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ON SOME INEQUALITIES OF SIMPSON'S TYPE VIA $h-$CONVEX FUNCTIONS

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Abstract

In this paper, we prove some new inequalities of Simpson's type for functions whose derivatives of absolute values are $h-$convex and $h-$concave functions. Some new estimations are obtained. Also we give some sophisticated results for some different kinds of convex functions.

Keywords: $h-$convex and $h-$concave functions, Simpson's Inequality, Hölder Inequality.

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1. Introduction

The following inequality is well known in the literature as Simpson's inequality;

$$(1.1) \qquad \frac{1}{b-a}\int_a^b f(x)\,dx - \frac{1}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right] \leq \frac{1}{2880}\left\|f^{(4)}\right\|_\infty (b-a)^4,$$

where the mapping $f:[a,b]\to\mathbb{R}$ is assumed to be four times continuously differentiable on the interval and $f^{(4)}$ to be bounded on (a,b) , that is,

$$\left\|f^{(4)}\right\|_\infty = \sup_{t\in(a,b)}\left|f^{(4)}(t)\right| < \infty.$$

For some results which generalize, improve and extend the inequality (1.1) see the papers [1]-[3].

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1.1. Definition. [5] *We say that $f : I \to \mathbb{R}$ is Godunova-Levin function or that f belongs to the class $Q(I)$ if f is non-negative and for all $x, y \in I$ and $t \in (0,1)$ we have*

$$f(tx + (1-t)y) \leq \frac{f(x)}{t} + \frac{f(y)}{1-t}. \tag{1.2}$$

The class $Q(I)$ was firstly described in [5] by Godunova-Levin. Among others, it is noted that nonnegative monotone and nonnegative convex functions belong to this class of functions.

1.2. Definition. [4] *We say that $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a $P-$function or that f belongs to the class $P(I)$ if f is nonnegative and for all $x, y \in I$ and $t \in [0,1]$, we have*

$$f(tx + (1-t)y) \leq f(x) + f(y). \tag{1.3}$$

1.3. Definition. [8] *Let* $s \in (0,1]$ be a fixed real number. *A function $f : [0,\infty) \to [0,\infty)$ is said to be $s-$convex (in the second sense) if*

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y), \tag{1.4}$$

for all $x, y \in [0,\infty)$ and $t \in [0,1]$. This class of s-convex functions is usually denoted by K_s^2.

In 1978, Breckner introduced $s-$convex functions as a generalization of convex functions [8]. Also, in that one work Breckner proved the important fact that the setvalued map is $s-$convex only if the associated support function is s-convex function [9]. Of course, $s-$convexity means just convexity when $s = 1$.

1.4. Definition. [7] *Let $h : J \subseteq \mathbb{R} \to \mathbb{R}$ be a positive function . We say that $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is $h-$convex function, or that f belongs to the class $SX(h, I)$, if f is nonnegative and for all $x, y \in I$ and $t \in (0,1)$ we have*

$$f(tx + (1-t)y) \leq h(t) f(x) + h(1-t) f(y). \tag{1.5}$$

If inequality (1.5) is reversed, then f is said to be $h-$concave, i.e. $f \in SV(h, I)$. Obviously, if $h(t) = t$, then all nonnegative convex functions belong to $SX(h, I)$ and all nonnegative concave functions belong to $SV(h, I)$; if $h(t) = \frac{1}{t}$, then $SX(h, I) = Q(I)$; if $h(t) = 1$, then $SX(h, I) \supseteq P(I)$; and if $h(t) = t^s$, where $s \in (0,1)$, then $SX(h, I) \supseteq K_s^2$.

1.5. Remark. [7] Let h be a non-negative function such that

$$h(\alpha) \geq \alpha$$

for all $\alpha \in (0,1)$. If f is a non-negative convex function on I , then for $x, y \in I$, $\alpha \in (0,1)$ we have

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha) f(y) \leq h(\alpha) f(x) + h(1-\alpha) f(y).$$

So, $f \in SX(h, I)$. Similarly, if the function h has the property: $h(\alpha) \leq \alpha$ for all $\alpha \in (0,1)$, then any non-negative concave function f belongs to the class $SV(h, I)$.

1.6. Definition. [7] A function $h : J \to \mathbb{R}$ is said to be a supermultiplicative function if

$$h(xy) \geq h(x) h(y) \tag{1.6}$$

for all $x, y \in J$.

If inequality (1.6) is reversed, then h is said to be a submultiplicative function. If equality held in (1.6), then h is said to be a multiplicative function.

In [1], Sarıkaya *et.al* established the following Simpson-type inequality for convex functions:

1.7. Theorem. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$, be differentiable mapping on I° such that $f' \in L_1[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|$ is a convex on $[a,b]$, then the following inequality holds:*

$$\left|\frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{5(b-a)}{72}\left[\left|f'(a)\right| + \left|f'(b)\right|\right] \tag{1.7}$$

In [10], Sarıkaya *et.al* established the following Hadmard-type inequality for $h-$convex functions:

1.8. Theorem. *Let $f \in SX(h,I)$, $a,b \in I$, with $a < b$ and $f \in L_1([a,b])$. Then*

$$\frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq [f(a) + f(b)]\int_0^1 h(t)\,dt. \tag{1.8}$$

For recent results and generalizations concerning $h-$convex functions see [7], [10].

The aim of this paper is to establish new inequalities for functions whose derivatives in absolute value are $h-$convex and $h-$concave functions.

2. Inequalities for $h-$convex and $h-$concave functions

To prove our new result we need the following lemma (see [3]).

2.1. Lemma. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be an absolutely continuous mapping on I^0 where $a, b \in I$ with $a < b$. Then the following equality holds:*

$$\frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{1}{b-a}\int_a^b f(x)\,dx = (b-a)\int_0^1 k(t)\,f'(ta + (1-t)b)\,dt,$$

where

$$k(t) = \begin{cases} t - \frac{1}{6}, & t \in \left[0, \frac{1}{2}\right) \\ t - \frac{5}{6}, & t \in \left[\frac{1}{2}, 1\right] \end{cases}.$$

2.2. Theorem. *Let $h : J \subset \mathbb{R} \to \mathbb{R}$ be a non-negative function, $f : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable function on I° such that $h^q, f' \in L[a,b]$ where $a, b \in I^\circ$ with $a < b$ and $h(\alpha) \geq \alpha$. If $|f'|$ is $h-$convex on I, then*

$$\begin{aligned} &\left|\frac{1}{b-a}\int_a^b f(x)\,dx - \frac{1}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right]\right| \\ &\leq \frac{(b-a)}{3}\left(\frac{1+2^{p+1}}{6(p+1)}\right)^{\frac{1}{p}} \times \left\{\left|f'(a)\right|\left[\left(\int_0^{\frac{1}{2}} h^q(t)\,dt\right)^{\frac{1}{q}} + \left(\int_{\frac{1}{2}}^1 h^q(t)\,dt\right)^{\frac{1}{q}}\right]\right. \\ &\left. + \left|f'(b)\right|\left[\left(\int_0^{\frac{1}{2}} h^q(1-t)\,dt\right)^{\frac{1}{q}} + \left(\int_{\frac{1}{2}}^1 h^q(1-t)\,dt\right)^{\frac{1}{q}}\right]\right\}. \end{aligned} \tag{2.1}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 2.1, $h-$convexity of $|f'|$ and properties of absolute value, we have

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)\,dx-\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]\right| \\
&=(b-a)\left|\int_0^1 k(t)\,f'(ta+(1-t)b)\,dt\right| \\
&\leq (b-a)\left(\int_0^{\frac{1}{2}}\left|t-\frac{1}{6}\right|\left|f'(ta+(1-t)b)\right|dt+\int_{\frac{1}{2}}^1\left|t-\frac{5}{6}\right|\left|f'(ta+(1-t)b)\right|dt\right) \\
&\leq (b-a)\left\{\int_0^{\frac{1}{2}}\left|t-\frac{1}{6}\right|\left(h(t)\left|f'(a)\right|+h(1-t)\left|f'(b)\right|\right)dt\right. \\
&\left.+\int_{\frac{1}{2}}^1\left|t-\frac{5}{6}\right|\left(h(t)\left|f'(a)\right|+h(1-t)\left|f'(b)\right|\right)dt\right\} \\
&=(b-a)\left|f'(a)\right|\left\{\int_0^{\frac{1}{2}}\left|t-\frac{1}{6}\right|h(t)\,dt+\int_{\frac{1}{2}}^1\left|t-\frac{5}{6}\right|h(t)\,dt\right\} \\
&+(b-a)\left|f'(b)\right|\left\{\int_0^{\frac{1}{2}}\left|t-\frac{1}{6}\right|h(1-t)\,dt+\int_{\frac{1}{2}}^1\left|t-\frac{5}{6}\right|h(1-t)\,dt\right\}.
\end{aligned}$$

By Hölder's inequality, we get

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)\,dx-\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]\right| \\
&\leq (b-a)\left|f'(a)\right|\left\{\left(\int_0^{\frac{1}{2}}\left|t-\frac{1}{6}\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}h^q(t)\,dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_{\frac{1}{2}}^1\left|t-\frac{5}{6}\right|^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1 h^q(t)\,dt\right)^{\frac{1}{q}}\right\} \\
&+(b-a)\left|f'(b)\right|\left\{\left(\int_0^{\frac{1}{2}}\left|t-\frac{1}{6}\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}h^q(1-t)\,dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_{\frac{1}{2}}^1\left|t-\frac{5}{6}\right|^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1 h^q(1-t)\,dt\right)^{\frac{1}{q}}\right\}
\end{aligned}$$

Since

$$\int_0^{\frac{1}{2}}\left|t-\frac{1}{6}\right|^p dt=\int_0^{\frac{1}{6}}\left(\frac{1}{6}-t\right)^p dt+\int_{\frac{1}{6}}^{\frac{1}{2}}\left(t-\frac{1}{6}\right)^p dt=\frac{1}{p+1}\left(\frac{1}{6^{p+1}}+\frac{1}{3^{p+1}}\right)$$

and

$$\int_{\frac{1}{2}}^1\left|t-\frac{5}{6}\right|^p dt=\int_{\frac{1}{2}}^{\frac{5}{6}}\left(\frac{5}{6}-t\right)^p dt+\int_{\frac{5}{6}}^1\left(t-\frac{5}{6}\right)^p dt=\frac{1}{p+1}\left(\frac{1}{6^{p+1}}+\frac{1}{3^{p+1}}\right),$$

we obtain

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx-\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]\right|$$
$$\leq\frac{(b-a)}{3}\left(\frac{1+2^{p+1}}{6(p+1)}\right)^{\frac{1}{p}}\times\left\{\left|f'(a)\right|\left[\left(\int_0^{\frac{1}{2}}h^q(t)\,dt\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{2}}^1 h^q(t)\,dt\right)^{\frac{1}{q}}\right]\right.$$
$$\left.+\left|f'(b)\right|\left[\left(\int_0^{\frac{1}{2}}h^q(1-t)\,dt\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{2}}^1 h^q(1-t)\,dt\right)^{\frac{1}{q}}\right]\right\}.$$

which completes the proof. □

2.3. Corollary. *In Theorem 2.2, if we choose $p=q=2$, we obtain*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx-\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]\right|$$
$$\leq\frac{b-a}{3\sqrt{2}}\left\{\left|f'(a)\right|\left[\left(\int_0^{\frac{1}{2}}h\left(t^2\right)dt\right)^{\frac{1}{2}}+\left(\int_{\frac{1}{2}}^1 h\left(t^2\right)dt\right)^{\frac{1}{2}}\right]\right.$$
$$\left.+\left|f'(b)\right|\left[\left(\int_0^{\frac{1}{2}}h\left((1-t)^2\right)dt\right)^{\frac{1}{2}}+\left(\int_{\frac{1}{2}}^1 h\left((1-t)^2\right)dt\right)^{\frac{1}{2}}\right]\right\}$$

where h is supermultiplicative.

2.4. Corollary. *In Theorem 2.2, if $f(a)=f\left(\frac{a+b}{2}\right)=f(b)$ and $h(t)=1$, then we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx-f\left(\frac{a+b}{2}\right)\right|$$
$$\leq\frac{2(b-a)}{3}\left(\frac{1+2^{p+1}}{6(p+1)}\right)^{\frac{1}{p}}\left(\frac{1}{2}\right)^{\frac{1}{q}}\left\{\left|f'(a)\right|+\left|f'(b)\right|\right\}$$
$$=\frac{b-a}{3}\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{\frac{1}{p}}\left\{\left|f'(a)\right|+\left|f'(b)\right|\right\}$$

2.5. Corollary. *In Theorem 2.2, if we choose $h(t)=t$, we obtain*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx-\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]\right|$$
$$\leq\frac{(b-a)}{3}\left(\frac{1+2^{p+1}}{6(p+1)}\right)^{\frac{1}{p}}\left\{\left|f'(a)\right|\left[\left(\frac{\left(\frac{1}{2}\right)^{q+1}}{q+1}\right)^{\frac{1}{q}}+\left(\frac{1}{2q+2}\left(2-\left(\frac{1}{2}\right)^q\right)\right)^{\frac{1}{q}}\right]\right.$$
$$\left.+\left|f'(b)\right|\left[\left(\frac{1}{2q+2}\left(2-\left(\frac{1}{2}\right)^q\right)\right)^{\frac{1}{q}}+\left(\frac{\left(\frac{1}{2}\right)^{q+1}}{q+1}\right)^{\frac{1}{q}}\right]\right\}$$
$$=\frac{(b-a)}{6}\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{\frac{1}{p}}\left(\frac{1}{q+1}\right)^{\frac{1}{q}}\left[\frac{\left|f'(a)\right|}{2}+\left|f'(b)\right|\left(2-\left(\frac{1}{2}\right)^q\right)^{\frac{1}{q}}\right].$$

2.6. Theorem. *Let $h:J\subset\mathbb{R}\to\mathbb{R}$ be a nonnegative supermultiplicative functions, $f:I\subset[0,\infty)\to\mathbb{R}$ be a differentiable function on I° such that $f'\in L[a,b]$ where $a,b\in I^\circ$*

with $a<b$ and $h(\alpha)\geq\alpha$. If $|f'|$ is $h-$convex on I, then

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)\,dx-\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]\right| \\
&\leq(b-a)\left\{\left|f'(a)\right|[A]+\left|f'(b)\right|[B]\right\} \qquad (2.2)
\end{aligned}$$

where

$$\begin{aligned}
A&=\int_0^{\frac{1}{6}}h\left(t\left[\frac{1}{6}-t\right]\right)dt+\int_{\frac{1}{6}}^{\frac{1}{2}}h\left(t\left[t-\frac{1}{6}\right]\right)dt \\
&\quad+\int_{\frac{1}{2}}^{\frac{5}{6}}h\left(t\left[\frac{5}{6}-t\right]\right)dt+\int_{\frac{5}{6}}^{1}h\left(t\left[t-\frac{5}{6}\right]\right)dt \\
B&=\int_0^{\frac{1}{6}}h\left[(1-t)\left(\frac{1}{6}-t\right)\right]dt+\int_{\frac{1}{6}}^{\frac{1}{2}}h\left[(1-t)\left(t-\frac{1}{6}\right)\right]dt \\
&\quad+\int_{\frac{1}{2}}^{\frac{5}{6}}h\left[(1-t)\left(\frac{5}{6}-t\right)\right]dt+\int_{\frac{5}{6}}^{1}h\left[(1-t)\left(t-\frac{5}{6}\right)\right]dt.
\end{aligned}$$

Proof. From Lemma 2.1, $h-$convexity of $|f'|$ and properties of absolute value, we have

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)\,dx-\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]\right| \\
&=(b-a)\left|\int_0^1 k(t)\,f'(ta+(1-t)b)\,dt\right| \\
&\leq(b-a)\left\{\int_0^{\frac{1}{2}}\left|t-\frac{1}{6}\right|\left|f'(ta+(1-t)b)\right|dt+\int_{\frac{1}{2}}^{1}\left|t-\frac{5}{6}\right|\left|f'(ta+(1-t)b)\right|dt\right\} \\
&\leq(b-a)\left\{\int_0^{\frac{1}{2}}\left|t-\frac{1}{6}\right|\left(h(t)\left|f'(a)\right|+h(1-t)\left|f'(b)\right|\right)dt\right. \\
&\left.+\int_{\frac{1}{2}}^{1}\left|t-\frac{5}{6}\right|\left(h(t)\left|f'(a)\right|+h(1-t)\left|f'(b)\right|\right)dt\right\} \\
&=(b-a)\left\{\int_0^{\frac{1}{6}}\left(\frac{1}{6}-t\right)\left(h(t)\left|f'(a)\right|+h(1-t)\left|f'(b)\right|\right)dt\right. \\
&+\int_{\frac{1}{6}}^{\frac{1}{2}}\left(t-\frac{1}{6}\right)\left(h(t)\left|f'(a)\right|+h(1-t)\left|f'(b)\right|\right)dt \\
&+\int_{\frac{1}{2}}^{\frac{5}{6}}\left(\frac{5}{6}-t\right)\left(h(t)\left|f'(a)\right|+h(1-t)\left|f'(b)\right|\right)dt \\
&\left.+\int_{\frac{5}{6}}^{1}\left(t-\frac{5}{6}\right)\left(h(t)\left|f'(a)\right|+h(1-t)\left|f'(b)\right|\right)dt\right\}.
\end{aligned}$$

By properties of function h, we can write

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)\,dx-\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]\right| \\
&\leq (b-a)\left\{\int_0^{\frac{1}{6}}\left[h\left(t\left(\frac{1}{6}-t\right)\right)\left|f'(a)\right|+h\left((1-t)\left(\frac{1}{6}-t\right)\right)\left|f'(b)\right|\right]dt\right. \\
&+\int_{\frac{1}{6}}^{\frac{1}{2}}\left[h\left(t\left(t-\frac{1}{6}\right)\right)\left|f'(a)\right|+h\left((1-t)\left(t-\frac{1}{6}\right)\right)\left|f'(b)\right|\right]dt \\
&+\int_{\frac{1}{2}}^{\frac{5}{6}}\left[h\left(t\left(\frac{5}{6}-t\right)\right)\left|f'(a)\right|+h\left((1-t)\left(\frac{5}{6}-t\right)\right)\left|f'(b)\right|\right]dt \\
&\left.+\int_{\frac{5}{6}}^{1}\left[h\left(t\left(t-\frac{5}{6}\right)\right)\left|f'(a)\right|+h\left((1-t)\left(t-\frac{5}{6}\right)\right)\left|f'(b)\right|\right]dt\right\}
\end{aligned}$$

$$\begin{aligned}
&=(b-a)\left|f'(a)\right|\left\{\int_0^{\frac{1}{6}}h\left(t\left[\frac{1}{6}-t\right]\right)dt+\int_{\frac{1}{6}}^{\frac{1}{2}}h\left(t\left[t-\frac{1}{6}\right]\right)dt\right. \\
&\left.+\int_{\frac{1}{2}}^{\frac{5}{6}}h\left(t\left[\frac{5}{6}-t\right]\right)dt+\int_{\frac{5}{6}}^{1}h\left(t\left[t-\frac{5}{6}\right]\right)dt\right\} \\
&+(b-a)\left|f'(b)\right|\left\{\int_0^{\frac{1}{6}}h\left((1-t)\left(\frac{1}{6}-t\right)\right)dt+\int_{\frac{1}{6}}^{\frac{1}{2}}h\left((1-t)\left(t-\frac{1}{6}\right)\right)dt\right. \\
&\left.+\int_{\frac{1}{2}}^{\frac{5}{6}}h\left[(1-t)\left(\frac{5}{6}-t\right)\right]dt+\int_{\frac{5}{6}}^{1}h\left[(1-t)\left(t-\frac{5}{6}\right)\right]dt\right\}
\end{aligned}$$

which completes the proof. □

2.7. Corollary. *In Theorem 2.6, if* $f(a)=f\left(\frac{a+b}{2}\right)=f(b)$ *and* $h(t)=1$, *then we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx-f\left(\frac{a+b}{2}\right)\right|\leq(b-a)\left\{\left|f'(a)\right|+\left|f'(b)\right|\right\}.$$

2.8. Remark. In Theorem 2.6, if we choose $h(t)=t$, then we obtain the inequality (1.7).

2.9. Theorem. *Let* $h:J\subset\mathbb{R}\to\mathbb{R}$ *be non-negative functions,* $f:I\subset[0,\infty)\to\mathbb{R}$ *be a differentiable function on* I° *such that* $f'\in L[a,b]$ *where* $a,b\in I^{\circ}$ *with* $a<b$. *If* $|f'|$ *is* $h-$*concave on* I, *then*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)\,dx-\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]\right| \\
&\leq\frac{b-a}{12}\left(\frac{2+2^{p+2}}{p+1}\right)^{\frac{1}{p}}\left[\frac{1}{h\left(\frac{1}{2}\right)}\right]^{\frac{1}{q}}\left|f'\left(\frac{a+b}{2}\right)\right|.
\end{aligned}$$

Proof. From Lemma 2.1 and using the Hölder inequality, we have

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(x)\,dx-\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]\right|\\&\leq(b-a)\left|\int_0^1 k(t)\,f'(ta+(1-t)b)\,dt\right|\\&\leq(b-a)\left(\int_0^1|k(t)|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}\end{aligned}$$

Since $|f'|$ is h-concave on I, by inequalities (1.8) we have

$$\int_0^1\left|f'(ta+(1-t)b)\right|^q dt\leq\frac{1}{2h\left(\frac{1}{2}\right)}\left|f'\left(\frac{a+b}{2}\right)\right|^q.$$

Therefore, we get

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(x)\,dx-\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]\right|\\&\leq(b-a)\left\{\int_0^{\frac{1}{2}}\left|t-\frac{1}{6}\right|^p dt+\int_{\frac{1}{2}}^1\left|t-\frac{5}{6}\right|^p dt\right\}^{\frac{1}{p}}\left[\frac{1}{2h\left(\frac{1}{2}\right)}\left|f'\left(\frac{a+b}{2}\right)\right|^q\right]^{\frac{1}{q}}.\end{aligned}$$

Since

$$\int_0^{\frac{1}{2}}\left|t-\frac{1}{6}\right|^p dt=\int_0^{\frac{1}{6}}\left(\frac{1}{6}-t\right)^p dt+\int_{\frac{1}{6}}^{\frac{1}{2}}\left(t-\frac{1}{6}\right)^p dt=\frac{1}{p+1}\left(\frac{1}{6^{p+1}}+\frac{1}{3^{p+1}}\right)$$

and

$$\int_{\frac{1}{2}}^1\left|t-\frac{5}{6}\right|^p dt=\int_{\frac{1}{2}}^{\frac{5}{6}}\left(\frac{5}{6}-t\right)^p dt+\int_{\frac{5}{6}}^1\left(t-\frac{5}{6}\right)^p dt=\frac{1}{p+1}\left(\frac{1}{6^{p+1}}+\frac{1}{3^{p+1}}\right),$$

we obtain

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(x)\,dx-\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]\right|\\&\leq\frac{b-a}{12}\left(\frac{2+2^{p+2}}{p+1}\right)^{\frac{1}{p}}\left[\frac{1}{h\left(\frac{1}{2}\right)}\right]^{\frac{1}{q}}\left|f'\left(\frac{a+b}{2}\right)\right|.\end{aligned}$$

This completes the proof. □

2.10. Corollary. *In Theorem 2.9, if we choose $h(t)=t$, then we obtain*

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(x)\,dx-\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]\right|\\&\leq\frac{b-a}{6}\left(\frac{1+2^{p+1}}{p+1}\right)^{\frac{1}{p}}\left|f'\left(\frac{a+b}{2}\right)\right|.\end{aligned}$$

3. APPLICATIONS TO SPECIAL MEANS

We now consider the means for arbitrary real numbers α,β $(\alpha\neq\beta)$. We take

(1) *Arithmetic mean* :

$$A(\alpha,\beta)=\frac{\alpha+\beta}{2},\quad \alpha,\beta\in\mathbb{R}^+.$$

(2) *Logarithmic mean*:

$$L(\alpha,\beta)=\frac{\alpha-\beta}{\ln|\alpha|-\ln|\beta|},\qquad |\alpha|\neq|\beta|,\ \alpha,\beta\neq 0,\ \alpha,\beta\in\mathbb{R}^{+}.$$

(3) *Generalized* $log-mean$:

$$L_n(\alpha,\beta)=\left[\frac{\beta^{n+1}-\alpha^{n+1}}{(n+1)(\beta-\alpha)}\right]^{\frac{1}{n}},\qquad n\in\mathbb{Z}\backslash\{-1,0\},\ \alpha,\beta\in\mathbb{R}^{+}.$$

Now using the results of Section 2, we give some applications for special means of real numbers.

3.1. Proposition. *Let* $a,b\in\mathbb{R}^{+}$, $0<a<b$ *and* $n\in\mathbb{N}, n>1$. *Then, we have*

$$\begin{aligned}&\left|L_n^n(a,b)-\frac{1}{3}\left[A(a^n,b^n)-A^n(a,b)\right]\right|\\&\leq n\frac{(b-a)}{6}\left(\frac{1+2^{p+1}}{3\,(p+1)}\right)^{\frac{1}{p}}\left(\frac{1}{q+1}\right)^{\frac{1}{q}}\left[\frac{a^{n-1}}{2}+b^{n-1}\left(2-\left(\frac{1}{2}\right)^{q}\right)^{\frac{1}{q}}\right].\end{aligned}$$

Proof. The assertion follows from Corollary 2.5 applied for $f(x)=x^n$, $x\in\mathbb{R}$, $n\in\mathbb{N}$. □

3.2. Proposition. *Let* $a,b\in\mathbb{R}^{+}$, $a<b$. *Then, we have*

$$|L_n^n(a,b)-A^n(a,b)|\leq(b-a)\left[\frac{1}{a^2}+\frac{1}{b^2}\right].$$

Proof. The assertion follows from Corollary 2.7 applied for $f(x)=\frac{1}{x}$, $x\in[a,b]$. □

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