Hacettepe Journal of Mathematics and Statistics
Volume 43 (2) (2014), 173 – 182

Generalized uniformly close-to-convex functions of order γ and type β

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Abstract

In this paper, a class of analytic functions f defined on the open unit disc satisfying

$$\operatorname{Re}\left\{\frac{z(D_\lambda^{n,\alpha}f(z))'}{D_\lambda^{n,\alpha}g(z)}\right\} > \beta\left|\frac{z(D_\lambda^{n,\alpha}f(z))'}{D_\lambda^{n,\alpha}g(z)} - 1\right| + \gamma,$$

is studied, where $\beta \geq 0, \quad -1 \leq \gamma < 1, \quad \beta + \gamma \geq 0.$ and g is a certain analytic function associated with conic domains.

Among other results, inclusion relations and the coefficients bound are studied. Various known special cases of these results are pointed out. A subclass of uniformly quasi-convex functions is also studied.

Keywords: Univalent functions, uniformly close-to-convex, uniformly quasi-convex, fractional differential operator.

2000 AMS Classification: 30C45.

1. Introduction

Let A denote the class of functions of the form

$$(1.1) \qquad f(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$

analytic in the unit disc $E = \{z \in \mathbb{C} : |z| < 1\}$, and let S denote the class of functions $f \in A$ which are univalent on E. Denote by $CV(\gamma), ST(\gamma), CC(\gamma)$, and $QC(\gamma)$, where $0 \leq \gamma < 1$, the well-known subclasses of S which are convex, starlike, close-to-convex and quasi-convex functions of order γ, respectively, and by CV, ST, CC, and QC, the corresponding classes when $\gamma = 0$.

Define the function $\varphi(a, c; z)$ by

$$\varphi(a, c; z) = z_2F_1(1, a; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} z^{k-1}, c \neq 0, -1, -2, \ldots, z \in E,$$

where $(\sigma)_k$ is Pochhammer symbol defined in terms of Gamma function.

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Owa and Srivastava [18] introduced the operator $\Omega^\alpha : A \to A$ where

$$\Omega^\alpha f(z) = \Gamma(2-\alpha) z^\alpha D_z^\alpha f(z), \quad \alpha \neq 2, 3, \ldots$$

$$= z + \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)} a_k z^k, \tag{1.2}$$

$$= \varphi(2, 2-\alpha; z) * f(z). \tag{1.3}$$

Note that $\Omega^0 f(z) = f(z)$.

The linear fractional differential operator $D_\lambda^{n,\alpha} f : A \to A, \quad 0 \le \alpha < 1, \quad \lambda \ge 0, \quad n \in N_0 = N \cup \{0\}$ is defined [5] as follows

$$D_\lambda^{n,\alpha} f(z) = z + \sum_{k=2}^{\infty} \psi_{k,n}(\alpha,\lambda) a_k z^k, \quad n \in N_0, \tag{1.4}$$

where

$$\psi_{k,n}(\alpha,\lambda) = \left[\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)} (1+\lambda(k-1)) \right]^n.$$

From (1.3), and (1.4), $D_\lambda^{n,\alpha} f(z)$ can be written, in terms of convolution, as

$$D_\lambda^{n,\alpha} f(z) = \underbrace{[\varphi(2, 2-\alpha; z) * h_\lambda(z) * \cdots * \varphi(2, 2-\alpha; z) * h_\lambda(z)]}_{n\text{-times}} * f(z), \tag{1.5}$$

where

$$h_\lambda(z) = \frac{z - (1-\lambda) z^2}{(1-z)^2} = z + \sum_{k=2}^{\infty} [1 + \lambda(k-1)] z^k.$$

Note that $D_\lambda^{n,0} = D_\lambda^n$ (Al-Oboudi differential operator [4]), $D_1^{n,0} = D^n$ (Salagean differential operator [23]) and $D_0^{1,\alpha} = \Omega^\alpha$ (Owa-Srivastava fractional differential operator [18]).

Using the operator $D_\lambda^{n,\alpha}$, the following classes are defined [5].

The classes $UCV_\lambda^{n,\alpha}(\beta,\gamma)$, $\beta \ge 0$, $-1 \le \gamma < 1$, $\beta + \gamma \ge 0$, and $SP_\lambda^{n,\alpha}(\beta,\gamma)$, satisfying

$$f \in UCV_\lambda^{n,\alpha}(\beta,\gamma) \text{ if and only if } zf' \in SP_\lambda^{n,\alpha}(\beta,\gamma).$$

Note that $f \in UCV_\lambda^{n,\alpha}(\beta,\gamma)(SP_\lambda^{n,\alpha}(\beta,\gamma))$ if and only if $D_\lambda^{n,\alpha} f \in UCV(\beta,\gamma)(SP(\beta,\gamma))$, where $UCV(\beta,\gamma)$, is the class of uniformly convex functions of order β and type γ and $SP(\beta,\gamma)$, is the class of functions of conic domains and related with$UCV(\beta,\gamma)$ by Alexander-type relation [7].

These classes generalize various other classes investigated earlier by Goodman [9], Ronning [20], [21], Kanas and Wisniowska [10], [11] Srivastava and Mishra [26] and others. Several basic and interesting results have been studied for these classes [5], [6], such as inclusion relations, convolution properties, coefficient bounds, subordination results.

The class $UCC(\beta,\gamma)$, of uniformly close-to-convex functions of order γ and type β is defined [3] as

$$\operatorname{Re}\left\{ \frac{zf'(z)}{g(z)} \right\} > \beta \left| \frac{zf'(z)}{g(z)} - 1 \right| + \gamma,$$

where $g \in SP(\beta,\gamma)$, $\beta \geq 0, -1 \leq \gamma < 1$, and $\beta+\gamma \geq 0$. It is clear that $UCC(0,\gamma) = CC(\gamma)$.

Since these functions are related to the uniformly convex functions UCV and with the class SP, they are called uniformly close-to-convex functions [8].

Denote by $UQC(\beta,\gamma)$, the class of uniformly quasi-convex functions of order γ and type β [3], where

$$f \in UQC(\beta,\gamma), \text{ if and only if } zf' \in UCC(\beta,\gamma).$$

Note that

$$UCV(\beta,\gamma) \subset UQC(\beta,\gamma) \subset UCC(\beta,\gamma).$$

The classes of uniformly close-to-convex and quasi-convex functions of order γ and type β had been studied by a number of authors under different operators, for example Acu [1], Acu and Blezu [2], Blezu [8], Kumar and Ramesha [13], Noor et al [16], Srivastava and Mishra [25] and Srivastava et al [26].

In the following, we use the operator $D_\lambda^{n,\alpha}$ to define generalized classes of uniformly close-to-convex functions and uniformly quasi-convex functions of order γ and type β.

1.1. Definition. A function $f \in A$ is in the class $UCC_\lambda^{n,\alpha}(\beta,\gamma)$ if and only if, there exist a function $g \in SP_\lambda^{n,\alpha}(\beta,\gamma)$ such that $z \in E$,

$$\text{(1.6)} \qquad \operatorname{Re}\left\{\frac{z(D_\lambda^{n,\alpha}f(z))'}{D_\lambda^{n,\alpha}g(z)}\right\} > \beta\left|\frac{z(D_\lambda^{n,\alpha}f(z))'}{D_\lambda^{n,\alpha}g(z)} - 1\right| + \gamma,$$

where $\beta \geq 0,\ -1 \leq \gamma < 1,\ \beta+\gamma \geq 0$. Note that $D_\lambda^{n,\alpha}f \in UCC(\beta,\gamma)$, and that $SP_\lambda^{n,\alpha}(\beta,\gamma) \subset UCC_\lambda^{n,\alpha}(\beta,\gamma)$.

1.2. Definition. A function $f \in A$ is in the class $U_\lambda^{n,\alpha}QC(\beta,\gamma)$ if and only if, there exists a function $g \in UCV_\lambda^{n,\alpha}(\beta,\gamma)$ such that for $z \in E$,

$$\text{(1.7)} \qquad \operatorname{Re}\left\{\frac{(z(D_\lambda^{n,\alpha}f(z))')'}{(D_\lambda^{n,\alpha}g(z))'}\right\} > \beta\left|\frac{(z(D_\lambda^{n,\alpha}f(z))')'}{(D_\lambda^{n,\alpha}g(z)} - 1\right| + \gamma,$$

where $\beta \geq 0,\ -1 \leq \gamma < 1,\ \beta+\gamma \geq 0$. Note that $D_\lambda^{n,\alpha}f \in UQC(\beta,\gamma)$.

It is clear that

$$\text{(1.8)} \qquad f \in UQC_\lambda^{n,\alpha}(\beta,\gamma) \text{ if and only if } zf' \in UCC_\lambda^{n,\alpha}(\beta,\gamma),$$

and that

$$\text{(1.9)} \qquad UCV_\lambda^{n,\alpha}(\beta,\gamma) \subset UQC_\lambda^{n,\alpha}(\beta,\gamma) \subset UCC_\lambda^{n,\alpha}(\beta,\gamma).$$

We may rewrite the condition (1.6)((1.7)), in the form

$$\text{(1.10)} \qquad p \prec P_{\beta,\gamma},$$

where $p(z) = \frac{z(D_\lambda^{n,\alpha}f(z))'}{D_\lambda^{n,\alpha}g(z)}\left(\frac{(z(D_\lambda^{n,\alpha}f(z))')'}{D(_\lambda^{n,\alpha}g(z))'}\right)$ and the function $P_{\beta,\gamma}$ is given in [5].

By virtue of (1.6), (1.7) and the properties of the domain $R_{\beta,\gamma}$, we have respectively

$$\text{(1.11)} \qquad \operatorname{Re}\left\{\frac{z(D_\lambda^{n,\alpha}f(z))'}{D_\lambda^{n,\alpha}g(z)}\right\} > \frac{\beta+\gamma}{1+\beta},$$

and

$$\text{(1.12)} \quad \operatorname{Re}\left\{\frac{(z(D_\lambda^{n,\alpha}f(z))')'}{D(_\lambda^{n,\alpha}g(z))'}\right\} > \frac{\beta+\gamma}{1+\beta},$$

which means that

$$f \in UCC(\beta,\gamma) \text{ implies } D_\lambda^{n,\alpha}f \in CC\left(\frac{\beta+\gamma}{1+\beta}\right) \subseteq CC,$$

and

$$f \in UQC(\beta,\gamma) \text{ implies } D_\lambda^{n,\alpha}f \in QC\left(\frac{\beta+\gamma}{1+\beta}\right) \subseteq QC.$$

Definitions 1.1, and 1.2, includes various classes introduced earlier by Al-Oboudi and Al-Amoudi [4], Blezu [8], Acu and Bezu [2], Aghalary and Azadi [3], Subramanian et al [27],.Kumar and Ramesha [13],.Kaplan [12], and Noor and Thomas [15]

In this paper, basic results for the classes $UCC_\lambda^{n,\alpha}(\beta,\gamma)$ and $UQC_\lambda^{n,\alpha}(\beta,\gamma)$ such as inclusion relations, the coefficients bound and sufficient condition, will be studied. Various known special cases of these results are pointed out.

2. Inclusion Relations

The inclusion relations of the classes $UCC_\lambda^{n,\alpha}(\beta,\gamma)$ and $UQC_\lambda^{n,\alpha}(\beta,\gamma)$ for different values of the parameters n, α, β and γ will be studied. It will also be shown that the classes $UQC_\lambda^{n,\alpha}(\beta,\gamma)$ and $SP_\lambda^{n,\alpha}(\beta,\gamma)$ are not related with set inclusion. To derive our results we need the following.

2.1. Lemma. *[22] Let $f, g \in A$ be univalent starlike of order $\frac{1}{2}$. Then, for every function $F \in A$, we have*

$$\frac{f(z)*g(z)F(z)}{f(z)*g(z)} \in \overline{co}F(z), \quad z \in E,$$

where $\overline{co}$ denotes the closed convex hull.

2.2. Lemma. [14] *Let P be analytic function in E, with Re $P(z) > 0$ for $z \in E$, and let h be a convex function in E. If p is analytic in E, with $p(0) = h(0)$ and if $p(z) + P(z)zp'(z) \prec h(z)$, then $p(z) \prec h(z)$.*

Following the same method of [5, Lemma 2.5], we obtain.

2.3. Lemma. *Let $\Omega^\alpha f$ be in the class $UCC_\lambda^{n,\alpha}(\beta,\gamma)(UQC_\lambda^{n,\alpha}(\beta,\gamma))$, then so is f.*

2.4. Theorem. *Let $0 \le \lambda \le \frac{1+\beta}{1-\gamma}$. Then*

$$UCC_\lambda^{n+1,\alpha}(\beta,\gamma) \subset UCC_\lambda^{n,\alpha}(\beta,\gamma).$$

Proof. Let $f \in UCC_\lambda^{n+1,\alpha}(\beta,\gamma)$. Then by (1.10)

$$\text{(2.1)} \quad \frac{z(D_\lambda^{n+1,\alpha}f(z))'}{D_\lambda^{n+1,\alpha}g(z)} \prec P_{\beta,\gamma}(z),$$

where the function $P_{\beta,\gamma}$ is given in [5], and $g \in SP_\lambda^{n+1,\alpha}(\beta,\gamma)$. From [5, proof of Theorem 2.4], $\Omega^\alpha g(z) \in SP_\lambda^{n,\alpha}(\beta,\gamma)$, for $0 \le \lambda < \dfrac{1+\beta}{1-\gamma}$. Hence

$$\text{(2.2)} \qquad \frac{z(D_\lambda^{n,\alpha}\Omega^\alpha g(z))'}{D_\lambda^{n,\alpha}\Omega^\alpha g(z)} = q(z),$$

where $q(z) \prec P_{\beta,\gamma}(z)$.

By the definition of $D_\lambda^{n,\alpha} f$, we get

$$D_\lambda^{n+1,\alpha} f(z) = (1-\lambda) D_\lambda^{n,\alpha}\Omega^\alpha f(z) + \lambda z (D_\lambda^{n,\alpha}\Omega^\alpha f(z))'$$

and

$$D_\lambda^{n+1,\alpha} g(z) = (1-\lambda) D_\lambda^{n,\alpha}\Omega^\alpha g(z) + \lambda z (D_\lambda^{n,\alpha}\Omega^\alpha g(z))'.$$

Using (2.1), (2.2) and the above equalities, with the notation $p(z) = \dfrac{z(D_\lambda^{n,\alpha}\Omega^\alpha f(z))'}{D_\lambda^{n,\alpha}\Omega^\alpha g(z)}$, we obtain

$$\text{(2.3)} \qquad \frac{z(D_\lambda^{n+1,\alpha} f(z))'}{D_\lambda^{n+1,\alpha} g(z)} = p(z) + \frac{\lambda z p'(z)}{(1-\lambda) q(z)} .$$

For $\lambda = 0$, $\Omega^\alpha f \in UCC_\lambda^{n,\alpha}(\beta,\gamma)$, from (2.1) and (2.3). Hence by Lemma 2.2 $f \in UCC_\lambda^{n,\alpha}(\beta,\gamma)$.

For $\lambda \neq 0$, (2.3) can be written, using (2.1), as

$$\text{(2.4)} \qquad p(z) + \frac{zp'(z)}{\frac{(1-\lambda)}{\lambda} q(z)} \prec P_{\beta,\gamma}.$$

Hence by Lemma 2.2 and (1.11), we have $p(z) \prec P_{\beta,\gamma}(z)$ for $0 < \lambda \le \dfrac{1+\beta}{1-\gamma}$. Thus $\Omega^\alpha f \in UCC_\lambda^{n,\alpha}(\beta,\gamma)$, which implies that $f \in UCC_\lambda^{n,\alpha}(\beta,\gamma)$, using Lemma 2.3. □

2.5. Corollary. *Let* $0 \le \lambda \le \dfrac{1+\beta}{1-\gamma}$. *Then*

$$UQC_\lambda^{n+1,\alpha}(\beta,\gamma) \subset UQC_\lambda^{n,\alpha}(\beta,\gamma).$$

Proof. Let $f \in UQC_\lambda^{n+1,\alpha}(\beta,\gamma)$, $0 \le \lambda \le \dfrac{1+\beta}{1-\gamma}$. Then by (1.8) $z\acute{f} \in UCC_\lambda^{n+1,\alpha}(\beta,\gamma)$. Which implies, by Theorem 2.4, that

$$z\acute{f} \in UCC_\lambda^{n,\alpha}(\beta,\gamma)$$

Hence, by (1.8), $f \in UQC_\lambda^{n,\alpha}(\beta,\gamma)$. □

2.6. Corollary. *Let* $0 \le \lambda \le \dfrac{1+\beta}{1-\lambda}$. *Then*

$$UCC_\lambda^{n,\alpha}(\beta,\gamma) \subset UCC_\lambda^{0,\alpha}(\beta,\gamma) \equiv UCC(\beta,\gamma) \subset CC,$$

and

$$UQC_\lambda^{n,\alpha}(\beta,\gamma) \subset UQC_\lambda^{0,\alpha}(\beta,\gamma) \equiv UQC(\beta,\gamma) \subset CC.$$

This means that, for $0 < \lambda \leq \dfrac{1+\beta}{1-\gamma}$ *functions in* $UCC_\lambda^{n,\alpha}(\beta,\gamma)$ *and* $UQC_\lambda^{n,\alpha}(\beta,\gamma)$*, are close-to-convex and hence univalent.*

2.7. Remark. If we put $\lambda = 1$ and $\alpha = 0$, in Theorem 2.4, then we get the result of Blezu [8].

In view of the relations

$$UCV_\lambda^{n,\alpha}(\beta,\gamma) \subset SP_\lambda^{n,\alpha}(\beta,\gamma) \subset UCC_\lambda^{n,\alpha}(\beta,\gamma),$$

and

$$UCV_\lambda^{n,\alpha}(\beta,\gamma) \subset UQC_\lambda^{n,\alpha}(\beta,\gamma) \subset UCC_\lambda^{n,\alpha}(\beta,\gamma),$$

one may ask whether the classes $SP_\lambda^{n,\alpha}(\beta,\gamma)$ and $UQC_\lambda^{n,\alpha}(\beta,\gamma)$ are related with set inclusion? The answer is negative. The function f_0, defined by

$$f_0(z) = \frac{1-i}{2}\frac{z}{1-z} - \frac{1+i}{2}\log(1-z).$$

belongs to $UQC_\lambda^{n,\alpha}(\beta,\gamma)$, but not to $SP_\lambda^{n,\alpha}(\beta,\gamma)$. In fact, Silverman and Telage [24], have shown that $f_0 \notin ST \equiv SP_\lambda^{0,\alpha}(1,0)$ and that $f_0 \in QC \equiv UQC_\lambda^{0,\alpha}(1,0)$..Also, the Koebe function $K(z) = \dfrac{z}{(1-z)^2} \in SP_\lambda^{0,\alpha}(1,0)$ and $K(z) \notin UQC_\lambda^{0,\alpha}(1,0)$.

In the following we prove the inclusion relation with respect to α.

2.8. Theorem. *Let* $0 \leq \mu \leq \alpha < 1$*. Then*

$$UCC_\lambda^{n,\alpha}(\beta,\gamma) \subset UCC_\lambda^{n,\mu}(\beta,\gamma),$$

where $\left(0 \leq \beta < 1 \text{ and } \dfrac{1}{2} \leq \gamma < 1\right)$ *or* $(\beta \geq 1$ *and* $0 \leq \gamma < 1)$*.*

Proof. Let $f \in UCC_\lambda^{n,\alpha}(\beta,\gamma)$. Then by (1.5) and the convolution properties, we have

$$z(D_\lambda^{n,\mu}f(z))' = \underbrace{\varphi(2-\alpha,2-\mu;z) * \cdots * \varphi(2-\alpha,2-\mu;z)}_{n\text{-times}} * z(D_\lambda^{n,\alpha}f(z))'.$$

Hence

$$\frac{z(D_\lambda^{n,\mu}f(z))'}{D_\lambda^{n,\mu}g(z)} = \frac{\underbrace{\varphi(2-\alpha,2-\mu;z) * \cdots * \varphi(2-\alpha,2-\mu;z)}_{n\text{-times}} * \dfrac{z(D_\lambda^{n,\alpha}f(z))'}{D_\lambda^{n,\alpha}g(z)} D_\lambda^{n,\alpha}g(z)}{\underbrace{\varphi(2-\alpha,2-\mu;z) * \cdots * \varphi(2-\alpha,2-\mu;z)}_{n\text{-times}} * D_\lambda^{n,\alpha}g(z)}.$$

It has been shown [5] that the function $\underbrace{\varphi(2-\alpha,2-\mu;z) * \cdots * \varphi(2-\alpha,2-\mu;z)}_{n\text{-times}} \in ST\left(\dfrac{1}{2}\right)$ and $D_\lambda^{n,\alpha}g(z)$ is a starlike function of order $\dfrac{1}{2}$ for $\left(0 \leq \beta < 1 \text{ and } \dfrac{1}{2} \leq \gamma < 1\right)$ or $(\beta \geq 1$ and $0 \leq \gamma < 1)$. Applying Lemma 2.1, we get the required result. $\square$

The next result follows using (1.8).

2.9. Corollary. *Let* $0 \le \mu \le \alpha < 1$. *Then*

$$UQC_\lambda^{n,\alpha}(\beta,\gamma) \subset UQC_\lambda^{n,\mu}(\beta,\gamma),$$

where $\left(0 \le \beta < 1 \text{ and } \dfrac{1}{2} \le \gamma < 1\right)$ *or* $(\beta \ge 1 \text{ and } 0 \le \gamma < 1)$.

The inclusion relation with respect to β and γ follows directly by (1.6) and (1.7).

2.10. Theorem. *Let* $\beta_1 \ge \beta_2$, *and* $\gamma_1 \ge \gamma_2$. *Then*

(i) $UCC_\lambda^{n,\alpha}(\beta_1,\gamma_1) \subset UCC_\lambda^{n,\alpha}(\beta_2,\gamma_2)$.
(ii) $UQC_\lambda^{n,\alpha}(\beta_1,\gamma_1) \subset UQC_\lambda^{n,\alpha}(\beta_2,\gamma_2)$.

2.11. Remark. If we put $\lambda = 1$ and $\alpha = 0$, in Theorem 2.10 (i), we get the result of Blezu [8].

3. Coefficients Bound

To derive our results we need the folowing.

3.1. Lemma. *[5] If a function* $f \in A$, *of the form (1.1) is in* $SP_\lambda^{n,\alpha}(\beta,\gamma)$, *then*

$$|a_k| \le \frac{1}{\psi_{k,n}(\alpha,\lambda)} \cdot \frac{(P_1)_{k-1}}{(1)_{k-1}}\ , \quad k \ge 2,$$

where

$$(3.1) \qquad P_1 = P_1(\beta,\gamma) = \begin{cases} \dfrac{8(1-\gamma)(\cos^{-1}\beta)^2}{\pi^2(1-\beta^2)}\ , & 0 \le \beta < 1, \\ \dfrac{8}{\pi^2}(1-\gamma)\ , & \beta = 1 \\ \dfrac{\pi^2(1-\gamma)}{4 \subseteq t(\beta^2-1)k^2(t)(1+t)}\ , & \beta > 1,\ 0 < t < 1, \end{cases}$$

3.2. Lemma. *[19] Let* $h(z) = 1 + \sum_{k=1}^{\infty} c_k z^k$ *be subordinate to* $H(z) = 1 + \sum_{k=1}^{\infty} C_k z^k$ *in* E. *If* $H(z)$ *is univalent in* E *and* $H(E)$ *is convex, then* $|c_k| \le |C_1|$, $\quad k \ge 1$.

3.3. Theorem. *Let* $f \in UCC_\lambda^{n,\alpha}(\beta,\gamma)$, *and given by (1.1). Then*

$$|a_k| \le \frac{1}{\psi_{k,n}(\alpha,\lambda)} \cdot \frac{(P_1)_{k-1}}{(1)_{k-1}}\ , \quad k \ge 2,$$

where P_1 *is given by (3.1).*

Proof. Since $f \in UCC_\lambda^{n,\alpha}(\beta,\gamma)$, then

$$(3.2) \qquad \frac{z(D_\lambda^{n,\alpha} f(z))'}{D_\lambda^{n,\alpha} g(z)} = p(z) \prec P_{\beta,\gamma}\ ,$$

where $p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k$, $g \in SP_{\lambda}^{n,\alpha}(\beta,\gamma)$, and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$. The function $P_{\beta,\gamma}$ is univalent in E and $P_{\beta,\gamma}(E)$, the conic domain is a convex domain, hence, applying Lemma 3.2, we obtain

$$|c_k| \le P_1, \quad k \ge 1.$$

where P_1 is given by (3.1).

From (3.2) and (1.4), we get

$$z + \sum_{k=2}^{\infty} \psi_{k,n}(\alpha,\lambda) k a_k z^k = \left(z + \sum_{k=2}^{\infty} \psi_{k,n}(\alpha,\lambda) b_k z^k\right)\left(1 + \sum_{k=1}^{\infty} c_k z^k\right). \tag{3.3}$$

Equating the coefficients of z^k in (3.3), we get

$$\begin{aligned}\psi_{k,n}(\alpha,\lambda) k a_k &= \sum_{j=1}^{k-1}[c_{k-j} b_j \psi_{j,n}(\alpha,\lambda)] + b_k \psi_{k,n}(\alpha,\lambda), \quad c_0 = 1\\ &= c_{k-1} + \sum_{j=2}^{k-1}[c_{k-j} b_j \psi_{j,n}(\alpha,\lambda)] + b_k \psi_{k,n}(\alpha,\lambda), b_1 = \psi_{1,n}(\alpha,\lambda) = 1.\end{aligned}$$

Hence

$$\psi_{k,n}(\alpha,\lambda) k |a_k| \le |c_{k-1}| + \sum_{j=2}^{k-1}[|c_{k-j}|\ |b_j| \psi_{j,n}(\alpha,\lambda)] + |b_k| \psi_{k,n}(\alpha,\lambda).$$

Using Lemmas 3.1 and 3.2, we obtain

$$\psi_{k,n}(\alpha,\lambda) k |a_k| \le P_1 \left\{1 + \sum_{j=2}^{k-1}\left[\frac{(P_1)_{j-1}}{(1)_{j-1}}\right]\right\} + \frac{(P_1)_{k-1}}{(1)_{k-1}}. \tag{3.4}$$

Applying mathematical induction, we can see that

$$1 + \sum_{j=2}^{k-1}\left[\frac{(P_1)_{j-1}}{(1)_{j-1}}\right] = \frac{(P_1)_{k-1}}{P_1 (1)_{k-2}}. \tag{3.5}$$

Using (3.5) in (3.4), we get

$$\begin{aligned}\psi_{k,n}(\alpha,\lambda) k |a_k| &\le \frac{(P_1)_{k-1}}{(1)_{k-2}} + \frac{(P_1)_{k-1}}{(1)_{k-1}}\\ &= \frac{(P_1)_{k-1}}{(1)_{k-1}}\, k,\end{aligned}$$

which is the required result. □

From (1.8) and Theorem 3.3, we immediately have

3.4. Corollary. *Let $f \in UQC_{\lambda}^{n,\alpha}(\beta,\gamma)$. Then*

$$|a_k| \le \frac{1}{\psi_{k,n}(\alpha,\lambda)} \cdot \frac{(P_1)_{k-1}}{(1)_k}, \quad k \ge 2,$$

where P_1 is given by (3.1).

3.5. Remark. The results of Theorem 3.3 and Corollary 3.4 are sharp for $k = 2$.

3.6. Remark. In special cases, Theorem 3.1 reduces to the results of Acu and Blezu [2], Subramanian et al [27], Kaplan [12] and Noor and Thomas [15].

Next we give a sufficient condition for a function to be in the class $UCC_{\lambda}^{n,\alpha}(\beta,\gamma)$.

3.7. Theorem. *If*

$$\sum_{k=2}^{\infty} k|a_k|\psi_{k,n}(\alpha,\lambda) \leq \frac{(1-\gamma)}{1+\beta}\ , \tag{3.6}$$

then a function f, given by (1.1), is in $UCC_{\lambda}^{n,\alpha}(\beta,\gamma)$.

Proof. Let $g(z) = z$. Then $D_{\lambda}^{n,\alpha}g(z) = z$, and

$$\frac{z(D_{\lambda}^{n,\alpha}f(z))'}{D_{\lambda}^{n,\alpha}g(z)} = z(D_{\lambda}^{n,\alpha}f(z))' = \sum_{k=2}^{\infty} k\psi_{k,n}(\alpha,\lambda)a_k z^k.$$

It is sufficient to show that

$$\beta\left|\frac{z(D_{\lambda}^{n,\alpha}f(z))'}{D_{\lambda}^{n,\alpha}g(z)} - 1\right| - \mathrm{Re}\left\{\frac{z(D_{\lambda}^{n,\alpha}f(z))'}{D_{\lambda}^{n,\alpha}g(z)} - 1\right\} < (1-\gamma).$$

Now

$$\begin{aligned}\beta\left|\frac{z(D_{\lambda}^{n,\alpha}f(z))'}{D_{\lambda}^{n,\alpha}g(z)} - 1\right| &- \mathrm{Re}\left\{\frac{z(D_{\lambda}^{n,\alpha}f(z))'}{D_{\lambda}^{n,\alpha}g(z)} - 1\right\} \\ &\leq (1+\beta)\left|\frac{z(D_{\lambda}^{n,\alpha}f(z))'}{D_{\lambda}^{n,\alpha}g(z)} - 1\right| \\ &\leq (1+\beta)\left|\sum_{k=2}^{\infty} k\psi_{k,n}(\alpha,\lambda)a_k z^{k-1}\right| \\ &\leq (1+\beta)\sum_{k=2}^{\infty} k\psi_{k,n}(\alpha,\lambda)a_k.\end{aligned}$$

The last expression is bounded above by $(1-\gamma)$, if (3.6) is satisfied. □

From (1.8) and Theorem 3.7, we get

3.8. Corollary. *A function f of the form (1.1) is in $UQC_{\lambda}^{n,\alpha}(\beta,\gamma)$ if*

$$\sum_{k=2}^{\infty} k^2|a_k|\psi_{k,n}(\alpha,\lambda) \leq \frac{(1-\gamma)}{1+\beta}\ .$$

3.9. Remark. Theorem 3.7 and Corollary 3.8, reduces to a result of Subramanian et al [27].

References

[1] M. Acu, On a subclass of n-uniformly close to convex functions, General Mathematics, **14**(1) (2006), 55–64.

[2] M. Acu and D. Blezu, Bounds of the coefficients for uniformly close-to-convex functions, Libertas Matematica **XXII** (2002), 81–86.

[3] R. Aghalary and GH. Azadi, The Dziok-Srivastava operator and k-uniformly starlike functions, J. Inequal. Pure Appl. Math. **6**(2) (2005), 1–7, Article 52 (electronic).

[4] F.M. Al-Oboudi, On univalent functions defined by a generalized Salagean operator, Int. J. Math. Math. Sci. **27** (2004), 1429–1436.

[5] F.M. Al-Oboudi and K.A. Al-Amoudi On classes of analytic functions related to conic domains, J. Math. Anal. Appl. **339**(1) (2008), 655–667.

[6] F.M. Al-Oboudi and K.A. Al-Amoudi, Subordination results for classes of analytic functions related to conic domains defined by a fractional operator, J. Math. Anal. Appl. **354**(2) (2009), 412–420.

[7] R. Bharti, R. Parvatham and A. Swaminathan, On subclasses of uniformly convex functions and corresponding class of starlike functions, Tamkang J. Math. **28**(1) (1997), 17–32.

[8] D. Blezu, On the n-uniformly close to convex functions with respect to a convex domain, General Mathematics **9**(3-4) (2001), 3–10.

[9] A.W. Goodman, On uniformly convex functions, Ann. Polon. Math. **56** (1991), 87–92.

[10] S. Kanas and A. Wisniowska, Conic regions and k-uniform convexity, II, Folia Sci. Univ. Tehn. Resov. **170** (1998), 65–78.

[11] S. Kanas and A. Wisniowska, Conic regions and k-uniform convexity, Comput. Appl. Math. **105** (1999), 327–336.

[12] W. Kaplan, Close-to-convex Schlicht functions, Mich. Math. J. **15** (1968), 277-282.

[13] S. Kumar and C. Ramesha, Subordination properties of uniformly convex and uniformly close to convex functions, J. Ramanujan Math. Soc. **9**(2) (1994), 203–214.

[14] S.S. Milller and PT. Mocanu, General second order inequalities in the complex plane. "Babes-Bolya" Univ. Fac. of Math. Research Seminars, Seminar on Geometric Function Theory, **4** (1982), 96–114.

[15] K.I. Noor and D.K. Thomas, Quasi-convex univalent functions, Int. J. Math. & Math. Sci. **3** (1980), 255–266.

[16] K.I. Noor, M. Arif, and W. Ul-Haq, On k-uniformly close-to-convex functions of complex order, Applied Mathematics and Computation, **215** (2009), 629–635.

[17] S. Owa, On the distortion theorems, I, Kyungpook Math. J. **18**(1) (1978), 53–59.

[18] S. Owa and H.M. Srivastava, Univalent and starlike generalized hypergeometric functions, Canad. J. Math. **39**(5) (1987), 1057–1077.

[19] W. Rogosinski, On the coefficients of subordinate functions, Proc. London Math. Soc. **48** (1943), 48–82.

[20] F. Rønning, On starlike functions associated with parabolic regions, Ann. Univ. Mariae Curie-Sklodowska Sect. A **45**(14) (1991), 117–122.

[21] F. Rønning, Uniformly convex functions and a corresponding class of starlike functions, Proc. Amer. Math. Soc. **118**(1) (1993), 189–196.

[22] St. Ruscheweyh, Convolutions in Geometric Function Theory, Sem. Math. Sup., vol. **83**, Presses Univ. de Montreal, 1982.

[23] G.S. Salagean, Subclasses of univalent functions, in: Complex Analysis – Fifth Romanian-Finish Seminar, Part 1, Bucharest, 1981, in: Lecture Notes in Math., vol. **1013**, Springer, Berlin, 1983, pp. 362–372.

[24] H. Silverman and D.N. Telage, Extreme points of subclasses of close-to-convex functions, Proc. Amer. Math. Soc. **74** (1979), 59–65.

[25] H.M. Srivastava and A.K. Mishra, Applications of fractional calculus to parabolic starlike and uniformly convex functions, J. Comput. Math. Appl. **39**(3/4) (2000), 57–69.

[26] H.M. Srivastava, Shu-Hai Li and Huo Tang, Certain classes of K-uniformly close-to-convex functions and other related functions defined by using the Dziok-Srivastava operator, Bull. Math. Anal. **1**(3) (2009), 49–63.

[27] K.G. Subramanian, T.V. Sudharsan and H. Silverman, On uniformly close-to-convex functions and uniformly quasi-convex functions, IJMMS. **48** (2003), 3053–3058.