

ON SOME INEQUALITIES OF SIMPSON'S TYPE VIA h -CONVEX FUNCTIONS

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Abstract

In this paper, we prove some new inequalities of Simpson's type for functions whose derivatives of absolute values are h -convex and h -concave functions. Some new estimations are obtained. Also we give some sophisticated results for some different kinds of convex functions.

Keywords: h -convex and h -concave functions, Simpson's Inequality, Hölder Inequality.

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1. Introduction

The following inequality is well known in the literature as Simpson's inequality;

$$(1.1) \quad \frac{1}{b-a} \int_a^b f(x) dx - \frac{1}{3} \left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right) \right] \leq \frac{1}{2880} \|f^{(4)}\|_{\infty} (b-a)^4,$$

where the mapping $f : [a, b] \rightarrow \mathbb{R}$ is assumed to be four times continuously differentiable on the interval and $f^{(4)}$ to be bounded on (a, b) , that is,

$$\|f^{(4)}\|_{\infty} = \sup_{t \in (a,b)} |f^{(4)}(t)| < \infty.$$

For some results which generalize, improve and extend the inequality (1.1) see the papers [1]-[3].

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1.1. Definition. [5] We say that $f : I \rightarrow \mathbb{R}$ is Godunova-Levin function or that f belongs to the class $Q(I)$ if f is non-negative and for all $x, y \in I$ and $t \in (0, 1)$ we have

$$(1.2) \quad f(tx + (1-t)y) \leq \frac{f(x)}{t} + \frac{f(y)}{1-t}.$$

The class $Q(I)$ was firstly described in [5] by Godunova-Levin. Among others, it is noted that nonnegative monotone and nonnegative convex functions belong to this class of functions.

1.2. Definition. [4] We say that $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is a P -function or that f belongs to the class $P(I)$ if f is nonnegative and for all $x, y \in I$ and $t \in [0, 1]$, we have

$$(1.3) \quad f(tx + (1-t)y) \leq f(x) + f(y).$$

1.3. Definition. [8] Let $s \in (0, 1]$ be a fixed real number. A function $f : [0, \infty) \rightarrow [0, \infty)$ is said to be s -convex (in the second sense) if

$$(1.4) \quad f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y),$$

for all $x, y \in [0, \infty)$ and $t \in [0, 1]$. This class of s -convex functions is usually denoted by K_s^2 .

In 1978, Breckner introduced s -convex functions as a generalization of convex functions [8]. Also, in that one work Breckner proved the important fact that the setvalued map is s -convex only if the associated support function is s -convex function [9]. Of course, s -convexity means just convexity when $s = 1$.

1.4. Definition. [7] Let $h : J \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a positive function. We say that $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is h -convex function, or that f belongs to the class $SX(h, I)$, if f is nonnegative and for all $x, y \in I$ and $t \in (0, 1)$ we have

$$(1.5) \quad f(tx + (1-t)y) \leq h(t)f(x) + h(1-t)f(y).$$

If inequality (1.5) is reversed, then f is said to be h -concave, i.e. $f \in SV(h, I)$. Obviously, if $h(t) = t$, then all nonnegative convex functions belong to $SX(h, I)$ and all nonnegative concave functions belong to $SV(h, I)$; if $h(t) = \frac{1}{t}$, then $SX(h, I) = Q(I)$; if $h(t) = 1$, then $SX(h, I) \supseteq P(I)$; and if $h(t) = t^s$, where $s \in (0, 1)$, then $SX(h, I) \supseteq K_s^2$.

1.5. Remark. [7] Let h be a non-negative function such that

$$h(\alpha) \geq \alpha$$

for all $\alpha \in (0, 1)$. If f is a non-negative convex function on I , then for $x, y \in I$, $\alpha \in (0, 1)$ we have

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y) \leq h(\alpha)f(x) + h(1-\alpha)f(y).$$

So, $f \in SX(h, I)$. Similarly, if the function h has the property: $h(\alpha) \leq \alpha$ for all $\alpha \in (0, 1)$, then any non-negative concave function f belongs to the class $SV(h, I)$.

1.6. Definition. [7] A function $h : J \rightarrow \mathbb{R}$ is said to be a supermultiplicative function if

$$(1.6) \quad h(xy) \geq h(x)h(y)$$

for all $x, y \in J$.

If inequality (1.6) is reversed, then h is said to be a submultiplicative function. If equality held in (1.6), then h is said to be a multiplicative function.

In [1], Sarikaya *et.al* established the following Simpson-type inequality for convex functions:

1.7. Theorem. *Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$, be differentiable mapping on I° such that $f' \in L_1[a, b]$, where $a, b \in I$ with $a < b$. If $|f'|$ is a convex on $[a, b]$, then the following inequality holds:*

$$(1.7) \quad \left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ \leq \frac{5(b-a)}{72} [|f'(a)| + |f'(b)|]$$

In [10], Sarikaya *et.al* established the following Hadmard-type inequality for h -convex functions:

1.8. Theorem. *Let $f \in SX(h, I)$, $a, b \in I$, with $a < b$ and $f \in L_1([a, b])$. Then*

$$(1.8) \quad \frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq [f(a) + f(b)] \int_0^1 h(t) dt.$$

For recent results and generalizations concerning h -convex functions see [7], [10].

The aim of this paper is to establish new inequalities for functions whose derivatives in absolute value are h -convex and h -concave functions.

2. Inequalities for h -convex and h -concave functions

To prove our new result we need the following lemma (see [3]).

2.1. Lemma. *Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be an absolutely continuous mapping on I^0 where $a, b \in I$ with $a < b$. Then the following equality holds:*

$$\frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \\ = (b-a) \int_0^1 k(t) f'(ta + (1-t)b) dt,$$

where

$$k(t) = \begin{cases} t - \frac{1}{6}, & t \in \left[0, \frac{1}{2}\right) \\ t - \frac{5}{6}, & t \in \left[\frac{1}{2}, 1\right] \end{cases}.$$

2.2. Theorem. *Let $h : J \subset \mathbb{R} \rightarrow \mathbb{R}$ be a non-negative function, $f : I \subset [0, \infty) \rightarrow \mathbb{R}$ be a differentiable function on I° such that $h^q, f' \in L[a, b]$ where $a, b \in I^\circ$ with $a < b$ and $h(\alpha) \geq \alpha$. If $|f'|$ is h -convex on I , then*

(2.1)

$$\left| \frac{1}{b-a} \int_a^b f(x) dx - \frac{1}{3} \left[\frac{f(a) + f(b)}{2} + 2f\left(\frac{a+b}{2}\right) \right] \right| \\ \leq \frac{(b-a)}{3} \left(\frac{1+2^{p+1}}{6(p+1)} \right)^{\frac{1}{p}} \times \left\{ |f'(a)| \left[\left(\int_0^{\frac{1}{2}} h^q(t) dt \right)^{\frac{1}{q}} + \left(\int_{\frac{1}{2}}^1 h^q(t) dt \right)^{\frac{1}{q}} \right] \right. \\ \left. + |f'(b)| \left[\left(\int_0^{\frac{1}{2}} h^q(1-t) dt \right)^{\frac{1}{q}} + \left(\int_{\frac{1}{2}}^1 h^q(1-t) dt \right)^{\frac{1}{q}} \right] \right\}.$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. From Lemma 2.1, h -convexity of $|f'|$ and properties of absolute value, we have

$$\begin{aligned}
& \left| \frac{1}{b-a} \int_a^b f(x) dx - \frac{1}{3} \left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right) \right] \right| \\
&= (b-a) \left| \int_0^1 k(t) f'(ta + (1-t)b) dt \right| \\
&\leq (b-a) \left(\int_0^{\frac{1}{2}} \left| t - \frac{1}{6} \right| |f'(ta + (1-t)b)| dt + \int_{\frac{1}{2}}^1 \left| t - \frac{5}{6} \right| |f'(ta + (1-t)b)| dt \right) \\
&\leq (b-a) \left\{ \int_0^{\frac{1}{2}} \left| t - \frac{1}{6} \right| (h(t)|f'(a)| + h(1-t)|f'(b)|) dt \right. \\
&\quad \left. + \int_{\frac{1}{2}}^1 \left| t - \frac{5}{6} \right| (h(t)|f'(a)| + h(1-t)|f'(b)|) dt \right\} \\
&= (b-a) |f'(a)| \left\{ \int_0^{\frac{1}{2}} \left| t - \frac{1}{6} \right| h(t) dt + \int_{\frac{1}{2}}^1 \left| t - \frac{5}{6} \right| h(t) dt \right\} \\
&\quad + (b-a) |f'(b)| \left\{ \int_0^{\frac{1}{2}} \left| t - \frac{1}{6} \right| h(1-t) dt + \int_{\frac{1}{2}}^1 \left| t - \frac{5}{6} \right| h(1-t) dt \right\}.
\end{aligned}$$

By Hölder's inequality, we get

$$\begin{aligned}
& \left| \frac{1}{b-a} \int_a^b f(x) dx - \frac{1}{3} \left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right) \right] \right| \\
&\leq (b-a) |f'(a)| \left\{ \left(\int_0^{\frac{1}{2}} \left| t - \frac{1}{6} \right|^p dt \right)^{\frac{1}{p}} \left(\int_0^{\frac{1}{2}} h^q(t) dt \right)^{\frac{1}{q}} \right. \\
&\quad \left. + \left(\int_{\frac{1}{2}}^1 \left| t - \frac{5}{6} \right|^p dt \right)^{\frac{1}{p}} \left(\int_{\frac{1}{2}}^1 h^q(t) dt \right)^{\frac{1}{q}} \right\} \\
&\quad + (b-a) |f'(b)| \left\{ \left(\int_0^{\frac{1}{2}} \left| t - \frac{1}{6} \right|^p dt \right)^{\frac{1}{p}} \left(\int_0^{\frac{1}{2}} h^q(1-t) dt \right)^{\frac{1}{q}} \right. \\
&\quad \left. + \left(\int_{\frac{1}{2}}^1 \left| t - \frac{5}{6} \right|^p dt \right)^{\frac{1}{p}} \left(\int_{\frac{1}{2}}^1 h^q(1-t) dt \right)^{\frac{1}{q}} \right\}
\end{aligned}$$

Since

$$\int_0^{\frac{1}{2}} \left| t - \frac{1}{6} \right|^p dt = \int_0^{\frac{1}{6}} \left(\frac{1}{6} - t \right)^p dt + \int_{\frac{1}{6}}^{\frac{1}{2}} \left(t - \frac{1}{6} \right)^p dt = \frac{1}{p+1} \left(\frac{1}{6^{p+1}} + \frac{1}{3^{p+1}} \right)$$

and

$$\int_{\frac{1}{2}}^1 \left| t - \frac{5}{6} \right|^p dt = \int_{\frac{1}{2}}^{\frac{5}{6}} \left(\frac{5}{6} - t \right)^p dt + \int_{\frac{5}{6}}^1 \left(t - \frac{5}{6} \right)^p dt = \frac{1}{p+1} \left(\frac{1}{6^{p+1}} + \frac{1}{3^{p+1}} \right),$$

we obtain

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(x) dx - \frac{1}{3} \left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right) \right] \right| \\ & \leq \frac{(b-a)}{3} \left(\frac{1+2^{p+1}}{6(p+1)} \right)^{\frac{1}{p}} \times \left\{ |f'(a)| \left[\left(\int_0^{\frac{1}{2}} h^q(t) dt \right)^{\frac{1}{q}} + \left(\int_{\frac{1}{2}}^1 h^q(t) dt \right)^{\frac{1}{q}} \right] \right. \\ & \quad \left. + |f'(b)| \left[\left(\int_0^{\frac{1}{2}} h^q(1-t) dt \right)^{\frac{1}{q}} + \left(\int_{\frac{1}{2}}^1 h^q(1-t) dt \right)^{\frac{1}{q}} \right] \right\}. \end{aligned}$$

which completes the proof. \square

2.3. Corollary. In Theorem 2.2, if we choose $p = q = 2$, we obtain

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(x) dx - \frac{1}{3} \left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right) \right] \right| \\ & \leq \frac{b-a}{3\sqrt{2}} \left\{ |f'(a)| \left[\left(\int_0^{\frac{1}{2}} h(t^2) dt \right)^{\frac{1}{2}} + \left(\int_{\frac{1}{2}}^1 h(t^2) dt \right)^{\frac{1}{2}} \right] \right. \\ & \quad \left. + |f'(b)| \left[\left(\int_0^{\frac{1}{2}} h((1-t)^2) dt \right)^{\frac{1}{2}} + \left(\int_{\frac{1}{2}}^1 h((1-t)^2) dt \right)^{\frac{1}{2}} \right] \right\} \end{aligned}$$

where h is supermultiplicative.

2.4. Corollary. In Theorem 2.2, if $f(a) = f\left(\frac{a+b}{2}\right) = f(b)$ and $h(t) = 1$, then we have

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \right| \\ & \leq \frac{2(b-a)}{3} \left(\frac{1+2^{p+1}}{6(p+1)} \right)^{\frac{1}{p}} \left(\frac{1}{2} \right)^{\frac{1}{q}} \{ |f'(a)| + |f'(b)| \} \\ & = \frac{b-a}{3} \left(\frac{1+2^{p+1}}{3(p+1)} \right)^{\frac{1}{p}} \{ |f'(a)| + |f'(b)| \} \end{aligned}$$

2.5. Corollary. In Theorem 2.2, if we choose $h(t) = t$, we obtain

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(x) dx - \frac{1}{3} \left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right) \right] \right| \\ & \leq \frac{(b-a)}{3} \left(\frac{1+2^{p+1}}{6(p+1)} \right)^{\frac{1}{p}} \left\{ |f'(a)| \left[\left(\frac{\left(\frac{1}{2}\right)^{q+1}}{q+1} \right)^{\frac{1}{q}} + \left(\frac{1}{2q+2} \left(2 - \left(\frac{1}{2}\right)^q \right) \right)^{\frac{1}{q}} \right] \right. \\ & \quad \left. + |f'(b)| \left[\left(\frac{1}{2q+2} \left(2 - \left(\frac{1}{2}\right)^q \right) \right)^{\frac{1}{q}} + \left(\frac{\left(\frac{1}{2}\right)^{q+1}}{q+1} \right)^{\frac{1}{q}} \right] \right\} \\ & = \frac{(b-a)}{6} \left(\frac{1+2^{p+1}}{3(p+1)} \right)^{\frac{1}{p}} \left(\frac{1}{q+1} \right)^{\frac{1}{q}} \left[\frac{|f'(a)|}{2} + |f'(b)| \left(2 - \left(\frac{1}{2}\right)^q \right)^{\frac{1}{q}} \right]. \end{aligned}$$

2.6. Theorem. Let $h : J \subset \mathbb{R} \rightarrow \mathbb{R}$ be a nonnegative supermultiplicative functions, $f : I \subset [0, \infty) \rightarrow \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a, b]$ where $a, b \in I^\circ$

with $a < b$ and $h(\alpha) \geq \alpha$. If $|f'|$ is h -convex on I , then

$$(2.2) \quad \left| \frac{1}{b-a} \int_a^b f(x) dx - \frac{1}{3} \left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right) \right] \right| \leq (b-a) \{ |f'(a)| [A] + |f'(b)| [B] \}$$

where

$$\begin{aligned} A &= \int_0^{\frac{1}{6}} h\left(t\left[\frac{1}{6}-t\right]\right) dt + \int_{\frac{1}{6}}^{\frac{1}{2}} h\left(t\left[t-\frac{1}{6}\right]\right) dt \\ &\quad + \int_{\frac{1}{2}}^{\frac{5}{6}} h\left(t\left[\frac{5}{6}-t\right]\right) dt + \int_{\frac{5}{6}}^1 h\left(t\left[t-\frac{5}{6}\right]\right) dt \\ B &= \int_0^{\frac{1}{6}} h\left[(1-t)\left(\frac{1}{6}-t\right)\right] dt + \int_{\frac{1}{6}}^{\frac{1}{2}} h\left[(1-t)\left(t-\frac{1}{6}\right)\right] dt \\ &\quad + \int_{\frac{1}{2}}^{\frac{5}{6}} h\left[(1-t)\left(\frac{5}{6}-t\right)\right] dt + \int_{\frac{5}{6}}^1 h\left[(1-t)\left(t-\frac{5}{6}\right)\right] dt. \end{aligned}$$

Proof. From Lemma 2.1, h -convexity of $|f'|$ and properties of absolute value, we have

$$\begin{aligned} &\left| \frac{1}{b-a} \int_a^b f(x) dx - \frac{1}{3} \left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right) \right] \right| \\ &= (b-a) \left| \int_0^1 k(t) f'(ta + (1-t)b) dt \right| \\ &\leq (b-a) \left\{ \int_0^{\frac{1}{2}} \left| t - \frac{1}{6} \right| |f'(ta + (1-t)b)| dt + \int_{\frac{1}{2}}^1 \left| t - \frac{5}{6} \right| |f'(ta + (1-t)b)| dt \right\} \\ &\leq (b-a) \left\{ \int_0^{\frac{1}{2}} \left| t - \frac{1}{6} \right| (h(t)|f'(a)| + h(1-t)|f'(b)|) dt \right. \\ &\quad \left. + \int_{\frac{1}{2}}^1 \left| t - \frac{5}{6} \right| (h(t)|f'(a)| + h(1-t)|f'(b)|) dt \right\} \\ &= (b-a) \left\{ \int_0^{\frac{1}{6}} \left(\frac{1}{6}-t\right) (h(t)|f'(a)| + h(1-t)|f'(b)|) dt \right. \\ &\quad + \int_{\frac{1}{6}}^{\frac{1}{2}} \left(t-\frac{1}{6}\right) (h(t)|f'(a)| + h(1-t)|f'(b)|) dt \\ &\quad + \int_{\frac{1}{2}}^{\frac{5}{6}} \left(\frac{5}{6}-t\right) (h(t)|f'(a)| + h(1-t)|f'(b)|) dt \\ &\quad \left. + \int_{\frac{5}{6}}^1 \left(t-\frac{5}{6}\right) (h(t)|f'(a)| + h(1-t)|f'(b)|) dt \right\}. \end{aligned}$$

By properties of function h , we can write

$$\begin{aligned}
& \left| \frac{1}{b-a} \int_a^b f(x) dx - \frac{1}{3} \left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right) \right] \right| \\
& \leq (b-a) \left\{ \int_0^{\frac{1}{6}} \left[h\left(t\left(\frac{1}{6}-t\right)\right) |f'(a)| + h\left((1-t)\left(\frac{1}{6}-t\right)\right) |f'(b)| \right] dt \right. \\
& \quad + \int_{\frac{1}{6}}^{\frac{1}{2}} \left[h\left(t\left(t-\frac{1}{6}\right)\right) |f'(a)| + h\left((1-t)\left(t-\frac{1}{6}\right)\right) |f'(b)| \right] dt \\
& \quad + \int_{\frac{1}{2}}^{\frac{5}{6}} \left[h\left(t\left(\frac{5}{6}-t\right)\right) |f'(a)| + h\left((1-t)\left(\frac{5}{6}-t\right)\right) |f'(b)| \right] dt \\
& \quad \left. + \int_{\frac{5}{6}}^1 \left[h\left(t\left(t-\frac{5}{6}\right)\right) |f'(a)| + h\left((1-t)\left(t-\frac{5}{6}\right)\right) |f'(b)| \right] dt \right\} \\
& = (b-a) |f'(a)| \left\{ \int_0^{\frac{1}{6}} h\left(t\left[\frac{1}{6}-t\right]\right) dt + \int_{\frac{1}{6}}^{\frac{1}{2}} h\left(t\left[t-\frac{1}{6}\right]\right) dt \right. \\
& \quad \left. + \int_{\frac{1}{2}}^{\frac{5}{6}} h\left(t\left[\frac{5}{6}-t\right]\right) dt + \int_{\frac{5}{6}}^1 h\left(t\left[t-\frac{5}{6}\right]\right) dt \right\} \\
& \quad + (b-a) |f'(b)| \left\{ \int_0^{\frac{1}{6}} h\left((1-t)\left(\frac{1}{6}-t\right)\right) dt + \int_{\frac{1}{6}}^{\frac{1}{2}} h\left((1-t)\left(t-\frac{1}{6}\right)\right) dt \right. \\
& \quad \left. + \int_{\frac{1}{2}}^{\frac{5}{6}} h\left((1-t)\left(\frac{5}{6}-t\right)\right) dt + \int_{\frac{5}{6}}^1 h\left((1-t)\left(t-\frac{5}{6}\right)\right) dt \right\}
\end{aligned}$$

which completes the proof. \square

2.7. Corollary. In Theorem 2.6, if $f(a) = f\left(\frac{a+b}{2}\right) = f(b)$ and $h(t) = 1$, then we have

$$\left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \right| \leq (b-a) \{|f'(a)| + |f'(b)|\}.$$

2.8. Remark. In Theorem 2.6, if we choose $h(t) = t$, then we obtain the inequality (1.7).

2.9. Theorem. Let $h : J \subset \mathbb{R} \rightarrow \mathbb{R}$ be non-negative functions, $f : I \subset [0, \infty) \rightarrow \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a, b]$ where $a, b \in I^\circ$ with $a < b$. If $|f'|$ is h -concave on I , then

$$\begin{aligned}
& \left| \frac{1}{b-a} \int_a^b f(x) dx - \frac{1}{3} \left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right) \right] \right| \\
& \leq \frac{b-a}{12} \left(\frac{2+2^{p+2}}{p+1} \right)^{\frac{1}{p}} \left[\frac{1}{h\left(\frac{1}{2}\right)} \right]^{\frac{1}{q}} \left| f'\left(\frac{a+b}{2}\right) \right|.
\end{aligned}$$

Proof. From Lemma 2.1 and using the Hölder inequality, we have

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(x) dx - \frac{1}{3} \left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right) \right] \right| \\ & \leq (b-a) \left| \int_0^1 k(t) f'(ta+(1-t)b) dt \right| \\ & \leq (b-a) \left(\int_0^1 |k(t)|^p dt \right)^{\frac{1}{p}} \left(\int_0^1 |f'(ta+(1-t)b)|^q dt \right)^{\frac{1}{q}} \end{aligned}$$

Since $|f'|$ is h -concave on I , by inequalities (1.8) we have

$$\int_0^1 |f'(ta+(1-t)b)|^q dt \leq \frac{1}{2h(\frac{1}{2})} \left| f'\left(\frac{a+b}{2}\right) \right|^q.$$

Therefore, we get

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(x) dx - \frac{1}{3} \left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right) \right] \right| \\ & \leq (b-a) \left\{ \int_0^{\frac{1}{2}} \left| t - \frac{1}{6} \right|^p dt + \int_{\frac{1}{2}}^1 \left| t - \frac{5}{6} \right|^p dt \right\}^{\frac{1}{p}} \left[\frac{1}{2h(\frac{1}{2})} \left| f'\left(\frac{a+b}{2}\right) \right|^q \right]^{\frac{1}{q}}. \end{aligned}$$

Since

$$\int_0^{\frac{1}{2}} \left| t - \frac{1}{6} \right|^p dt = \int_0^{\frac{1}{6}} \left(\frac{1}{6} - t \right)^p dt + \int_{\frac{1}{6}}^{\frac{1}{2}} \left(t - \frac{1}{6} \right)^p dt = \frac{1}{p+1} \left(\frac{1}{6^{p+1}} + \frac{1}{3^{p+1}} \right)$$

and

$$\int_{\frac{1}{2}}^1 \left| t - \frac{5}{6} \right|^p dt = \int_{\frac{1}{2}}^{\frac{5}{6}} \left(\frac{5}{6} - t \right)^p dt + \int_{\frac{5}{6}}^1 \left(t - \frac{5}{6} \right)^p dt = \frac{1}{p+1} \left(\frac{1}{6^{p+1}} + \frac{1}{3^{p+1}} \right),$$

we obtain

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(x) dx - \frac{1}{3} \left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right) \right] \right| \\ & \leq \frac{b-a}{12} \left(\frac{2+2^{p+2}}{p+1} \right)^{\frac{1}{p}} \left[\frac{1}{h(\frac{1}{2})} \right]^{\frac{1}{q}} \left| f'\left(\frac{a+b}{2}\right) \right|. \end{aligned}$$

This completes the proof. \square

2.10. Corollary. In Theorem 2.9, if we choose $h(t) = t$, then we obtain

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(x) dx - \frac{1}{3} \left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right) \right] \right| \\ & \leq \frac{b-a}{6} \left(\frac{1+2^{p+1}}{p+1} \right)^{\frac{1}{p}} \left| f'\left(\frac{a+b}{2}\right) \right|. \end{aligned}$$

3. APPLICATIONS TO SPECIAL MEANS

We now consider the means for arbitrary real numbers α, β ($\alpha \neq \beta$). We take

(1) *Arithmetic mean* :

$$A(\alpha, \beta) = \frac{\alpha + \beta}{2}, \quad \alpha, \beta \in \mathbb{R}^+.$$

(2) *Logarithmic mean:*

$$L(\alpha, \beta) = \frac{\alpha - \beta}{\ln|\alpha| - \ln|\beta|}, \quad |\alpha| \neq |\beta|, \alpha, \beta \neq 0, \alpha, \beta \in \mathbb{R}^+.$$

(3) *Generalized log - mean:*

$$L_n(\alpha, \beta) = \left[\frac{\beta^{n+1} - \alpha^{n+1}}{(n+1)(\beta - \alpha)} \right]^{\frac{1}{n}}, \quad n \in \mathbb{Z} \setminus \{-1, 0\}, \alpha, \beta \in \mathbb{R}^+.$$

Now using the results of Section 2, we give some applications for special means of real numbers.

3.1. Proposition. *Let $a, b \in \mathbb{R}^+$, $0 < a < b$ and $n \in \mathbb{N}, n > 1$. Then, we have*

$$\begin{aligned} & \left| L_n^n(a, b) - \frac{1}{3} [A(a^n, b^n) - A^n(a, b)] \right| \\ & \leq n \frac{(b-a)}{6} \left(\frac{1+2^{p+1}}{3(p+1)} \right)^{\frac{1}{p}} \left(\frac{1}{q+1} \right)^{\frac{1}{q}} \left[\frac{a^{n-1}}{2} + b^{n-1} \left(2 - \left(\frac{1}{2} \right)^q \right)^{\frac{1}{q}} \right]. \end{aligned}$$

Proof. The assertion follows from Corollary 2.5 applied for $f(x) = x^n$, $x \in \mathbb{R}$, $n \in \mathbb{N}$. \square

3.2. Proposition. *Let $a, b \in \mathbb{R}^+$, $a < b$. Then, we have*

$$|L_n^n(a, b) - A^n(a, b)| \leq (b-a) \left[\frac{1}{a^2} + \frac{1}{b^2} \right].$$

Proof. The assertion follows from Corollary 2.7 applied for $f(x) = \frac{1}{x}$, $x \in [a, b]$. \square

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