(c\) is a constant value that can be 0.20, 0.50, or a minimum value which is different from zero \([1]\).

4. Simulation Study

A simulation study is performed to compare the proposed adjusted degree of distinguishability with the classical one. It is also aimed to develop a table to interpret the adjusted degree of distinguishability.

To generate \(2 \times 2\) contingency tables, we used the method presented by Goktas and Isci \([8]\). Bivariate standard normal distribution is used. At the first step, two identically independently distributed random variables \((X_1\) and \(X_2\)) are generated. Equations (4.1) and (4.2) is used to generate two random variables \((X\) and \(Y\)) from bivariate normal distribution with certain correlation \((\rho)\).

\[
X = aX_1 + bX_2
\]
\[ Y = bX_1 + aX_2 \]

where

\[ a = \frac{\sqrt{1 + \rho} + \sqrt{1 - \rho}}{2}, \]

and

\[ b = \frac{\sqrt{1 + \rho} - \sqrt{1 - \rho}}{2}. \]

Then, \( X \) and \( Y \) variables are categorized into two equal intervals and crossed to have \( 2 \times 2 \) tables. The sample sizes \( (n) \) of the table are considered as 30, 50, 70, 100, and 300. \( \rho \) values are taken as 0.20, 0.50, and 0.80. The kappa coefficient, classical and adjusted degree of distinguishabilities are calculated for each table. All the results are based on 50,000 replications of each sample.

Table 4 shows the minimum, maximum values, median, mean, and standard errors of the classical and adjusted degree of distinguishabilities for different sample sizes and the different values of correlation. While some of the minimum values classical degrees of distinguishability are negative, \( ADD \) lies between 0 and 1. In that case, \( ADD \) should be used instead of \( DD \).

Table 4. The descriptive statistics of the classical and adjusted degree of distinguishabilities for different sample sizes and the different values of correlation

<table>
<thead>
<tr>
<th>( \rho )</th>
<th>( n )</th>
<th>Min</th>
<th>Med</th>
<th>Max</th>
<th>Mean</th>
<th>S.E.</th>
<th>Min</th>
<th>Med</th>
<th>Max</th>
<th>Mean</th>
<th>S.E.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.20</td>
<td>30</td>
<td>-29.18</td>
<td>0.4167</td>
<td>0.99</td>
<td>0.2643</td>
<td>0.0036</td>
<td>0.00</td>
<td>0.4557</td>
<td>0.99</td>
<td>0.4712</td>
<td>0.0011</td>
</tr>
<tr>
<td>50</td>
<td>-7.33</td>
<td>0.4000</td>
<td>0.96</td>
<td>0.2985</td>
<td>0.0021</td>
<td>0.00</td>
<td>0.4356</td>
<td>0.96</td>
<td>0.4268</td>
<td>0.0010</td>
<td></td>
</tr>
<tr>
<td>70</td>
<td>-5.55</td>
<td>0.4119</td>
<td>0.94</td>
<td>0.3336</td>
<td>0.0016</td>
<td>0.00</td>
<td>0.4271</td>
<td>0.94</td>
<td>0.4092</td>
<td>0.0010</td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>-2.72</td>
<td>0.4023</td>
<td>0.92</td>
<td>0.3549</td>
<td>0.0012</td>
<td>0.00</td>
<td>0.4092</td>
<td>0.92</td>
<td>0.3937</td>
<td>0.0009</td>
<td></td>
</tr>
<tr>
<td>300</td>
<td>-0.63</td>
<td>0.4023</td>
<td>0.79</td>
<td>0.3885</td>
<td>0.0006</td>
<td>0.00</td>
<td>0.4023</td>
<td>0.79</td>
<td>0.3905</td>
<td>0.0006</td>
<td></td>
</tr>
<tr>
<td>0.50</td>
<td>30</td>
<td>-8.00</td>
<td>0.7976</td>
<td>1.00</td>
<td>0.6843</td>
<td>0.0014</td>
<td>0.00</td>
<td>0.7576</td>
<td>1.00</td>
<td>0.7064</td>
<td>0.0010</td>
</tr>
<tr>
<td>50</td>
<td>-1.72</td>
<td>0.7656</td>
<td>0.99</td>
<td>0.7159</td>
<td>0.0009</td>
<td>0.00</td>
<td>0.7656</td>
<td>0.99</td>
<td>0.7189</td>
<td>0.0008</td>
<td></td>
</tr>
<tr>
<td>70</td>
<td>-0.81</td>
<td>0.7586</td>
<td>0.99</td>
<td>0.7266</td>
<td>0.0007</td>
<td>0.00</td>
<td>0.7586</td>
<td>0.99</td>
<td>0.7272</td>
<td>0.0007</td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>-0.28</td>
<td>0.7567</td>
<td>0.97</td>
<td>0.7338</td>
<td>0.0005</td>
<td>0.00</td>
<td>0.7567</td>
<td>0.97</td>
<td>0.7338</td>
<td>0.0005</td>
<td></td>
</tr>
<tr>
<td>300</td>
<td>0.34</td>
<td>0.7513</td>
<td>0.92</td>
<td>0.7453</td>
<td>0.0003</td>
<td>0.34</td>
<td>0.7513</td>
<td>0.92</td>
<td>0.7453</td>
<td>0.0003</td>
<td></td>
</tr>
<tr>
<td>0.90</td>
<td>30</td>
<td>-0.91</td>
<td>0.9441</td>
<td>1.00</td>
<td>0.9190</td>
<td>0.0004</td>
<td>0.00</td>
<td>0.9441</td>
<td>1.00</td>
<td>0.9191</td>
<td>0.0004</td>
</tr>
<tr>
<td>50</td>
<td>0.15</td>
<td>0.9394</td>
<td>1.00</td>
<td>0.9257</td>
<td>0.0003</td>
<td>0.15</td>
<td>0.9394</td>
<td>1.00</td>
<td>0.9257</td>
<td>0.0003</td>
<td></td>
</tr>
<tr>
<td>70</td>
<td>0.41</td>
<td>0.9385</td>
<td>1.00</td>
<td>0.9285</td>
<td>0.0002</td>
<td>0.41</td>
<td>0.9385</td>
<td>1.00</td>
<td>0.9285</td>
<td>0.0002</td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>0.62</td>
<td>0.9372</td>
<td>1.00</td>
<td>0.9301</td>
<td>0.0002</td>
<td>0.62</td>
<td>0.9372</td>
<td>1.00</td>
<td>0.9301</td>
<td>0.0002</td>
<td></td>
</tr>
<tr>
<td>300</td>
<td>0.78</td>
<td>0.9350</td>
<td>0.98</td>
<td>0.9320</td>
<td>0.0001</td>
<td>0.78</td>
<td>0.9350</td>
<td>0.98</td>
<td>0.9320</td>
<td>0.0001</td>
<td></td>
</tr>
</tbody>
</table>

While there is a medium correlation between raters where \( n = 300 \) and while there is high correlation between raters where \( n > 30 \), the classical and adjusted degrees of distinguishabilities are equal. The results in Table 4 show that when the correlation between raters increases, the classical and adjusted degrees of distinguishabilities also increase. While there is a low correlation and the sample size increases, the classical and adjusted degree of distinguishabilities decrease. However, while there is medium or high correlation, the classical and adjusted degree of distinguishabilities are not affected by
the sample sizes. As expected, when the sample size increases, standard error decreases.

The scatter plots of the classical and adjusted degree of distinguishabilities for different sample sizes and the different values of correlation are given in Figure 1. Figure 1 shows that the negative values of degree of distinguishabilities are relocated to $[0,1]$ interval. When the negative values of $DD$ diverge from 0, the values of $ADD$ converge to perfect agreement.

![Figure 1](image_url). The scatter plots of the classical and adjusted degrees of distinguishabilities for different values of sample size and correlation

The kappa coefficients calculated for the tables generated randomly are classified into six categories considering Landis and Koch [10] intervals. Then, the minimum, maximum values, and medians of the adjusted degree of distinguishabilities are calculated for each kappa interval. The results are summarized in Table 5 and Figure 2. The aim of this study is to investigate the kappa coefficient and the adjusted category distinguishability together. Besides, it is purposed to investigate the $ADD$ intervals according to the kappa intervals.
Table 5 shows that when there is a poor agreement and the correlation between raters decreases, ADD increases. Except for the poor agreement, when the correlation between raters increases, the value of ADD also increases and converges to 1.

| n   | ρ   | Min | Med | Max | Min | Med | Max | Min | Med | Max | Min | Med | Max | Min | Med | Max | Min | Med | Max | Min | Med | Max | Min | Med | Max |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 30  | 0.20| 0.04| 0.3330|0.97 | 0.00| 0.3750|0.80 | 0.60| 0.7270|0.93 | 0.84| 0.8750|0.97 | 0.95| 0.9650|0.99 | –   | –   | –   | –   | –   | –   | –   | –   |
|     | 0.50| 0.04| 0.2180|0.89 | 0.00| 0.4440|0.82 | 0.60| 0.7500|0.95 | 0.84| 0.9000|0.98 | 0.95| 0.9730|0.99 | 0.99| 0.9940|1.00 | –   | –   | –   | –   | –   | –   |
|     | 0.80| 0.04| 0.1270|0.48 | 0.00| 0.5120|0.78 | 0.60| 0.7960|0.95 | 0.84| 0.9210|0.98 | 0.95| 0.9760|0.99 | 0.99| 0.9950|1.00 | –   | –   | –   | –   | –   | –   |
| 50  | 0.20| 0.03| 0.2560|0.86 | 0.00| 0.3740|0.71 | 0.57| 0.6860|0.95 | 0.83| 0.8610|0.96 | 0.95| 0.9570|0.96 | –   | –   | –   | –   | –   | –   | –   | –   |
|     | 0.50| 0.03| 0.1540|0.63 | 0.00| 0.4710|0.71 | 0.57| 0.7410|0.91 | 0.83| 0.8800|0.99 | 0.95| 0.9600|0.99 | –   | –   | –   | –   | –   | –   | –   | –   |
|     | 0.80| –   | –   | –   | 0.15| 0.5410|0.60 | 0.57| 0.7970|0.91 | 0.83| 0.9200|0.99 | 0.95| 0.9670|0.99 | 0.99| 0.9940|1.00 | –   | –   | –   | –   | –   | –   | –   | –   |
| 70  | 0.20| 0.03| 0.2060|0.78 | 0.00| 0.3650|0.71 | 0.58| 0.6670|0.95 | 0.83| 0.8540|0.94 | –   | –   | –   | –   | –   | –   | –   | –   | –   | –   | –   | –   | –   |
|     | 0.50| 0.03| 0.1250|0.43 | 0.00| 0.4930|0.64 | 0.58| 0.7350|0.96 | 0.83| 0.8700|0.97 | 0.94| 0.9560|0.99 | –   | –   | –   | –   | –   | –   | –   | –   | –   |
|     | 0.80| –   | –   | –   | –   | 0.41| 0.5190|0.57 | 0.59| 0.8040|0.88 | 0.83| 0.9160|0.98 | 0.94| 0.9650|0.99 | 0.99| 0.9920|1.00 | –   | –   | –   | –   | –   | –   | –   | –   |
| 100 | 0.20| 0.02| 0.1490|0.73 | 0.00| 0.3710|0.65 | 0.57| 0.6500|0.85 | 0.83| 0.8480|0.92 | –   | –   | –   | –   | –   | –   | –   | –   | –   | –   | –   | –   | –   |
|     | 0.50| 0.03| 0.0850|0.22 | 0.00| 0.5100|0.62 | 0.57| 0.7370|0.92 | 0.83| 0.8630|0.96 | 0.94| 0.9520|0.97 | –   | –   | –   | –   | –   | –   | –   | –   | –   |
|     | 0.80| –   | –   | –   | –   | 0.62| 0.8660|0.85 | 0.83| 0.9190|0.97 | 0.94| 0.9590|0.99 | 0.99| 0.9910|1.00 | –   | –   | –   | –   | –   | –   | –   | –   | –   |
| 300 | 0.20| 0.02| 0.0748|0.39 | 0.00| 0.3888|0.58 | 0.57| 0.6025|0.79 | 0.83| 0.8440|0.92 | –   | –   | –   | –   | –   | –   | –   | –   | –   | –   | –   | –   | –   |
|     | 0.50| –   | –   | –   | 0.34| 0.5420|0.58 | 0.57| 0.7450|0.84 | 0.82| 0.8380|0.92 | –   | –   | –   | –   | –   | –   | –   | –   | –   | –   | –   | –   |
|     | 0.80| –   | –   | –   | –   | –   | –   | 0.78| 0.8150|0.83 | 0.82| 0.9240|0.95 | 0.94| 0.9500|0.98 | –   | –   | –   | –   | –   | –   | –   | –   | –   |

Table 5. The minimum, maximum values, and medians of the adjusted degree of distinguishabilities for each kappa interval.
By means of the results in Table 5, it is possible to develop a mixed table of the kappa coefficient and adjusted degree of distinguishability. As the minimum, maximum values, and medians in Table 5 are considered, we suggested the interpretation levels for $ADD$. The results for $2 \times 2$ tables are summarized in Table 6.

<table>
<thead>
<tr>
<th>$\kappa$</th>
<th>$ADD$</th>
<th>Strength of $ADD$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.81-1.00</td>
<td>&gt;0.99</td>
<td>Perfect</td>
</tr>
<tr>
<td>0.61-0.80</td>
<td>0.94-0.99</td>
<td>Substantial</td>
</tr>
<tr>
<td>0.41-0.60</td>
<td>0.82-0.93</td>
<td>Moderate</td>
</tr>
<tr>
<td>0.21-0.40</td>
<td>0.57-0.81</td>
<td>Fair</td>
</tr>
<tr>
<td>&lt;0.20</td>
<td>0.00-0.56</td>
<td>Poor</td>
</tr>
</tbody>
</table>

In order to test the validity of the defined intervals, a simulation study is performed with 50,000 replications for different sample sizes and correlation levels. Then, the kappa coefficient and the adjusted degree of distinguishability are calculated for each replication. The percentages of correct classifications are calculated for each scenario and given in Table 7. The percentages of correct classifications in Table 7 change between 0.73 and 0.97.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\rho$</th>
<th>$0.20$</th>
<th>$0.50$</th>
<th>$0.80$</th>
</tr>
</thead>
<tbody>
<tr>
<td>30</td>
<td>0.85</td>
<td>0.78</td>
<td>0.73</td>
<td></td>
</tr>
<tr>
<td>50</td>
<td>0.94</td>
<td>0.85</td>
<td>0.78</td>
<td></td>
</tr>
<tr>
<td>70</td>
<td>0.96</td>
<td>0.87</td>
<td>0.81</td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>0.97</td>
<td>0.90</td>
<td>0.82</td>
<td></td>
</tr>
<tr>
<td>300</td>
<td>0.97</td>
<td>0.93</td>
<td>0.75</td>
<td></td>
</tr>
</tbody>
</table>

5. Illustrative Examples

In this section, we revisit three examples that will be used to illustrate the kappa coefficient, classical and adjusted degrees of distinguishabilities.

**Example 1:** To illustrate the calculation of kappa coefficient and adjusted degrees of distinguishability, let us consider the $2 \times 2$ contingency tables in Table 8 and 9. The data is taken from Shoukri [16] who examined 197 patients with prostate cancer. A modified TNM (tumor, node, metastasis) staging system is used to categorized MRI (magnetic resonance imaging), ultrasound, and pathological finding.
Table 8. Ultrasonography vs. pathological analysis for prostate cancer differentiation

<table>
<thead>
<tr>
<th>Stage in ultrasound</th>
<th>Stage in pathological study</th>
<th>Advanced</th>
<th>Localized</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Advanced</td>
<td></td>
<td>45</td>
<td>50</td>
<td>95</td>
</tr>
<tr>
<td>Localized</td>
<td></td>
<td>60</td>
<td>90</td>
<td>150</td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td>105</td>
<td>140</td>
<td>245</td>
</tr>
</tbody>
</table>

Table 9. MRI vs. pathological analysis for prostate cancer differentiation

<table>
<thead>
<tr>
<th>Stage in MRI</th>
<th>Stage in pathological study</th>
<th>Advanced</th>
<th>Localized</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Advanced</td>
<td></td>
<td>51</td>
<td>28</td>
<td>79</td>
</tr>
<tr>
<td>Localized</td>
<td></td>
<td>30</td>
<td>88</td>
<td>118</td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td>81</td>
<td>116</td>
<td>197</td>
</tr>
</tbody>
</table>

For Table 8 and 9, kappa coefficients are calculated as 0.07 and 0.39, respectively. The adjusted degree of distinguishabilities are calculated as 0.26 and 0.81.

While it is possible to infer a slight agreement between pathological and ultrasound results, it can be said that the distinguishability of the advance and localized categories is also at a slight level. While it is possible to infer a fair agreement between pathological and MRI results, it can be said that the distinguishability of the advance and localized categories is also at a fair level. Furthermore, when the pathological analysis results accept as reference, it can be said that MRI is more able to distinguish the categories than ultrasound.

Example 2: The radiographs of each of 60 patients are shown to two groups of doctors (two trauma surgeons and two radiologists). The data is taken from Oh [13]. To illustrate the calculation of linearly weighted kappa coefficient, classical and adjusted degrees of distinguishabilities, we consider the $4 \times 4$ ordered square contingency table in Table 10.

Table 10. The ratings given by Trauma surgeons and radiologists

<table>
<thead>
<tr>
<th>Trauma surgeons</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3</td>
<td>15</td>
<td>1</td>
<td>2</td>
<td>21</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>11</td>
<td>13</td>
<td>1</td>
<td>26</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>5</td>
<td>4</td>
<td>2</td>
<td>12</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>Total</td>
<td>5</td>
<td>31</td>
<td>9</td>
<td>5</td>
<td>60</td>
</tr>
</tbody>
</table>

For Table 10, linearly weighted kappa coefficient is calculated as 0.11. The classical and adjusted degrees of distinguishabilities are:
602

<table>
<thead>
<tr>
<th></th>
<th>0-1</th>
<th>1-2</th>
<th>2-3</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>DD</strong></td>
<td>0.42</td>
<td>-0.43</td>
<td>-0.67</td>
</tr>
<tr>
<td><strong>ADD</strong></td>
<td>0.42</td>
<td>0.30</td>
<td>0.40</td>
</tr>
</tbody>
</table>

While it is possible to infer a slight agreement between doctors’ decisions, it is possible to infer poor distinguishabilities of all the pairs of categories. Furthermore, the distinguishabilities of the adjacent categories are homogenous.

**Example 3**: 190 patients’ slides with advanced and nonadvanced adenomas is identified in a case-control study of adenomatous polyps conducted in NYC colonoscopy-based practices. The slides were classified into 5 categories in 1988 and 1998 by a pathologist. The data taken from Terry *et al* [17] is given in Table 11.

<table>
<thead>
<tr>
<th>1998</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>8</td>
<td>13</td>
<td>4</td>
<td>1</td>
<td>1</td>
<td>27</td>
</tr>
<tr>
<td>2</td>
<td>9</td>
<td>16</td>
<td>12</td>
<td>2</td>
<td>0</td>
<td>39</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>13</td>
<td>8</td>
<td>1</td>
<td>1</td>
<td>24</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>19</td>
<td>12</td>
<td>9</td>
<td>6</td>
<td>48</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>6</td>
<td>11</td>
<td>6</td>
<td>27</td>
<td>52</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td>22</td>
<td>67</td>
<td>47</td>
<td>19</td>
<td>35</td>
<td>190</td>
</tr>
</tbody>
</table>

For Table 11, linearly weighted kappa coefficient is calculated as 0.38. The adjusted degrees of distinguishabilities are:

<table>
<thead>
<tr>
<th></th>
<th>1-2</th>
<th>2-3</th>
<th>3-4</th>
<th>4-5</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>ADD</strong></td>
<td>0.09</td>
<td>0.17</td>
<td>0.77</td>
<td>0.84</td>
</tr>
</tbody>
</table>

While it is possible to infer a fair agreement between doctors’ decisions, it is possible to infer poor category distinguishability of categories (1) and (2), and categories (2) and (3). While categories (3) and (4) are fairly distinguishable, categories (4) and (5) are moderately distinguishable. Furthermore, the distinguishabilities of the adjacent categories are non-homogenous.

Because there are poor distinguishabilities of categories (1) and (2), and categories (2) and (3), it is suggested to be reclassified the categories. We can reclassify the categories as 3 alternatives. Linearly weighted kappa coefficients and the adjusted degrees of distinguishabilities are calculated for the reclassified tables.
Alternative 1: (1+2), (3), (4), (5)

<table>
<thead>
<tr>
<th>ADD_{1+2,3}</th>
<th>ADD_{3,4}</th>
<th>ADD_{1,5}</th>
<th>κ_w</th>
</tr>
</thead>
<tbody>
<tr>
<td>Estimate</td>
<td>0.39</td>
<td>0.83</td>
<td>0.85</td>
</tr>
<tr>
<td>Level</td>
<td>Poor</td>
<td>Moderate</td>
<td>Moderate</td>
</tr>
</tbody>
</table>

Alternative 2: (1), (2+3), (4), (5)

<table>
<thead>
<tr>
<th>ADD_{1,2+3}</th>
<th>ADD_{2+3,4}</th>
<th>ADD_{4,5}</th>
<th>κ_w</th>
</tr>
</thead>
<tbody>
<tr>
<td>Estimate</td>
<td>0.57</td>
<td>0.79</td>
<td>0.85</td>
</tr>
<tr>
<td>Level</td>
<td>Fair</td>
<td>Fair</td>
<td>Moderate</td>
</tr>
</tbody>
</table>

The linearly weighted kappas increase to 0.40 after the reclassifications 1 and 2.

Alternative 3: (1+2+3), (4), (5)

<table>
<thead>
<tr>
<th>ADD_{1+2+3,4}</th>
<th>ADD_{4,5}</th>
<th>κ_w</th>
</tr>
</thead>
<tbody>
<tr>
<td>Estimate</td>
<td>0.83</td>
<td>0.85</td>
</tr>
<tr>
<td>Level</td>
<td>Moderate</td>
<td>Moderate</td>
</tr>
</tbody>
</table>

For the third alternative, the adjusted degree of distinguishabilities and linearly weighted kappa coefficient increase to moderate levels.

6. Conclusions

When working on square contingency tables, firstly the agreement between the row and column variables is investigated. The variables of a square contingency table can be possibly two different raters who rate the same subjects or two different time points which is rated by the same rater. In the agreement studies, it is expected that the decisions of the raters are correspond to each other. If the agreement between raters is not high enough, there could be many reasons. One of these reasons is that the raters cannot distinguish the categories and because of this cannot classify the subjects correctly. Incorrect classification may affect the level of the agreement. In that case, it will be useful to use the degree of distinguishability to detect if the categories are distinguishable or not.

In practice, there occurs some problems to the use of the degree of distinguishability. In this article, we purposed to solve the problem of calculating the degree of distinguishability falls outside the defined range and to discuss the terms agreement and category distinguishability together. We proposed to use adjusted degree of distinguishability instead of the classical one.

The simulation results show that the adjusted degrees of distinguishability increases when the the correlation between raters increases. Besides, when there is medium or high correlation, it is not affected by the sampling size changes. While there is a low correlation and the sample size increases, the adjusted degree of distinguishability decreases.

It is easy to interpret ADD by use of the classification in Table 6. If the distinguishability between the categories is less than “moderate” level, then it is proposed to combine the categories. However, because of the definition of the categories, sometimes it is not logical to combine the categories. In that case, it is reasonable to reclassify the categories and repeat the study.
References

Limit theorem for a semi - Markovian stochastic model of type (s,S)

Zulfiye Hanalioglu* and Tahir Khaniyev† ‡

Abstract
In this study, a semi-Markovian inventory model of type (s, S) is considered and the model is expressed by means of renewal-reward process \( X(t) \) with an asymmetric triangular distributed interference of chance and delay. The ergodicity of the process \( X(t) \) is proved and the exact expression for the ergodic distribution is obtained. Then, two-term asymptotic expansion for the ergodic distribution is found for standardized process \( W(t) \equiv \frac{2X(t)}{S - s} \). Finally, using this asymptotic expansion, the weak convergence theorem for the ergodic distribution of the process \( W(t) \) is proved and the explicit form of the limit distribution is found.

Keywords: Inventory model of type \((s, S)\), Renewal-reward process, Weak convergence, Asymmetric triangular distribution, Asymptotic expansion.

Mathematics Subject Classification (2010): Primary 60K15, Secondary 60K05, 60K20.

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1. Introduction
A number of very interesting problems arising in the theories of inventory, stock control, queuing theory, reliability, mathematical insurance, stochastic finance etc., can be expressed by means of renewal processes, renewal-reward processes, random walk processes and their modifications. There are many important theoretical results about these subjects in literature (Borovkov (1984), Brown and Solomon (1975), Feller (1971), Gihman and Skorohod (1975), Janssen and Leuwaarden (2007), Khaniyev et al. (2008, 2008), Karabuk University, Department of Actuarial Sciences and Risk Management, 78050, Karabuk, Turkey. Email: zulfiyyamammadova@karabuk.edu.tr
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‡ Corresponding Author.
2013), Lotov (1996), Rogozin (1964), etc.). The results of these studies have complex mathematical structures and they are not useful for applied problems.

To avoid this difficulty, in recent years, the asymptotic methods have started to be applied to these problems. There are several valuable studies about using asymptotic methods, as well (e.g., Janssen and Leuwaarden (2007), Lotov (1996), Khaniyev (2005), Khaniyev and Atalay (2010), Khaniyev and Mammadova (2006), Khaniyev et al. (2008, 2013)).

Lately, the inventory models of type (s,S) have been extensively considered and some of their characteristics have been investigated in the literature (see, e.g., Khaniyev and Atalay (2010), Khaniyev and Mammadova (2006), Khaniyev et al. (2013)). Especially, in the studies Khaniyev and Atalay (2010) and Khaniyev et al. (2013), an inventory model of type (s,S) with triangular distributed interference of chance is tackled. In Khaniyev and Atalay (2010), the weak convergence theorem for the considered process is proved and in Khaniyev et al. (2013), three-term asymptotic expansions for the moments of ergodic distribution are obtained. These results are not only remarkable from the theoretical point of view but are also very useful in the application. Unfortunately, in these both studies, discrete interference has a symmetric triangular distribution and it is assumed that the lead time is zero. Note that, the processes having this restricted properties cannot adequately express real world problems arising in applied sciences. It can be observed in the following example which is given by Khaniyev et al. (2013).

1.1. The Real-World Model. A company operating in the energy sector produces, stores, fills, and distributes liquefied petroleum gas (LPG). Domestic LPG distribution is carried out via pipelines and transported from the LPG production center (a city in Turkey) to the 30 dealers by tankers with the capacities of 22m$^3$ (approximately 10-11 tons) and 35m$^3$ (approximately 17-18 tons). The tankers are kept under surveillance with the GPS (Global Positioning Systems) 24 hours a day and 7 days a week. After delivering the needed amount of gas to the dealer, if more than 10% of the capacity of the tanker is left over, the tanker waits in its position until the next order of any dealer. Each dealer has a storage capacity of $S = 30$ m$^3$. Random amounts of LPG ($\eta_n$) are sold from these storage tanks at random times $T_n = \sum_{i=1}^{n} \xi_i$. When at random moments $\tau_n$, $n \geq 1$ the level of LPG in the tank of the dealer falls below the control level $s = S/5$, a demand signal is automatically sent online to the production center. As a response to this demand, the nearest tanker to the dealer is directed to the demanding dealer. If there is no tanker near to the dealer, a full tanker is sent from the production center. For safety concerns (in order not to allow the gas pressure to reach its maximum value), the dealers, most of the time, fill about 85% of the capacity (S) of their tanks. However, with a low probability, by taking a risk the dealers fill their tanks to the full capacity when the need arises. On the other hand, even if the amount of gas in the tanker does not meet 85% of the dealer’s tank, the existing amount of gas in the tanker is loaded into the dealer’s tank.

The concept of filling the depot approximately 85% indicates the necessity of using an asymmetric triangular distributed interference of chance for modeling this problem. Therefore, in our opinion, the process that expresses the working principle of the depot can best be modelled as a stochastic process with an asymmetric triangular distributed interference of chance.

Moreover, the existing studies in literature, assumes that the lead time is equal to zero. However, in real world problems, it is not possible to refill a depot immediately everytime. This delay time may be due to transportation, trying to provide demands from suppliers, etc. Therefore, to solve certain real world problems, the following assumptions should be satisfied.
1. The random variable which represents the discrete interference of chance has an asymmetric triangular distribution.

2. Lead time takes positive values.

The studies existing in literature, unfortunately do not satisfy these assumptions. To fill this gap, in this study, a semi-Markovian inventory model of type \((s, S)\) is considered under these two assumptions. In Section 2, the stochastic process \(X(t)\) which expresses this model is constructed mathematically. Next, under some weak conditions, the ergodicity of the process is proved and the exact form of the ergodic distribution is found in Section 3. Finally, in Section 4, the standardized process \(W(t)\) is defined and two-term asymptotic expansion for the ergodic distribution of \(W(t)\) is obtained. Then, weak convergence theorem which is the main aim of the study is proved. Additionally, the explicit form of the limit distribution is found. Before stating these results, let us first define the process mathematically.

2. Mathematical Construction of the Process \(X(t)\)

Let \(\{\xi_n\}, \{\eta_n\}, \{\zeta_n\}, \{\theta_n\}, \ n = 1, 2, \ldots\) be four sequences of random variables defined on the same probability space \((\Omega, \mathcal{F}, P)\) such that variables in each sequences are independent and identically distributed. Additionally, \(\xi_n, \eta_n, \zeta_n\) and \(\theta_n\) are also mutually independent and can take only non-negative values. Denote the distribution functions of \(\xi_1, \eta_1, \zeta_1, \theta_1\) by

\[
\Phi(t) = P(\xi_1 \leq t); \quad F(x) = P(\eta_1 \leq x); \quad \pi(z) = P(\zeta_1 \leq z); \quad H(t) = P(\theta_1 \leq t),
\]

respectively. Define the renewal sequences \(\{T_n\}\) and \(\{Y_n\}\) using \(\{\xi_n\}\) and \(\{\eta_n\}\) as follows:

\[
T_0 = Y_0 = 0; \quad T_n = \sum_{i=1}^{n} \xi_i; \quad Y_n = \sum_{i=1}^{n} \eta_i; \quad n = 1, 2, \ldots
\]

and a sequence of integer-valued random variables \(\{N_n\}; \ n = 0, 1, 2, \ldots\) as:

\[
N_0 = 0; \quad N_1 = N_1(z) = \inf \{k \geq 1 : z - Y_k < s\}; \quad z \in [s, S];
\]

\[
N_{n+1} = N_{n+1}(\zeta_n) = \inf \{k \geq N_n + 1 : \zeta_n - (Y_k - Y_{N_n}) \leq s\}; \quad n = 1, 2, \ldots
\]

Define

\[
\tau_0 = 0; \quad \zeta_0 = z \in [s, S]; \quad \tau_n = \tau_n(\zeta_{n-1}) = \sum_{i=0}^{N_n} \xi_i;
\]

\[
\gamma_n = \tau_n + \theta_n, \quad n = 1, 2, \ldots
\]

\[
\nu(t) = \max \{n \geq 0 : T_n \leq t\}, \quad t > 0.
\]

Now let us construct the stochastic process \(X(t)\):

\[
X(t) = \sum_{n=0}^{\infty} \max \left\{s, \zeta_n - (Y_{\nu(t)} - Y_{N_n})\right\} I_{[\gamma_n, \gamma_{n+1})}(t).
\]

Here, indicator function \(I_A(t)\) of the set \(A\) is defined as

\[
I_A(t) = \begin{cases} 1, & t \in A \\ 0, & t \notin A \end{cases}
\]

Considered process \(X(t)\) is known as “Renewal-reward process with a discrete interference of chance” in literature. In this study, it is assumed that the random variable \(\zeta_1\) has asymmetric triangular distribution with parameters \((s, m, S)\). For this reason, this
process can be called “Renewal-reward process with an asymmetric triangular distributed interference of chance”. A sample trajectory of this process is shown in Figure 1.

![Figure 1. A trajectory of the process X(t)](image)

3. The Ergodicity of the Process X(t)

In order to study the stationary characteristics of the process, it is required to show that the process is ergodic under some weak conditions. For this aim, first of all, we are going to prove the ergodicity of the process.

3.1. Proposition. Let the initial sequences of the random variables \( \{\xi_n\}, \{\eta_n\}, \{\zeta_n\}, \{\theta_n\}, n = 1, 2, ... \) satisfy the following supplementary conditions:

(i) \( 0 < E(\xi_1) < +\infty \);
(ii) \( 0 \leq E(\theta_1) < +\infty \);
(iii) \( E(\eta_1) > 0 \);
(iv) \( \eta_1 \) is a non-arithmetic random variable;
(v) random variable \( \zeta_1 \) has asymmetric triangular distribution with parameters \( (s, m, S) \), \( 0 \leq s < m < S < +\infty \).

Then, the process \( X(t) \) is ergodic and the following relation holds with probability 1, for each bounded and measurable function \( f(x) \) \((f : [0, +\infty) \rightarrow R)\):

\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t f(X(u)) \, du = \frac{1}{E(\gamma_1)} \int_{z=s}^{S} \int_{v=s}^{S} \int_{t=0}^{t=+\infty} f(v) P_k \{ \gamma_1 > t; X(t) \in dv \} \, dt \, d\pi(z).
\]

Proof. The process \( X(t) \) belongs to a wide class of processes is called “stochastic processes with a discrete interference of chance”. This notion is first introduced to literature by A. N. Kolmogorov. For this class, the general ergodic theorem of type Smith’s “key renewal theorem” exists in the literature (Gihman and Skorohod (1975), p.243). According to this theorem, to prove the ergodicity of the processes with a discrete interference of chance, it is sufficient to show that the following two assumptions hold.

Assumption 1. Choosing a sequence of ascending random times is required such that the values of the process \( X(t) \) at these times form an embedded Markov chain
which is ergodic. For this purpose, it is sufficient to choose the sequence of random times \( \{\gamma_n, n \geq 0\} \), defined in Section 2. The values of the process \( X(t) \) at these times are equal to \( \zeta_n = X(\gamma_n), n \geq 1 \) which form an embedded Markov chain. In our case, the embedded Markov chain \( \{\zeta_n\}, n \geq 1 \) is ergodic with a stationary distribution \( \pi(z) = \lim_{n \to \infty} P\{\zeta_n \leq z\} = P\{\zeta_1 \leq z\} \), because the random variables \( \{\zeta_n\}, n = 1, 2, \ldots \) are independent and identically distributed random variables in the interval \([s, S]\). Therefore, the first assumption of the general ergodic theorem (Gihman and Skorohod (1975)) is satisfied.

**Assumption 2.** The expected value of the times between successive stopping times \( \{\gamma_n\}, n = 1, 2, \ldots \) should be finite, that is \( E(\gamma_n - \gamma_{n-1}) < \infty, n = 1, 2, \ldots \). For this aim, it is sufficient to prove that

\[
E(\tau_1(z)) = E((\tau_1(z) + \theta_1)) = E(\tau_1(z)) + E(\theta_1) < \infty;
\]

\[
E(\gamma_n - \gamma_{n-1}) = E(\gamma_1(\zeta_1)) = E(\tau_1(\zeta_1)) + E(\theta_1) < \infty; \quad n = 2, 3, \ldots
\]

By the conditions of Proposition 3.1, \( E(\xi_1) < \infty \) and \( E(\theta_1) < \infty \). From Wald identity (Feller (1971)), it is possible to prove that \( E(\tau_1(z)) = E\left(\sum_{i=1}^{N_1(z)} \xi_i\right) = E(\xi_1)E(\xi_1) \). Note that, \( E(N_1(z)) \equiv U_\eta(z-s) < \infty \) for each finite \( z \) (Feller (1971), p.185). Here, \( U_\eta(x) \) is a renewal function generated by the sequence of the random variables \( \{\eta_n\}, n \geq 1 \). Therefore, \( E(\tau_1(z)) < \infty \). At the same time, the renewal function \( U_\eta(x) \) is a non-decreasing function. Therefore, for each \( z \in [s, S] \), \( U_\eta(z-s) < \infty \) is provided. Hence, we have the following relation:

\[
E(\tau_1(\zeta_1)) = E(\xi_1) \int_s^S U_\eta(z-s) d\pi(z) \leq E(\xi_1)U_\eta(S-s) < \infty
\]

Thus, for each \( n = 2, 3, \ldots; \) \( E(\gamma_n - \gamma_{n-1}) < \infty \). This shows that Assumption 2 is also satisfied. Thereby, the process \( X(t) \) is ergodic and the relation in Eq. (3.1) holds. This concludes the proof of Proposition 3.1.

From this theorem, many valuable results can be obtained. Some of them can be given as follows.

**3.2. Corollary.** Under the conditions Proposition 3.1, for each \( x \in [s, S] \), the exact expression of the ergodic distribution function of \( X(t) \) is given as follows:

\[
Q_X(x) = 1 - \frac{E(U_\eta(\xi_1-x))}{K + E(U_\eta(\xi_1-s))}
\]

Here, \( Q_X(x) \equiv \lim_{t \to \infty} P\{X(t) \leq x\} \) is the ergodic distribution of process \( X(t) \). Moreover, \( K = E(\theta_1)/E(\xi_1); \pi(z) = P(\zeta_1 \leq z) \) and

\[
\pi'(z) \equiv p(z) = \begin{cases} 
\frac{2(z-s)}{2S-z} & \text{for } s < x \leq m \\
\frac{2(m-x)+S-s}{2S-z} & \text{for } m < x \leq S
\end{cases}
\]

Let us denote \( \tilde{X}(t) \equiv X(t) - s \) and \( \tilde{\xi}_1 = \xi_1 - s \). Then, state the following corollary.

**3.3. Corollary.** Under the conditions of Proposition 3.1, the process \( \tilde{X}(t) \) is ergodic and the exact expression of its ergodic distribution \((Q_{\tilde{X}}(x))\) can be shown as follows:

\[
Q_{\tilde{X}}(x) = \lim_{t \to \infty} P\{\tilde{X}(t) \leq x\} = 1 - \frac{E(U_\eta(\tilde{\xi}_1-x))}{K + E(U_\eta(\tilde{\xi}_1-s))}; \quad x \in [0, S-s]
\]
4. Weak Convergence Theorem for the Ergodic Distribution of the Process $X(t)$

The aim of this section is to prove the weak convergence theorem for the ergodic distribution of standardized process $W(t) \equiv \bar{X}(t)/\beta$ where $\bar{X}(t) \equiv X(t) - s$ and $\beta \equiv (S - s)/2$, when $\beta \to \infty$. Denote the ergodic distribution of $W(t)$ with $Q_W(x)$. Then, the exact expression of $Q_W(x)$ can be written as follows ($x \in [0,2]$):

$$Q_W(x) \equiv \lim_{t \to \infty} P \{ W(t) \leq x \} = Q_X(\beta x) = 1 - \frac{E \left( U_\eta \left( \tilde{\xi}_1 - \beta x \right) \right)}{K + E \left( U_\eta \left( \tilde{\xi}_1 - x \right) \right)}.$$  

Here, $K = E(\theta)/E(\xi_1)$ is delay coefficient and random variable $\tilde{\xi}_1$ has the asymmetric triangular distribution with parameters $(0, m - s, S - s)$ and its probability density function $(\tilde{p}(z))$ is written as follows:

$$\tilde{p}(z) \equiv \tilde{\pi}'(z) = \pi'(s + z) = \begin{cases} \frac{z}{2\alpha \beta}, & 0 < z \leq 2\alpha \beta \\ \frac{2\beta - z}{2(\alpha - 1)\beta}, & 2\alpha \beta < x \leq 2\beta \end{cases},$$

where $\alpha = (m - s)/(S - s)$ and $\beta = (S - s)/2$.

Before proving the weak convergence theorem, define the following functions $A(z) = \int_0^z U_\eta(y)dy$ and $V(z) = \int_0^z yU_\eta(y)dy$, then give the following propositions.

**4.1. Proposition.** The Laplace transforms $\tilde{A}(\lambda)$ and $\tilde{V}(\lambda)$ of the function $A(z)$ and $V(z)$ can be represented as follows:

$$\tilde{A}(\lambda) = \frac{1}{\lambda^2 (1 - \varphi(\lambda))}; \quad \tilde{V}(\lambda) = \frac{1 - \varphi(\lambda) - \lambda \varphi'(\lambda)}{\lambda^3 (1 - \varphi(\lambda))^2}.$$  

Here, $\varphi(\lambda) = E(\exp(-\lambda \eta_1)); \quad \lambda > 0$.

**Proof.** Proof of the Proposition 4.1 can be derived from both definitions of $A(z), V(z)$ and properties of Laplace transform. \hfill \Box

**4.2. Proposition.** Assume that $m_3 \equiv E(\eta_3^3) < \infty$. Then, the following asymptotic expansions can be written, when $z \to \infty$:

$$A(z) = \frac{1}{2m_1} z^2 + \frac{m_2}{2m_1^2} z + \frac{A_1}{m_1} + o(1);$$

$$V(z) = \frac{1}{3m_1} z^3 + \frac{m_2}{4m_1^2} z^2 + o(z)$$

Here, $A_1 = m_{21}^2 - m_{31}/2$; $m_{k1} = m_k / (km_1)$; $m_k = E(\eta_1^k), \quad k = 1,2,...$

**Proof.** Since $m_3 < +\infty$ is satisfied, then the following asymptotic expansions for $\varphi(\lambda)$ and $\varphi'(\lambda)$ can be written, when $\lambda \to 0$ (Feller (1971)):

$$\varphi(\lambda) = E \left( e^{-\lambda \eta_1} \right) = 1 - \lambda m_1 + \frac{\lambda^2}{2} m_2 - \frac{\lambda^3}{6} m_3 + o(\lambda^3);$$

$$\varphi'(\lambda) = E(-\eta_1 e^{-\lambda \eta_1}) = -m_1 + \lambda m_2 - \frac{\lambda^2}{2} m_3 + o(\lambda^2).$$

Substituting Eq. (4.5) and Eq. (4.6) in Eq. (4.3) and applying Tauber-Abel Theorem to Eq. (4.3), the asymptotic expansions in Eq.(4.4) are obtained. This concludes the proof of Proposition 4.2. \hfill \Box
4.3. Lemma. In addition to the conditions of Proposition 3.1, let \( m_3 < \infty \) be also satisfied. Then, the asymptotic expansion of \( E \left( U_q \left( \tilde{z}_1 \right) \right) \) can be written as follows, when \( \beta \to \infty \):

\[
E \left( U_q \left( \tilde{z}_1 \right) \right) = \frac{2 + 2\alpha}{3m_1} \beta + \frac{m_2}{2n_1^2} + o \left( \frac{1}{\beta} \right)
\]

Here, \( m_k = E(\eta^k_1); \ k = 1, 2, 3. \)

Proof. Present \( E \left( U_q \left( \tilde{z}_1 \right) \right) \) as follows:

\[
E \left( U_q \left( \tilde{z}_1 \right) \right) = \int_0^{2\beta} U_q(z)\tilde{p}(z)dz = \int_0^{2\alpha\beta} U_q(z)\tilde{p}(z)dz + \int_{2\alpha\beta}^{2\beta} U_q(z)\tilde{p}(z)dz
\]

Using Proposition 4.2, calculate the first integral in Eq.(4.8) as follows:

\[
I_1 \equiv \int_0^{2\alpha\beta} U_q(z)\tilde{p}(z)dz = \frac{1}{2\alpha^2\beta^2} V(2\alpha\beta) = \left[ \frac{2\alpha^2}{3m_1} \beta + \frac{\alpha m_2}{2n_1^2} + o(1) \right]
\]

Now, with the help of Proposition 4.2, calculate the second integral in Eq.(4.8) as follows:

\[
I_2 \equiv \int_{2\alpha\beta}^{2\beta} U_q(z)\tilde{p}(z)dz = \frac{1}{2(1 - \alpha)\beta^2} [D(2\beta) - D(2\alpha\beta)]
\]

Here, \( D(x) \equiv \int_0^x U_q(z)(2\beta - z)dz \). From Proposition 4.2, \( D(2\beta) \) can be written as follows:

\[
D(2\beta) = 2\beta A(2\beta) - V(2\beta) = \frac{(2\beta)^3}{6m_1} + \frac{m_2(2\beta)^2}{4n_1^2} + \frac{2\beta A_1}{m_1} + o(\beta)
\]

Here, \( A_1 = m_2^2 - m_{31}/2; \ m_{k1} = m_k/(km_1); \ m_k = E(\eta^k_1), \ k = 1, 2,... \)

In a similar way, \( D(2\alpha\beta) \) can be presented as follows:

\[
D(2\alpha\beta) = 2\beta A(2\alpha\beta) - V(2\alpha\beta) = \frac{8\alpha^2(3 - 2\alpha)}{6m_1} + \frac{4\alpha m_2(2 - \alpha)}{4n_1^2} \beta^2 + \frac{2A_1}{m_1} \beta + o(\beta)
\]

Considering Eq.(4.11) and Eq.(4.12) into Eq.(4.10), the following asymptotic expansion can be obtained, when \( \beta \to \infty \):

\[
I_2 = \frac{1}{2(1 - \alpha)\beta^2} [D(2\beta) - D(2\alpha\beta)] = \frac{2(1 + \alpha - 2\alpha^2)}{3m_1} \beta + \frac{m_2(1 - \alpha)}{2n_1^2} + o(1)
\]

Substituting Eq.(4.9) and Eq.(4.13) in Eq. (4.8), the following asymptotic expansion is obtained, when \( \beta \to \infty \):

\[
E \left( U_q \left( \tilde{z}_1 \right) \right) = I_1 + I_2 = \frac{2 + 2\alpha}{3m_1} \beta + \frac{m_2}{2n_1^2} + o(1)
\]

Therefore, Lemma 4.3 is proved. \( \square \)

4.4. Lemma. Suppose that \( m_3 < +\infty \) is also satisfied in addition to the conditions of Proposition 3.1 Then, two-term asymptotic expansion for \( E \left( U_q \left( \tilde{z}_1 - \beta x \right) \right) \) can be written as follows, when \( \beta \to \infty \):

\[
E \left( U_q \left( \tilde{z}_1 - \beta x \right) \right) = \begin{cases}
\frac{8\alpha^2 + 8\alpha - 12\alpha x + x^3}{12m_1 \alpha} \beta + \frac{m_2(4\alpha - x^2)}{8m_1 \alpha} + o(1), & x \in (0, 2\alpha) \\
\frac{(2-x)^2}{12(1-\alpha)} \beta + \frac{m_2(2-x)^2}{8(1-\alpha)n_1^2} + o(1), & x \in (2\alpha, 2)
\end{cases}
\]

Here \( \alpha = (m - s)/(S - s) \).
Proof. For each $x \in (0, 2\alpha)$, write $E \left( U_n \left( \zeta_1 - \beta x \right) \right)$ as follows:

$$E \left( U_n \left( \zeta_1 - \beta x \right) \right) = J_{11}(x) + J_{12}(x)$$

Here,

$$J_{11}(x) \equiv \int_{\beta x}^{2\beta} U_n(z - \beta x) \hat{p}(z) dz; \quad J_{12}(x) \equiv \int_{2\alpha\beta}^{2\beta} U_n(z - \beta x) \hat{p}(z) dz;$$

First of all, using Proposition 4.2, calculate $J_{11}(x)$:

$$J_{11}(x) = \frac{1}{2\alpha\beta^2} \int_{\beta x}^{2\beta} U_n(z - \beta x) zdz$$

$$= \frac{1}{2\alpha\beta^2} \{ \beta x [ A((2\alpha - x)\beta) + V((2\alpha - x)\beta)] \}$$

$$= \frac{16\alpha^3 - 12\alpha^2 x + x^3}{12\alpha m_1} \beta + \frac{m_2(4\alpha^2 - x^2)}{8\alpha^2} + o(1)$$

Here, $A(z) = \int_0^z U_n(y)dy$ and $V(z) = \int_0^z y U_n(y)dy$. Now, with the similar method, calculate $J_{12}(x)$:

$$J_{12}(x) = \int_{2\alpha\beta}^{2\beta} U_n(z - \beta x) \hat{p}(z) dz = \frac{1}{2(1 - \alpha) \beta^2} \int_{2\alpha\beta}^{2\beta} U_n(z - \beta x) (2\beta - z) dz$$

$$= \int_{2\alpha\beta}^{2\beta} U_n(z - \beta x) dy$$

$$= \frac{1}{2(1 - \alpha) \beta^2} \{ \int_{(2\alpha - x)\beta}^{2\beta} U_n(y)(2\beta - \beta x - y) dy \}$$

$$- \int_{0}^{(2\alpha - x)\beta} U_n(y)(2\beta - \beta x - y) dy \}$$

$$= \frac{1}{2(1 - \alpha) \beta^2} [ B((2 - x)\beta) - B((2\alpha - x)\beta) ]$$

Here, $B(t) \equiv \int_1^t U_n(y)(2\beta - \beta x - y) dy$ for simplicity. With the help of Proposition 4.2, compute $B((2 - x)\beta)$ and $B((2\alpha - x)\beta)$, as follows:

$$B((2 - x)\beta) = (2\beta - \beta x) A((2 - x)\beta) - V((2 - x)\beta)$$

$$= \frac{(2 - x)^3}{6m_1} \beta^3 + \frac{m_2(2 - x)^2}{4m_1^2} \beta^2 + \frac{A_1}{m_1} \beta + o(\beta)$$

and

$$B((2\alpha - x)\beta) = (2 - x)\beta A((2\alpha - x)\beta) - V((2\alpha - x)\beta)$$

$$= \frac{(2\alpha - x)^2(6 - x - 4\alpha)}{6m_1} \beta^3 + \frac{m_2(2 - x)^3(4 - x - 2\alpha)}{4m_1^2} \beta^2$$

$$+ \frac{A_1(2 - x)}{m_1} \beta + o(\beta).$$

Here, $A_1 = \frac{m_2^2}{m_3} - \frac{m_{31}}{2}$; $m_{k1} = m_k / (km_1)$; $m_k = E(\eta_k^k)$, $k = 1, 2, ...$

By considering Eq.(4.19) and Eq.(4.20) in Eq.(4.18), $J_{12}(x)$ can be written as follows:

$$J_{12}(x) = \frac{1}{2(1 - \alpha) \beta^2} [ B((2 - x)\beta) - B((2\alpha - x)\beta) ]$$

$$= \frac{16\alpha^3 - 24\alpha^2 - 12\alpha^2 x + 24\alpha x - 12x + 8}{12(1 - \alpha)m_1} \beta + \frac{m_2(4\alpha^2 - x^2)}{8am_1^2} + o(1)$$
Hence, the first part of Eq.(4.15) holds. Now, we can obtain the second part of the Eq.(4.15) in a similar way. 
For each $x \in (2\alpha; 2)$, $\beta x \in (2\alpha \beta; 2\beta)$ holds. Then, using Proposition 4.2,
\[ E \left(U_\eta \left( \tilde{\zeta}_1 - \beta x \right) \right) \]
is calculated, as follows:
\[
E \left(U_\eta \left( \tilde{\zeta}_1 - \beta x \right) \right) = \int_{\beta x}^{2\beta} U_\eta(z - \beta x)\tilde{p}(z)dz \\
= \frac{1}{2(1 - \alpha)\beta^2} \int_{\beta x}^{2\beta} U_\eta(z - \beta x)(2\beta - z)dz \\
= \frac{1}{2(1 - \alpha)\beta^2} [(2 - x)\beta A((2 - x)\beta) - V((2 - x)\beta)] \\
= \frac{(2 - x)^3}{12(1 - \alpha)m_1} \beta + \frac{m_2(2 - x)^2}{8(1 - \alpha)m_1^2} + o(1).
\]
Here, $m_k = E(\eta_k^3)$, $k = 1, 2$. Thus, the second part of the Eq.(4.15) is obtained. 

With the help of the given propositions above, weak convergence theorem can be stated as follows.

**4.5. Theorem.** In addition to the conditions of Proposition 3.1, let $m_3 < \infty$ be also satisfied. Then, the following two-term asymptotic expansion for the ergodic distribution of $W(t)$ can be written, when $\beta \to \infty$, i.e.,
\[
Q_W(x) = \begin{cases}
R_1(x) + \frac{D_1(x)}{\beta} + o \left( \frac{1}{\beta} \right), & x \in (0; 2\alpha) \\
R_2(x) + \frac{D_2(x)}{\beta} + o \left( \frac{1}{\beta} \right), & x \in (2\alpha; 2)
\end{cases}
\]  
(4.22)

Here,
\[
\begin{align*}
R_1(x) &= \frac{12\alpha x - x^3}{8\alpha + (1 + \alpha)}; \\
R_2(x) &= 1 - \frac{(2 - x)^3}{8(1 - \alpha)^2}; \\
D_1(x) &= \frac{3m_2(4\alpha - x^2)}{8\alpha + (1 + \alpha)} - \frac{3(8\alpha^2 - 8\alpha - 12\alpha x + x^3)(Km_1 + m_2)}{16\alpha + (1 + \alpha)^2}; \\
D_2(x) &= \frac{3m_2(2 - x)^2 - 3(2 - x)^3(Km_1 + m_2)}{8(1 - \alpha)^2}; \\
K &= E(\eta_1)/E(\xi_1).
\end{align*}
\]

**Proof.** As in shown in Eq. (4.3), the exact expression for the ergodic distribution of the process $W(t)$ is as follows:
\[
Q_W(x) = 1 - \frac{E \left(U_\eta \left( \tilde{\zeta}_1 - \beta x \right) \right)}{K + E \left(U_\eta \left( \tilde{\zeta}_1 \right) \right)}
\]
(4.23)

Using the asymptotic expansion of $E \left(U_\eta \left( \tilde{\zeta}_1 \right) \right)$ in Eq.(4.14), the following asymptotic expansion can be written:
\[
\frac{1}{K + E \left(U_\eta \left( \tilde{\zeta}_1 \right) \right)} = \frac{3m_1}{2(1 + \alpha)\beta} \left[ 1 - \frac{3(2Km_1^2 + m_2)}{4(1 + \alpha)m_1} \frac{1}{\beta} + \frac{1}{2β} \right]
\]
(4.24)

With the help of the Eq.(4.15) and Eq.(4.24), asymptotic expansion for $Q_W(x)$ in Eq.(4.22) is found. Thus, the Theorem 4.5 is proved. 

Now, let us give the following proposition.
4.6. **Proposition.** Suppose that \( m_2 < \infty \) and \( K < \infty \) holds. Then, for each \( x \in (0; 2\alpha) \), the inequality \(|D_1(x)| \leq 5m_{21} + 4Km_1 < \infty \) and for each \( x \in (2\alpha; 2) \), the inequality \(|D_2(x)| \leq 3m_{21} + Km_1 < \infty \) are satisfied. Here, \( K = E(\theta_1)/E(\xi_1) \) is delay coefficient.

Proof. Using the inequality \(|a-b| \leq |a|+|b|\), since \( m_2 < \infty \); \( E(\theta_1) < \infty \) and \( 0 < E(\xi_1) < +\infty \) are satisfied, then the following inequalities can be written:

\[
|D_1(x)| \leq \frac{12\alpha m_{21}}{8\alpha(1+\alpha)} + \frac{3(8\alpha^2 + 8\alpha + 24\alpha^2 + 8\alpha^3)(Km_1 + m_{21})}{16\alpha(1+\alpha)} \\
\leq 5m_{21} + 4Km_1 < \infty;
\]

\[
|D_2(x)| \leq \frac{3m_{21}(2-x)^2}{8(1-\alpha^2)} + \frac{3(2-x)^3(Km_1 + m_{21})}{8(1-\alpha^2)} \leq 3m_{21} + Km_1 < \infty.
\]

Finally, let us give the following theorem which is the main goal of this study.

4.7. **Theorem.** (Weak Convergence Theorem) Assume that the conditions of Theorem 4.5 are satisfied. Then, the ergodic distribution \((Q_W(x))\) of \( W(t) \) weakly converges to limit distribution \( R(x) \), for each \( x \in (0; 2) \), when \( \beta \to \infty \), i.e.,

\[
\lim_{\beta \to \infty} Q_W(x) = R(x) = \begin{cases} R_1(x) & x \in (0, 2\alpha) \\ R_2(x) & x \in (2\alpha, 2) \end{cases}
\]

Here,

\[
R_1(x) = \frac{12\alpha x - x^3}{8\alpha(1+\alpha)}; \quad R_2(x) = 1 - \frac{(2-x)^3}{8(1-\alpha^2)}; \quad \alpha = \frac{m - s}{S - s}; \quad \beta = \frac{S - s}{2}.
\]

Proof. According to Lemma 4.3, \(|D_1(x)| < \infty \) and \(|D_2(x)| < \infty \) are satisfied. Then, from Theorem 4.5, the following inequalities can be obtained, when \( \beta \to \infty \):

a) For each \( x \in (0; 2\alpha) \)

\[
|Q_W(x) - R_1(x)| \leq \frac{|D_1(x)|}{\beta} + \left| \frac{1}{\beta} \right| \leq 2 \left( \frac{5m_{21} + 4Km_1}{\beta} \right).
\]

b) For each \( x \in (2\alpha; 2) \)

\[
|Q_W(x) - R_2(x)| \leq 2 \frac{|D_2(x)|}{\beta} \leq 2 \left( \frac{3m_{21} + Km_1}{\beta} \right).
\]

According to the conditions of Theorem 4.7, \( K < \infty \) and \( m_3 < \infty \). Therefore, the right hand sides of the inequalities in Eq.(4.26) and Eq.(4.27) are finite. Then, as \( \beta \to \infty \), the right hand sides of the inequalities in Eq.(4.26) and Eq.(4.27) converge to zero. Hence, the following relation holds:

\[
\lim_{\beta \to \infty} Q_W(x) = R(x) \equiv \begin{cases} R_1(x) & x \in (0, 2\alpha) \\ R_2(x) & x \in (2\alpha, 2) \end{cases}
\]

That is, as \( \beta \to \infty \), the ergodic distribution of the process \( W(t) \) weakly converges to the limit distribution \( R(x) \), for each \( x \in (0; 2) \). Thus, Theorem 4.7 is proved. \( \square \)
5. Conclusion

In this study, a semi-Markovian inventory model of type \((s, S)\) is considered. This model is expressed by means of the renewal-reward process \((X(t))\) with an asymmetric triangular distributed interference of chance. Ergodicity of this model is proved and the exact expression for the ergodic distribution function is found. Using the exact expression for the ergodic distribution of the process, two-term asymptotic expansion for the ergodic distribution is obtained and the weak convergence theorem is proved. As a result, the explicit form for the limit distribution function is found in Eq.(4.25). In the case when the interference has a symmetric triangular distribution, the parameter \(\alpha = 0.5\). Then, from Eq.(4.25), limit distribution \(R(x)\) can be extracted as follows:

\[
R(x) = \begin{cases} 
    x - \frac{x^3}{6} & ; \ x \in (0, 1] \\
    1 - \frac{(2-x)^3}{6} & ; \ x \in (1, 2) 
\end{cases}
\]

The result in Eq.(5.1) coincides with the limit distribution given in Khaniyev and Atalay (2010). It means that our result includes the results of Khaniyev and Atalay (2010) as a special case. On the other hand, in the real-world problem introduced in Section 1, the parameter \(\alpha\) which characterizes degree of asymmetry of the triangular distribution, is equal to 0.85. Hence, the limit distribution for real-world model in Section 1 can be written as follows:

\[
R(x) = \begin{cases} 
    \frac{10.2x - x^3}{12.58 - (2-x)^3} & ; \ x \in (0; 1.7] \\
    1 - \frac{(2-x)^3}{2.22} & ; \ x \in (1.7; 2) 
\end{cases}
\]

References

Adaptive kernel density estimation with generalized least square cross-validation

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Abstract

Adaptive kernel density estimator is an efficient estimator when the density to be estimated has long tail or multi-mode. They use varying bandwidths at each observation point by adapting a fixed bandwidth for data. It is well-known that bandwidth selection is too important for performance of kernel estimators. An efficient recent method is the generalized least square cross-validation which improves the least squares cross-validation. In this paper, performances of the adaptive kernel estimators obtained based on the generalized least square cross-validation are investigated. We performed a simulation study to inform about performances of the modified adaptive kernel estimators. For the simulation, we use also the bandwidth selection methods of normal reference, least squares cross-validation, biased cross-validation, and plug-in methods. Simulation study shows that the adaptive kernel estimators improve the performances of the kernel estimators with fixed bandwidth selected based on generalized least square cross-validation.

Keywords: Kernel density estimation, Adaptive bandwidth, Cross-validation.

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1. Introduction

The kernel density estimation (KDE) is the most popular non-parametric method to estimate density function of a distribution. Let \( X_1, X_2, \ldots, X_n \) be randomly chosen sample from a population with unknown probability density function \( f(x) \). The KDE for density function for any estimation point \( x \) is given as

\[
\hat{f}_h(x) = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{h} k \left( \frac{x - X_i}{h} \right)
\]
where \( h \) is called as bandwidth or smoothing parameter which controls the smoothness of function. The choice of \( h \) is crucial. In Equation (1.1), \( k(\cdot) \) is the kernel function which is assumed to satisfy following properties

\[
\int_{-\infty}^{\infty} k(u) du = 1, \quad \int_{-\infty}^{\infty} uk(u) du = 0, \quad \int_{-\infty}^{\infty} u^2 k(u) du = \mu_2(k) < \infty
\]

The selection of kernel function is not as important as the selection of bandwidth and such selection is made by taking into consideration of the ease of calculation and differentiability features. Some popular kernel functions are Gaussian, Epanechnikov, Triangular, Quartic, and Triweight [9].

The mean squared error (MSE), the mean integrated squared error (MISE), and the asymptotic MISE of KDE is follows as

\[
\text{MSE}(\hat{f}(x)) = \frac{f(x)R(k)}{nh} + \frac{h^4}{4} \left\{ f''(x)\mu_2(k) \right\}^2 + o((nh)^{-1}) + o(h^4)
\]

\[
\text{MISE}(\hat{f}) = \int \text{MSE}(\hat{f}(x)) dx = (nh)^{-1} R(k)
\]

\[
+ \frac{h^4}{4} \left\{ \mu_2(k) \right\}^2 \int_{-\infty}^{\infty} \{f''(x)\}^2 dx + o((nh)^{-1}) + o(h^4)
\]

\[
\text{AMISE}(\hat{f}(x)) = (nh)^{-1} R(k) + \frac{h^4}{4} \left\{ \mu_2(k) \right\}^2 \int_{-\infty}^{\infty} \{f''(x)\}^2 dx
\]

where \( R(k) = \int k^2(u) du [9, 13, 20] \). The optimal bandwidth value which minimizes the AMISE is obtained as follows,

\[
h_{opt} = \left[ \frac{R(k)}{n\mu_2^2(k)R(f''(x))} \right]^{1/5}
\]

To compute \( h_{opt} \) approximately, there are the most widely-used methods such normal reference (NR), least squares cross-validation (LSCV), biased cross validation (BCV), and plug-in. Basic idea of these approaches is to use the estimations of unknowns. The issue that which one is the best is still controversial. Generally, it is determined which method works well heuristically and through experience in practice. In section 2, the basic properties of the most common fixed bandwidth selection methods are given. In section 3, the adaptive bandwidth selectors are introduced. We will give comparisons of performances of the selectors based on Monte Carlo simulations in section 4. In section 5, a real-data example is presented. Section 6 gives the conclusions.

2. Fixed bandwidth selectors

The simplest method for selecting a bandwidth \( h \) is to use the normal reference band \( (h_{NR}) \). If \( f \) and \( k \) are assumed to be a normal distribution and a Gaussian kernel in Equation (1.5) respectively, then \( h_{opt} \) becomes \( h_{NR} \) as follows

\[
h_{NR} = 1.06\sigma n^{-1/5}
\]

or alternatively

\[
h_{NR} = 0.79 (IQR) n^{-1/5}
\]
where $\sigma$ and IQR are the standard deviation and the interquartile range of $X$, respectively [13, 16, 17]. By combining Equation (2.1) and Equation (2.2) and using the estimations of $\sigma$ and IQR, a better normal reference bandwidth is obtained as

$$(2.3) \quad \hat{h}_{NR} = 1.06 \min(\hat{\sigma}, I\hat{Q}R/1.34) n^{-1/5}$$

It is well-known that $\hat{h}_{NR}$ works well if $f$ approaches to normal distribution. Otherwise, it often obtains oversmooth estimations, specially in case of multi-modality [9, 16, 17, 20]. Recently, Zhang [21] proposed a robust simple and quick bandwidth selector $\hat{h}_{NR}(p)$ based on quantile for kernel density estimation. Even Zhang [21] states that $\hat{h}_{NR}(0.75)$ is a good choice of adaptive bandwidths by using the results of the simulation studies, but it is controversial.

As an automatic method, $LSCV$ which also is called as unbiased cross-validation (UCV) is a flexible and easy computable method. In $LSCV$, the optimal bandwidth

$$\hat{h}_{LSCV} = \arg \min_h LSCV(h)$$

which minimizes the following cross-validation function $LSCV(h)$ over $h$ is follows

$$(2.4) \quad LSCV(h) = \int \hat{f}_h^2(x)dx - \frac{2}{n} \sum_{i=1}^{n} \hat{f}_{h(i)}(X_i)$$

where

$$\int \hat{f}_h^2(x)dx = \frac{1}{n^2h^4} \sum_{i=1}^{n} \sum_{j=1}^{n} (k * k) \left( \frac{X_i - X_j}{h} \right)$$

In Equation (2.4),

$$\hat{f}_{h(i)}(X_i) = \frac{1}{(n-2)h} \sum_{j \neq i}^{n} k(\frac{X_i - X_j}{h})$$

is a leave-one-out kernel estimator that is computed from the sample points by ignoring $X_i$ [2, 9, 14, 20]. $LSCV$ bandwidth estimator is unbiased but highly variable depending on selected sample and often produces undersmooth estimations [4, 8, 12].

Differently from $LSCV$, $BCV$ method is based on AMISE. The BCV bandwidth

$$\hat{h}_{BCV} = \arg \min_h BCV(h)$$

is the minimizer of

$$BCV(h) = \frac{R(k)}{nh} + \frac{h^4}{4} \{\mu_2(k)\}^2 \hat{R}(f'')$$

where

$$\hat{R}(f'') = \frac{1}{(nh)^2} \sum_{i \neq j}^{n} (k'' * k'') \left( \frac{X_i - X_j}{h} \right)$$

is a estimator of $R(f'')$ and $k''$ is the second derivative of $k$ [9, 20]. Scott and Terrell [14] showed that $\hat{h}_{BCV}$ is more stable than $\hat{h}_{LSCV}$ but a biased estimator. Chiu [4] stated that $\hat{h}_{BCV}$ does not work for small sample sizes. Zhang [21]’s simulation studies showed that the minima of $LSCV$ and $BCV$ functions sometime occurs at extreme points of $h$, especially for sharp and multiple peaks.

Basic idea of plug-in bandwidth selectors is plugging in estimates of the unknown quantities in $h_{opt}$ [20]. Sheather and Jones [15] proposed a bandwidth selector $\hat{h}_{SJ}$ which is a plug-in approach. A version of $\hat{h}_{SJ}$ is ‘direct-plug-in’ method. Another version of $\hat{h}_{SJ}$
is ‘solve-the-equation’ method. Chiu [4] stated that procedure SJ performs quite well for
densities close to normal distribution. Loader [11] expressed that “the much touted plug-
in approaches have fared rather poorly, being tuned largely by arbitrary specification of
pilot bandwidths and being heavily biased when this specification is wrong”. Zhang [21]
showed \( \hat{h}_{SJ} \) performs well in all cases (unimodal and multimodal).

Recently, a generalized least squares cross-validation (GLSCV) method is proposed by
Zhang [22]. This method aims to improve the finite sample behavior of LSCV method.
Zhang [22] give the GLSCV function as

\[
L_{SCV}(h) = \frac{\phi(\sqrt{2}h(0))}{n} + \frac{2}{n(n-1)} \sum_{i<j} \left[ \frac{2}{g(g-2)} \phi(\sqrt{2}h(X_i - X_j)) - \left( \frac{1}{n} + \frac{1}{g-2} \right) \phi(\sqrt{2}h(X_i - X_j)) \right]
\]

where \( \Phi(\cdot) \) is Gaussian kernel and \( \Phi_h(u) = \Phi(u/h) \). Zhang [22] only discussed \( L_{SCV}(h) \) for Gaussian kernel. When \( g=1 \) then \( L_{SCV}(h) \) equals to \( LSCV(h) \). The generalized
\( LSCV \) bandwidth selector \( \hat{h}_{LSCV} \) is defined as the minimizer of \( L_{SCV}(h) \) over \( h \). Zhang [22] stated that “based on our simulation study, the poor finite sample behavior of
\( \hat{h}_{LSCV} \) can be dramatically improved by \( \hat{h}_{LSCV} \) with \( 3 \leq g \leq 4 \), where \( g=4 \) seems to be the best
choice for any sample size \( n \)”. Zhang [22] give a script for computing \( \hat{h}_{LSCV} \) in R code [5]. By using this code, \( L_{SCV}(h) \) is minimized over \( h \) within \([0.01 \hat{h}_{OS}, \hat{h}_{OS}]\). Here, \( \hat{h}_{OS} = 1.144n^{-1/5}S \) is the oversmoothed bandwidth selector for Gaussian Kernel where
\( S \) is sample standard deviation [20]. In this study, we use also Zhang’s codes located in
our R codes for the simulation study.

3. Adaptive kernel density estimators

It is well known that all the classical bandwidth selection methods perform well if
true density is close to normal distribution. Otherwise, they are problematic, specially
for long-tailed or multi-moded densities. While a kernel density estimator with fixed
bandwidth has performance well about the peak of a distribution, but performs poorly
at the tails. It is not easy to find only one bandwidth which is satisfied adequately at
peaks and tails of a density. As an efficient solution for handling this issue, it is to use
the kernel estimator which has a different bandwidth for each data point. These type of kernel
estimators are commonly called as adaptive kernel density estimators (AKDE). Van Kerm
[19] states that it is commonly preferred for decreasing the oversmooth/undersmooth
effects of the fixed bandwidth to use AKDE. Firstly, Breiman et. al. [3] introduced
AKDE as

\[
\hat{f}(x) = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{h(X_i)^d} k(x - X_i) / h(X_i)
\]

where \( h(X_i) \) is the variable bandwidth for each data point \( X_i \) and \( d \) is the number of
dimension. Breiman et. al. [3] suggested that \( h(X_i) \) must be taken as being proportional
to the distance from \( X_i \) to its \( k \)th nearest neighbor. Abramson [1] proposed that \( h(X_i) \)
must be proportional to \( f^{-1/2}(X_i) \), with \( f \) replaced by a pilot estimate, for all dimensions.
The mean squared error (MSE) of \( \hat{f}(x) \) with Abramson’s approach is derived by Jones
[10] for \( d=1 \) as follows:

\[
MSE(\hat{f}(x)) \simeq \frac{1}{576} \delta^2 h^8 A^7(x) + (nh)^{-1} S(k) f^{3/2}(x)
\]
where,
\[
A(x) = \frac{d^4}{dx^4} \left[ \frac{1}{f(x)} \right], \delta_k = \int x^4 k(x) dx, S(k) = \frac{3}{2} R(k) + \frac{1}{4} R(xk').
\]

Silverman [17] suggested that \( h(X_j) \) must be proportional to \((g/\hat{f}(X_j))^3/2\) where \( g \) is the geometric mean of \( \hat{f}(X_i) \) values. Silverman [17] suggested a three-stage algorithm to compute adaptive kernel estimations.

1. Compute a pilot estimation \( \hat{f}(X_i) \) by using KDE with a fixed bandwidth \( h \) for all data points.
2. Compute the local bandwidth factors as \( \lambda_i = \left\{ \frac{\hat{f}(X_i)}{g} \right\}^{-\alpha} \) and \( \alpha \) is the sensitivity parameter which is commonly preferred as 0.5 [1].
3. Compute the adaptive bandwidths as \( h(X_i) = h\lambda_i \) and estimate the adaptive kernel density as

\[
\hat{f}^g(x) = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{h(X_i)} K\left( \frac{x - X_i}{h(X_i)} \right) = \frac{1}{nh} \sum_{i=1}^{n} X_i K\left( \frac{x - X_i}{h\lambda_i} \right).
\]

Hall and Marron [7] and Terrell and Scott [18] showed that the AKDE have higher convergence rate than KDE’s.

Cula et al. [6] investigated the finite sample performances of the modified adaptive kernel density estimators \( \hat{f}^g(x) \), \( \hat{f}^a(x) \), and \( \hat{f}'(x) \). The modified adaptive kernel density estimators \( \hat{f}^g(x) \) and \( \hat{f}'(x) \) use average, \( a = \sum_{i=1}^{n} \hat{f}(X_i)/n \), and range \( r = \max \hat{f}(X_i) - \min \hat{f}(X_i) \), instead of geometric mean \( g \) in Equation (3.2). Cula et al. [6] used the only LSCV bandwidth selector as fixed bandwidth selector and showed that the modified adaptive kernel density estimators based on LSCV outperform the classical kernel density estimators.

Here, we define new modified adaptive kernel density estimators based on using the the fixed bandwidth selectors NR, BCV, SJ, and LSCV4 as \( \hat{f}_{NR}, \hat{f}_{BCV}, \hat{f}_{SJ}, \) and \( \hat{f}_{LSCV4} \), respectively.

1. Let \( \hat{f}^g_{NR}, \hat{f}^a_{LSCV4}, \hat{f}^g_{BCV}, \hat{f}^g_{SJ}, \) and \( \hat{f}^g_{LSCV4} \) denote the adaptive kernel density estimators based on adaptive bandwidths obtained by using geometric mean.
2. Let \( \hat{f}^g_{NR}, \hat{f}^{\alpha}_{LSCV4}, \hat{f}^{\alpha}_{BCV}, \hat{f}^{\alpha}_{SJ}, \) and \( \hat{f}^{\alpha}_{LSCV4} \) denote the adaptive kernel density estimators based on adaptive bandwidths obtained by using arithmetic mean.
3. Let \( \hat{f}^g_{NR}, \hat{f}^{\alpha}_{LSCV4}, \hat{f}^{\alpha}_{BCV}, \hat{f}^{\alpha}_{SJ}, \) and \( \hat{f}^{\alpha}_{LSCV4} \) denote the adaptive kernel density estimators based on adaptive bandwidths obtained by using range values.

We performed a simulation study to inform about performances of the all above modified estimators.

4. Finite sample performances of the modified adaptive bandwidth selectors

Because of theoretical difficulties of the kernel estimators, it is most common method to use Monte Carlo simulations for comparing their performances. We generate 1000 Monte Carlo samples of size \( n \) (50, 250, 1000) from the normal mixture model as follows

\[
f(x) = 0.5\phi(x) + 0.5\phi_{\mu}(x - \mu)
\]

where \( \mu = 0, 1, 5 \) and \( \sigma = 1, 0.5, 0.1 \) [22]. Following Zhang [22], we use the ‘direct-plug-in’ (dpi) method for \( \hat{h}_{SJ} \) and Gaussian kernel function for all estimations.
By using generated samples, the root mean integrated square error (RMISE) values of the fixed kernel estimations and the adaptive kernel estimations are computed. For each case, the average values of RMISE’s over 1000 samples are given in Table 1, Table 2, and Table 3. Figure 1 shows the behavior of the RMISE values graphically. It can be concluded the following comments.

As expected, the adaptive kernel density estimators significantly improve the classical kernel density estimators for all cases.

The classical and the adaptive BCV kernel density estimators perform poorly if the true density is sharp or two moded. Otherwise, they perform well. The adaptive BCV estimators often improve the classical BCV estimator.

<table>
<thead>
<tr>
<th>Estimator</th>
<th>50</th>
<th>250</th>
<th>1000</th>
<th>50</th>
<th>250</th>
<th>1000</th>
<th>50</th>
<th>250</th>
<th>1000</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \hat{f}_{NR} )</td>
<td>0.03314</td>
<td>0.01838</td>
<td>0.01079</td>
<td>0.04223</td>
<td>0.02311</td>
<td>0.01184</td>
<td>0.03666</td>
<td>0.02085</td>
<td>0.04478</td>
</tr>
<tr>
<td>( \hat{f}_{LSCV} )</td>
<td>0.03883</td>
<td>0.02078</td>
<td>0.01184</td>
<td>0.04859</td>
<td>0.02618</td>
<td>0.01508</td>
<td>0.11118</td>
<td>0.05876</td>
<td>0.03996</td>
</tr>
<tr>
<td>( \hat{f}_{BCV} )</td>
<td>0.03200</td>
<td>0.01818</td>
<td>0.01081</td>
<td>0.04196</td>
<td>0.02349</td>
<td>0.01402</td>
<td>0.10986</td>
<td>0.05790</td>
<td>0.03297</td>
</tr>
<tr>
<td>( \hat{f}_{SJ} )</td>
<td>0.03500</td>
<td>0.01879</td>
<td>0.01095</td>
<td>0.04428</td>
<td>0.02367</td>
<td>0.01402</td>
<td>0.10986</td>
<td>0.05790</td>
<td>0.03297</td>
</tr>
<tr>
<td>( \hat{f}_{LSCV}^4 )</td>
<td>0.03248</td>
<td>0.01833</td>
<td>0.01089</td>
<td>0.04236</td>
<td>0.02367</td>
<td>0.01402</td>
<td>0.10986</td>
<td>0.05790</td>
<td>0.03297</td>
</tr>
<tr>
<td>( \tilde{f}_{NR} )</td>
<td>0.03561</td>
<td>0.01957</td>
<td>0.01172</td>
<td>0.04308</td>
<td>0.02312</td>
<td>0.01389</td>
<td>0.09682</td>
<td>0.04624</td>
<td>0.02632</td>
</tr>
<tr>
<td>( \tilde{f}_{LSCV} )</td>
<td>0.04194</td>
<td>0.02264</td>
<td>0.01297</td>
<td>0.04967</td>
<td>0.02672</td>
<td>0.01572</td>
<td>0.10707</td>
<td>0.05975</td>
<td>0.03582</td>
</tr>
<tr>
<td>( \tilde{f}_{BCV} )</td>
<td>0.03250</td>
<td>0.01866</td>
<td>0.01136</td>
<td>0.03721</td>
<td>0.02183</td>
<td>0.01381</td>
<td>0.18457</td>
<td>0.05455</td>
<td>0.03318</td>
</tr>
<tr>
<td>( \tilde{f}_{SJ} )</td>
<td>0.03751</td>
<td>0.02006</td>
<td>0.01184</td>
<td>0.04600</td>
<td>0.02453</td>
<td>0.01465</td>
<td>0.09598</td>
<td>0.05538</td>
<td>0.03432</td>
</tr>
<tr>
<td>( \tilde{f}_{LSCV}^4 )</td>
<td>0.03315</td>
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<td>0.01146</td>
<td>0.03938</td>
<td>0.02255</td>
<td>0.01404</td>
<td>0.09263</td>
<td>0.05420</td>
<td>0.03552</td>
</tr>
<tr>
<td>( \tilde{f}_{NR} )</td>
<td>0.03483</td>
<td>0.01912</td>
<td>0.01147</td>
<td>0.04160</td>
<td>0.02419</td>
<td>0.01426</td>
<td>0.09894</td>
<td>0.04564</td>
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<tr>
<td>( \tilde{f}_{LSCV} )</td>
<td>0.04090</td>
<td>0.02209</td>
<td>0.01268</td>
<td>0.04803</td>
<td>0.02560</td>
<td>0.01501</td>
<td>0.10084</td>
<td>0.05349</td>
<td>0.03151</td>
</tr>
<tr>
<td>( \tilde{f}_{BCV} )</td>
<td>0.03187</td>
<td>0.01831</td>
<td>0.01116</td>
<td>0.03636</td>
<td>0.02097</td>
<td>0.01319</td>
<td>0.19306</td>
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<td>( \tilde{f}_{SJ} )</td>
<td>0.03609</td>
<td>0.01961</td>
<td>0.01159</td>
<td>0.04443</td>
<td>0.02345</td>
<td>0.01399</td>
<td>0.09132</td>
<td>0.04930</td>
<td>0.02996</td>
</tr>
<tr>
<td>( \tilde{f}_{LSCV}^4 )</td>
<td>0.03231</td>
<td>0.01854</td>
<td>0.01125</td>
<td>0.03834</td>
<td>0.02162</td>
<td>0.01341</td>
<td>0.08992</td>
<td>0.04840</td>
<td>0.02932</td>
</tr>
<tr>
<td>( \tilde{f}_{NR} )</td>
<td>0.03447</td>
<td>0.01827</td>
<td>0.01107</td>
<td>0.03912</td>
<td>0.02028</td>
<td>0.01213</td>
<td>0.10947</td>
<td>0.05617</td>
<td>0.02860</td>
</tr>
<tr>
<td>( \tilde{f}_{LSCV} )</td>
<td>0.03553</td>
<td>0.02079</td>
<td>0.01211</td>
<td>0.04483</td>
<td>0.02436</td>
<td>0.01364</td>
<td>0.09559</td>
<td>0.04724</td>
<td>0.02707</td>
</tr>
<tr>
<td>( \tilde{f}_{BCV} )</td>
<td>0.03090</td>
<td>0.01760</td>
<td>0.01089</td>
<td>0.03563</td>
<td>0.01953</td>
<td>0.01210</td>
<td>0.20320</td>
<td>0.04672</td>
<td>0.02520</td>
</tr>
<tr>
<td>( \tilde{f}_{SJ} )</td>
<td>0.03599</td>
<td>0.01876</td>
<td>0.01117</td>
<td>0.04166</td>
<td>0.02148</td>
<td>0.01276</td>
<td>0.09121</td>
<td>0.04418</td>
<td>0.02576</td>
</tr>
<tr>
<td>( \tilde{f}_{LSCV}^4 )</td>
<td>0.03148</td>
<td>0.01788</td>
<td>0.01096</td>
<td>0.03698</td>
<td>0.02003</td>
<td>0.01220</td>
<td>0.09468</td>
<td>0.04429</td>
<td>0.02542</td>
</tr>
</tbody>
</table>

The classical generalized LSCV and adaptive generalized LSCV density estimators perform well for all cases. The adaptive estimator \( \hat{f}_{LSCV}^4 \) has generally very attractive performance. Specially, it has increasing performance unless the sample size is large.
Table 2. Average RMISE for the case with $\mu = 1$

<table>
<thead>
<tr>
<th>Estimator</th>
<th>$\sigma = 1$</th>
<th>$\sigma = 0.5$</th>
<th>$\sigma = 1$</th>
<th>$\sigma = 0.5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f^{LSCV}$</td>
<td>0.03104</td>
<td>0.03294</td>
<td>0.03560</td>
<td>0.03684</td>
</tr>
<tr>
<td>$f^{LSCV}$</td>
<td>0.03660</td>
<td>0.03808</td>
<td>0.03640</td>
<td>0.03859</td>
</tr>
<tr>
<td>$f^{LSCV}$</td>
<td>0.03294</td>
<td>0.02926</td>
<td>0.02963</td>
<td>0.03452</td>
</tr>
<tr>
<td>$f^{LSCV}$</td>
<td>0.02974</td>
<td>0.03081</td>
<td>0.03042</td>
<td>0.03161</td>
</tr>
<tr>
<td>$f^{LSCV}$</td>
<td>0.02974</td>
<td>0.03081</td>
<td>0.03042</td>
<td>0.03161</td>
</tr>
<tr>
<td>$f^{LSCV}$</td>
<td>0.03192</td>
<td>0.03192</td>
<td>0.03192</td>
<td>0.03192</td>
</tr>
<tr>
<td>$f^{LSCV}$</td>
<td>0.03192</td>
<td>0.03192</td>
<td>0.03192</td>
<td>0.03192</td>
</tr>
<tr>
<td>$f^{LSCV}$</td>
<td>0.03192</td>
<td>0.03192</td>
<td>0.03192</td>
<td>0.03192</td>
</tr>
<tr>
<td>$f^{LSCV}$</td>
<td>0.03192</td>
<td>0.03192</td>
<td>0.03192</td>
<td>0.03192</td>
</tr>
<tr>
<td>$f^{LSCV}$</td>
<td>0.03192</td>
<td>0.03192</td>
<td>0.03192</td>
<td>0.03192</td>
</tr>
<tr>
<td>$f^{LSCV}$</td>
<td>0.03192</td>
<td>0.03192</td>
<td>0.03192</td>
<td>0.03192</td>
</tr>
<tr>
<td>$f^{LSCV}$</td>
<td>0.03192</td>
<td>0.03192</td>
<td>0.03192</td>
<td>0.03192</td>
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<tr>
<td>$f^{LSCV}$</td>
<td>0.03192</td>
<td>0.03192</td>
<td>0.03192</td>
<td>0.03192</td>
</tr>
<tr>
<td>$f^{LSCV}$</td>
<td>0.03192</td>
<td>0.03192</td>
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</tr>
<tr>
<td>$f^{LSCV}$</td>
<td>0.03192</td>
<td>0.03192</td>
<td>0.03192</td>
<td>0.03192</td>
</tr>
<tr>
<td>$f^{LSCV}$</td>
<td>0.03192</td>
<td>0.03192</td>
<td>0.03192</td>
<td>0.03192</td>
</tr>
</tbody>
</table>

Classical LSCV and the adaptive-LSCV kernel density estimators perform well if the true density is far from normal. Otherwise, the LSCV-type estimators perform poorly. The adaptive $f^{LSCV}$ estimator improves the classical LSCV estimator.

Table 3. Average RMISE for the case with $\mu = 5$

<table>
<thead>
<tr>
<th>Estimator</th>
<th>$\sigma = 1$</th>
<th>$\sigma = 0.5$</th>
<th>$\sigma = 1$</th>
<th>$\sigma = 0.5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f^{LSCV}$</td>
<td>0.04456</td>
<td>0.04344</td>
<td>0.04399</td>
<td>0.03319</td>
</tr>
<tr>
<td>$f^{LSCV}$</td>
<td>0.04456</td>
<td>0.04344</td>
<td>0.04399</td>
<td>0.03319</td>
</tr>
<tr>
<td>$f^{LSCV}$</td>
<td>0.04456</td>
<td>0.04344</td>
<td>0.04399</td>
<td>0.03319</td>
</tr>
<tr>
<td>$f^{LSCV}$</td>
<td>0.04456</td>
<td>0.04344</td>
<td>0.04399</td>
<td>0.03319</td>
</tr>
<tr>
<td>$f^{LSCV}$</td>
<td>0.04456</td>
<td>0.04344</td>
<td>0.04399</td>
<td>0.03319</td>
</tr>
<tr>
<td>$f^{LSCV}$</td>
<td>0.04456</td>
<td>0.04344</td>
<td>0.04399</td>
<td>0.03319</td>
</tr>
<tr>
<td>$f^{LSCV}$</td>
<td>0.04456</td>
<td>0.04344</td>
<td>0.04399</td>
<td>0.03319</td>
</tr>
<tr>
<td>$f^{LSCV}$</td>
<td>0.04456</td>
<td>0.04344</td>
<td>0.04399</td>
<td>0.03319</td>
</tr>
<tr>
<td>$f^{LSCV}$</td>
<td>0.04456</td>
<td>0.04344</td>
<td>0.04399</td>
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</tr>
<tr>
<td>$f^{LSCV}$</td>
<td>0.04456</td>
<td>0.04344</td>
<td>0.04399</td>
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<tr>
<td>$f^{LSCV}$</td>
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<td>0.04344</td>
<td>0.04399</td>
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</tr>
<tr>
<td>$f^{LSCV}$</td>
<td>0.04456</td>
<td>0.04344</td>
<td>0.04399</td>
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</tr>
<tr>
<td>$f^{LSCV}$</td>
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<td>0.04344</td>
<td>0.04399</td>
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<tr>
<td>$f^{LSCV}$</td>
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<td>0.04344</td>
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</tr>
<tr>
<td>$f^{LSCV}$</td>
<td>0.04456</td>
<td>0.04344</td>
<td>0.04399</td>
<td>0.03319</td>
</tr>
</tbody>
</table>

The adaptive $f^{LSCV}$ estimator improves the classical LSCV estimator.
The classical and the adaptive NR kernel density estimators perform generally poorly if true density is far from normal. Specially, they behave very poorly for two moded densities. The adaptive NR estimators improves the classical NR estimator for the most of such abnormal situations.

The classical and the adaptive SJ kernel density estimators perform well except for sharp densities. Again, the adaptive SJ estimators often improves the classical SJ estimator.

5. An Example

We realize an application for the all estimators with Gaussian kernel function. The application data is the durations (in minutes) of 272 eruptions of the Old Faithful geyser in Yellowstone National Park [9]. Fixed bandwidths for the kernel estimates are computed as $\hat{h}_{NR} = 0.394$, $\hat{h}_{LSCV} = 0.103$, $\hat{h}_{BCV} = 0.157$, $\hat{h}_{SJ} = 0.165$, and $\hat{h}_{LSCV4} = 0.128$. Figure 2 shows the data points and the considered all kernel estimates in this study. Figure 3 shows only the kernel estimates obtained based on selector GLSCV4.

All the estimates show clearly that the duration of eruption has a bimodal density.

The adaptive kernel estimates behave similar to their classical kernel estimates tend to get better slightly. Specially, they lead to improve the estimates about the peaks and valley between the two peaks.
Figure 2. Classical and adaptive kernel density estimates.

Figure 3. Adaptive kernel density estimates based on the bandwidth selector $LSCV_4$.

6. Conclusions

The adaptive kernel density estimators are often used for the estimation of densities far from normal distribution. The classical kernel density estimators are based on using the fixed bandwidths. The generalized LSCV estimator is a new efficient kernel density estimator which uses fixed bandwidth. It improves the finite sample behavior of the classical LSCV estimator.
The adaptive estimators use the different bandwidth for each observation point. Therefore, they are more robust to the existence of outliers or extremes. Here, we focused the adaptive variates of the generalized LSCV estimator. We also compared the performances of the other adaptive estimates. The results show that the adaptive estimators often significantly improves the classical estimators.

References