Abstract
Let $h(z)$ and $g(z)$ be analytic functions in the open unit disc $\mathbb{D} = \{ z \mid |z| < 1 \}$, with the normalization $h(0) = g(0) = 1$. The class of log-harmonic mappings of the form $f = zh(z)\overline{g(z)}$ is denoted by $S_{lh}$.

The aim of this paper is to investigate the class of Janowski starlike log-harmonic mappings, a subclass of the log-harmonic mappings.

Keywords: Log-harmonic univalent functions, Janowski starlike log-harmonic functions, Subordination principle, Distortion theorems.


1. Introduction

Let $H(D)$ be the the linear space of all analytic functions defined on the unit disc $\mathbb{D}$. A log-harmonic mapping $f$ is a solution of the non-linear elliptic partial differential equation
\[
\frac{\overline{\omega}}{f} = w(z)\frac{f'}{f},
\]
where the second dilatation function $w(z) \in H(D)$ is such that $|w(z)| < 1$ for all $z \in \mathbb{D}$. It has been shown that if $f$ is a non-vanishing log-harmonic mapping, then $f$ can be expressed as
\[
f = h(z)\overline{g(z)}
\]
where $h(z)$ and $g(z)$ are analytic functions in $\mathbb{D}$. On the other hand, if $f$ vanishes at $z = 0$ but is not identically zero, then $f$ admits the representation
\[
f = z|z|^{2\beta} h(z)\overline{g(z)},
\]
where $\Re \beta > -1/2$, $h(z)$ and $g(z)$ are analytic functions in $\mathbb{D}$, $g(0) = 1$ and $h(0) \neq 0$, [1]. Let us denote by $\Omega$ the family of functions $\phi(z)$ which are regular in $\mathbb{D}$ and satisfy the conditions $\phi(0) = 0$ and $|\phi(z)| < 1$ for all $z \in \mathbb{D}$. For arbitrary fixed real numbers $A$ and $B$, with $-1 \leq B < A \leq 1$ we use $P(A, B)$ to denote the family of functions
\begin{equation}
\phi(0) = 0 \quad \text{and} \quad |\phi(z)| < 1 \quad \text{for all} \quad z \in \mathbb{D}.
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\subsection*{2. Main Results}

For the proof of the main theorem, we need the following lemmas which were proved by I.S. Jack [3], Kozuo Kuroki and S. Owa [5], respectively.

\begin{Lemma}
Let $\phi(z)$ be a non-constant function and analytic in $\mathbb{D}$ with $\phi(0) = 0$. If $|\phi(z)|$ attains its maximum value on the circle $|z| = r < 1$ at a point $z_0 \in \mathbb{D}$, then we have
\begin{equation}
z_0 \phi'(z_0) = k \phi(z_0)
\end{equation}
for some real $k$ with $k \geq 1$.
\end{Lemma}

\begin{Lemma}
Let $p(z)$ be an element of $P(A, B)$ then
\begin{equation}
\Re p(z) > \frac{1 - A}{1 - B} \geq 0.
\end{equation}
\end{Lemma}

The following lemma was proved by H. Silverman and E. M. Silvia [6].

\begin{Lemma}
s(z) \in S^*(A, B) if and only if
\begin{equation}
\left| \frac{z s'(z)}{s(z)} - \frac{1 - AB}{1 - B^2} \right| < \frac{A - B}{1 - B^2}, \quad (z \in \mathbb{D}, \ B \neq -1).
\end{equation}
\end{Lemma}

\begin{Theorem}
Let $f = zh(z)g(z)$ be a log-harmonic mapping on $\mathbb{D}$ and $0 \notin hg(\mathbb{D})$. If
\begin{equation}
\frac{z h'(z)}{h(z)} - \frac{z g'(z)}{g(z)} < \begin{cases} (A - B)z \\ \frac{1 + Bz}{1 - Bz} \end{cases} = F_1(z), \quad B \neq 0; \\
Az = F_2(z), \quad B = 0;
\end{equation}
then $f \in S_{lh}^*(A, B)$. 
\end{Theorem}
2.5. Corollary. Marx-Strohhacker Inequality of $f$

Considering the relations (2.7), (2.8), (2.9) and Lemma 2.3 together, we obtain that $\phi(z)$ is analytic in $\mathbb{D}$ and $\phi(0) = 0$. If we take the logarithmic derivative

$$
\frac{zh'(z)}{h(z)} - \frac{zg'(z)}{g(z)} = \begin{cases} 
(A - B)z\phi'(z), & B \neq 0; \\
1 + B\phi(z), & B = 0.
\end{cases}
$$

Now it is easy to realize that the subordination (2.3) is equivalent to $|w(z)| < 1$ for all $z \in \mathbb{D}$. Indeed, assume the contrary. Then there is $z_0 \in \mathbb{D}$ such that $|\phi(z_0)| = 1$, so by lemma 2.1 $z_0\phi'z_0 = k\phi(z_0)$, $k \geq 1$, and for such $z_0 \in \mathbb{D}$ we have

$$
\frac{z_0h'(z_0)}{h(z_0)} - \frac{z_0g'(z_0)}{g(z_0)} = \begin{cases} 
k(A - B)\phi(z_0), & B \neq 0; \\
kA\phi(z_0) = F_2(\phi(z_0)) \notin F_2(\mathbb{D}), & B = 0;
\end{cases}
$$

but this contradicts (2.3); so our assumptions is wrong, i.e, $|w(z)| < 1$ for every $z \in \mathbb{D}$.

By using Condition (2.3) we get

$$
1 + \frac{zh'(z)}{h(z)} - \frac{zg'(z)}{g(z)} = \begin{cases} 
1 + A\phi(z), & B \neq 0; \\
1 + A\phi(z) = p(z), & B = 0;
\end{cases}
$$

and using Lemma 2.2, we have

$$
\text{Re} \left( 1 + \frac{zh'(z)}{h(z)} - \frac{zg'(z)}{g(z)} \right) = \text{Re}p(z) > \frac{1 - A}{1 - B}.
$$

On the other hand

$$
f = zh(z)g(z) \implies \frac{zf_s - \bar{z}f_s}{f} = 1 + \frac{zh'(z)}{h(z)} - \frac{zg'(z)}{g(z)}
$$

$$
\implies \text{Re} \left( \frac{zf_s - \bar{z}f_s}{f} \right) = \text{Re} \left( 1 + \frac{zh'(z)}{h(z)} - \frac{zg'(z)}{g(z)} \right)
$$

$$
= \text{Re} \left( 1 + \frac{zh'(z)}{h(z)} - \frac{zg'(z)}{g(z)} \right).
$$

Considering the relations (2.7), (2.8), (2.9) and Lemma 2.3 together, we obtain that $f \in S_{h_k}(A, B)$. \hfill \Box

The corollary below is a simple consequence of Theorem 2.3. It is known as the Marx-Strohhacker Inequality of $f$.

2.5. Corollary.

$$
\begin{align*}
\left| \frac{h(z)}{g(z)} \right| - 1 &< |B|, & B \neq 0; \\
\left| \log \left( \frac{h(z)}{g(z)} \right) \right| &< |A|, & B = 0.
\end{align*}
$$
Proof. Using (2.7) we have:
\[
\left( \frac{h(z)}{g(z)} \right)^{\alpha - \beta} - 1 = B\phi(z), \; B \neq 0 \implies \left| \left( \frac{h(z)}{g(z)} \right)^{\alpha - \beta} - 1 \right| < |B|, \; B \neq 0,
\]
\[
\log \left( \frac{h(z)}{g(z)} \right) = A\phi(z), \; B = 0 \implies \left| \log \left( \frac{h(z)}{g(z)} \right) \right| < |A|, \; B = 0.
\]
This completes the proof. \(\square\)

2.6. Theorem. If \(f \in S_{ih}(A, B)\) then
\[
(2.11) \begin{cases}
(1 - Br)^{\frac{A-B}{|A|}} \leq \left| \frac{h(z)}{g(z)} \right| \leq (1 + Br)^{\frac{A-B}{|A|}}, \quad B \neq 0; \\
e^{-Ar} \leq \left| \frac{h(z)}{g(z)} \right| \leq e^{Ar}, \quad B = 0.
\end{cases}
\]

Proof. Using Theorem 2.3 we have
\[
(2.12) \begin{cases}
\left| \frac{z h'(z)}{h(z)} - \frac{g'(z)}{g(z)} \right| - B(B - A)r^2 \\
\left| \frac{z h'(z)}{h(z)} - \frac{g'(z)}{g(z)} \right| \leq A, \quad B = 0.
\end{cases}
\]

The inequalities (2.12) can be rewritten in the following form:
\[
(2.13) \begin{cases}
\frac{-(A - B)}{1 + Br} \leq \frac{\partial}{\partial r} \log |h(z)| - \frac{\partial}{\partial r} \log |g(z)| \leq \frac{(A - B)}{1 - Br}, \quad B \neq 0; \\
-A \leq \frac{\partial}{\partial r} \log |h(z)| - \frac{\partial}{\partial r} \log |g(z)| \leq A, \quad B = 0.
\end{cases}
\]

Then, after integration we obtain (2.11). \(\square\)

2.7. Corollary. If \(f = h(z)g(z) \in S_{ih}(A, B)\), then
\[
(2.14) \begin{cases}
\left| \frac{|b_1| - |a_1| r}{|a_1| - |b_1| r} \right| \cdot \frac{1}{(1 + Br)^{\frac{A-B}{|A|}}} \leq \left| \frac{g'(z)}{h'(z)} \right| \leq \left| \frac{|b_1| + |a_1| r}{|a_1| + |b_1| r} \right| \cdot \frac{1}{(1 - Br)^{\frac{A-B}{|A|}}}, \quad B \neq 0; \\
\left| \frac{|b_1| - |a_1| r}{|a_1| - |b_1| r} \right| \cdot e^{-Ar} \leq \left| \frac{g'(z)}{h'(z)} \right| \leq \left| \frac{|b_1| + |a_1| r}{|a_1| + |b_1| r} \right| e^{-Ar}, \quad B = 0.
\end{cases}
\]

Proof. Since \(f = h(z)g(z)\) is the solution of the non-linear elliptic partial differential equation \(\frac{\partial}{\partial f} = w(z)\), \(f = w(z)\), we have
\[
(2.15) \quad w(z) = \frac{f w(z)}{f_z w(z)} = \frac{g'(z)}{h'(z)} = \frac{b_1}{a_1} + 1 \cdot \left( 2b_2 - b_1^2 - \frac{b_1}{a_1} \cdot (2a_2 - a_1^2) \right) z + \cdots.
\]

Therefore we define the function
\[
(2.16) \quad \phi(z) = \frac{w(z) - w(0)}{1 - w(0)w(z)}
\]
that satisfies the assumptions of Schwarz’s Lemma. Using the estimate of Schwarz’s Lemma we have \(|\phi(z)| \leq r\), which gives
\[
(2.17) \quad |w(z) - w(0)| \leq r \left| 1 - w(0)w(z) \right|.
\]
This inequality is equivalent to
\[
|w(z) - (1 - r^2) \frac{b_1}{a_1} | \leq \left(1 - \frac{b_1}{a_1} \right) \frac{r}{1 - \frac{b_1}{a_1} r^2},
\]
and equality holds only for the function
\[
w(z) = \frac{z + \frac{b_1}{a_1} z}{1 + \frac{b_1}{a_1} z}.
\]

From the inequality (2.18) we obtain
\[
|w(z)| \geq \left(1 - r^2\right) \frac{b_1}{a_1} r^2 - \left(1 - \frac{b_1}{a_1} \right) ^2 \frac{r}{1 - \frac{b_1}{a_1} r^2} = \frac{\frac{b_1}{a_1} - r}{1 - \frac{b_1}{a_1} r^2},
\]
\[
|w(z)| \leq \left(1 - r^2\right) \frac{b_1}{a_1} r^2 + \left(1 - \frac{b_1}{a_1} \right) ^2 \frac{r}{1 - \frac{b_1}{a_1} r^2} = \frac{\frac{b_1}{a_1} + r}{1 + \frac{b_1}{a_1} r^2}.
\]

Therefore we have
\[
\left(\frac{\frac{b_1}{a_1} - r}{1 - \frac{b_1}{a_1} r^2}\right) \leq |w(z)| \leq \left(\frac{\frac{b_1}{a_1} + r}{1 + \frac{b_1}{a_1} r^2}\right).
\]

Considering (2.15), (2.18) and Theorem 2.6 together we obtain (2.14). \qed

References


