Hacettepe Journal of Mathematics and Statistics
Volume 40 (5) (2011), 725 – 736

SUBORDINATION RESULTS OF MULTIVALENT FUNCTIONS DEFINED BY CONVOLUTION

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Received 23 : 09 : 2010 : Accepted 23 : 03 : 2011

Abstract

Using the method of differential subordination, we investigate some properties of certain classes of multivalent functions, which are defined by means of convolution.

Keywords: Analytic functions, Multivalent functions, Convolution product, Differential subordination.

2000 AMS Classification: 30 C 45.

1. Introduction

Let $A_n(p)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=n}^{\infty} a_{k+p} z^{k+p}, \quad p, n \in \mathbb{N} = \{1, 2, \dots\}, \tag{1.1}$$

which are analytic and p–valent in the unit disc $\mathrm{U} = \{z \in \mathbb{C} : |z| < 1\}$. If f and g are analytic functions in U, we say that f is subordinate to g, written $f(z) \prec g(z)$, if there exists a *Schwarz function* w, which (by definition) is analytic in U, with $w(0) = 0$, and $|w(z)| < 1$ for all $z \in \mathrm{U}$, such that $f(z) = g(w(z))$, $z \in \mathrm{U}$. Furthermore, if the function g is univalent in U, then we have the equivalence (cf., e.g., [18] and [19])

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(\mathrm{U}) \subset g(\mathrm{U}).$$

For functions f given by (1.1) and $g \in A_n(p)$ given by

$$g(z) = z^p + \sum_{k=n}^{\infty} b_{k+p} z^{k+p},$$

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the *Hadamard product (convolution)* of f and g is defined by

$$(f*g)(z)=z^p+\sum_{k=n}^{\infty}a_{k+p}b_{k+p}z^{k+p}=(g*f)(z).$$

For the functions $f,g\in A_n(p)$ we define the linear operator $D^m_{\lambda,p}:A_n(p)\to A_n(p)$, where $\lambda\geq 0$, $p\in\mathbb{N}$, $m\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$, by

$$D^0_{\lambda,p}h(z)=h(z),$$
$$D^1_{\lambda,p}h)(z)=(1-\lambda)h(z)+\frac{\lambda z}{p}\,(h(z))',$$

and

$$\begin{aligned} D^m_{\lambda,p}h(z)&=D^1_{\lambda,p}\left(D^{m-1}_{\lambda,p}h(z)\right)\\ &=z^p+\sum_{k=n}^{\infty}\left(\frac{p+\lambda k}{p}\right)^m a_{k+p}b_{k+p}z^{k+p},\ m\in\mathbb{N}, \end{aligned}\tag{1.2}$$

where $h=f*g$.

From (1.2) we may easily deduce that

$$\frac{\lambda z}{p}\left(D^m_{\lambda,p}(f*g)(z)\right)'=D^{m+1}_{\lambda,p}(f*g)(z)-(1-\lambda)D^m_{\lambda,p}(f*g)(z),\tag{1.3}$$

for $\lambda\geq 0$ and $m\in\mathbb{N}_0$.

For the special case $p=1$, the operator $D^m_{\lambda,p}(f*g)$ was introduced and studied by Aouf and Mostafa [4], while for different choices of the function g, the operator $D^m_{\lambda,p}(f*g)$ reduces to several interesting operators as follows:

(i) For $b_{k+p}=1$ for all $k\geq n$ $\left(\text{or}\quad \widetilde{g}(z)=z^p+\dfrac{z^{p+n}}{1-z}\right)$, we have
$D^m_{\lambda,p}(f*\widetilde{g})(z)\equiv D^m_{\lambda,p}f(z)=z^p+\sum_{k=n}^{\infty}\left(\frac{p+\lambda k}{p}\right)^m a_{k+p}z^{k+p},\ \lambda\geq 0.$
Taking in this special case $\lambda=1$, we have $D^m_{1,p}(f*\widetilde{g})\equiv D^m_p f$, where D^m_p is the p–valent Sălăgean operator introduced and studied by Kamali and Orhan [14] (see also [3]);

(ii) For $m=0$ and

$$g_*(z)=z^p+\sum_{k=n+p}^{\infty}\left[\frac{p+l+\lambda(k-p)}{p+l}\right]^s z^k,\ (\lambda\geq 0,\ p\in\mathbb{N},\ l,s\in\mathbb{N}_0),\tag{1.4}$$

we see that $D^0_{\lambda,p}(f*g_*)=f*g_*=\mathrm{I}_p(s,\lambda,l)f$, where $\mathrm{I}_p(s,\lambda,l)$ is the generalized multiplier transformation which was introduced and studied by Cătas [7]. The operator $\mathrm{I}_p(s,\lambda,l)$ contains as special cases the multiplier transformation (see [8]), the generalized Sălăgean operator introduced and studied by Al-Oboudi [1], which in turn contains as a special case the Sălăgean operator (see [24]).

For $p=1$ and
$g_{**}(z)=z+\sum_{k=2}^{\infty}\left[\dfrac{\Gamma(k+1)\Gamma(2-\beta)}{\Gamma(k+1-\beta)}\,(1+\lambda(k-1))\right]z^k,$
where $0\leq\beta<1$, $\lambda\geq 0$, we see that $D^m_{\lambda,1}(f*g_{**})\equiv D^{m,\beta}_{\lambda}f$ is the fractional differential multiplier operator defined and studied by Al-Oboudi and Al-Amoudi in [2].

(iii) For $m=0$ and

$$g^*(z)=z^p+\sum_{k=n}^{\infty}\frac{(\alpha_1)_k\cdot\ldots\cdot(\alpha_l)_k}{(\beta_1)_k\cdot\ldots\cdot(\beta_s)_k}\,\frac{z^{k+p}}{(1)_k},\tag{1.5}$$

where $\alpha_i \in \mathbb{C}$, $i=\overline{1,l}$, and $\beta_j \in \mathbb{C}\setminus\{0,-1,-2,\dots\}$, $j=\overline{1,s}$, with $l \le s+1$, $l,s \in \mathbb{N}_0$, we see that $D^0_{\lambda,p}(f*g^*) = f*g^* \equiv \mathrm{H}^p_{l,s}(\alpha_1,\dots\alpha_l;\beta_1,\dots\beta_s)f = \mathrm{H}^p_{l,s}[\alpha_1]f$, where $\mathrm{H}^p_{l,s}[\alpha_1]$ is the Dziok-Srivastava operator introduced and studied by Dziok and Srivatava [9] (see also [10] and [11]).

The operator $\mathrm{H}^p_{l,s}[\alpha_1]$ contains in turn many interesting operators, such as the Hohlov linear operator (see [13]), the Carlson-Shaffer linear operator (see [6] and [23]), the Ruscheweyh derivative operator (see [22]), the Bernardi-Libera-Livingston operator (see [5], [15] and [16]), and the Owa-Srivastava fractional derivative operator (see [20]).

Using the linear operator $D^m_{\lambda,p}$, we define a new subclass of the class $A_n(p)$ as follows:

1.1. Definition. For fixed parameters A and B, with $-1 \le B < A \le 1$, for $\lambda > 0$, $p \in \mathbb{N}$, $m \in \mathbb{N}_0$ and $g \in A_n(p)$, we say that a function $f \in A_n(p)$ is in the class $T^m_{p,n}(\lambda; A, B)$, if it satisfies the following subordination condition

$$\frac{\left(D^m_{\lambda,p}(f*g)(z)\right)'}{pz^{p-1}} \prec \frac{1+Az}{1+Bz}. \tag{1.6}$$

A function f analytic in U is said to be *convex of order* η, $\eta < 1$, if $f'(0) \neq 0$ and

$$\mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) > \eta,\ z \in \mathrm{U}.$$

If $\eta = 0$, then the function f is *convex*.

It is easy to check that, if $h(z) = \dfrac{1+Az}{1+Bz}$, then $h'(0) \neq 0$ and $\mathrm{Re}\left(1+\dfrac{zh''(z)}{h'(z)}\right) = \mathrm{Re}\,\dfrac{1-Bz}{1+Bz} > 0$, $z \in \mathrm{U}$, whenever $|B| \le 1$ and $A \neq B$, hence h is convex in the unit disc. If $B \neq -1$, from the fact that $h(\overline{z}) = \overline{h(z)}$, $z \in \mathrm{U}$, we deduce that the image $h(\mathrm{U})$ is symmetric with respect to the real axis, and that h maps the unit disc U onto the disc $\left|w - \dfrac{1-AB}{1-B^2}\right| < \dfrac{A-B}{1-B^2}$. If $B=-1$, the function h maps the unit disc U onto the half plane $\mathrm{Re}\, w > \dfrac{1-A}{2}$, hence we obtain:

1.2. Remark. The function $f \in A(p)$ is in the class $T^m_{p,n}(\lambda; A, B)$ if and only if

$$\left|\frac{\left(D^m_{\lambda,p}(f*g)(z)\right)'}{pz^{p-1}} - \frac{1-AB}{1-B^2}\right| < \frac{A-B}{1-B^2},\ z \in \mathrm{U},\ \text{for } B \neq -1,$$

and

$$\mathrm{Re}\,\frac{\left(D^m_{\lambda,p}(f*g)(z)\right)'}{pz^{p-1}} > \frac{1-A}{2},\ z \in \mathrm{U},\ \text{for } B=-1.$$

Denoting by $T^m_{p,n}(\lambda;\gamma)$ the class of functions $f \in A_n(p)$ that satisfy the inequality

$$\mathrm{Re}\,\frac{(D^m_{\lambda,p}(f*g)(z))'}{z^{p-1}} > \gamma,\ z \in \mathrm{U} \quad (0 \le \gamma < p),$$

where $g \in A_n(p)$, we have $T^m_{p,n}(\lambda;\gamma) = T^m_{p,n}\left(\lambda; 1-\frac{2\gamma}{p}, -1\right)$.

In the present paper, we derive several inclusion relationships for the function class $T^m_{p,n}(\lambda; A, B)$.

2. Preliminaries

To prove our main results, we need the following lemmas.

2.1. Lemma. [12] *Let h be a convex function in* U *with $h(0) = 1$. Suppose also that the function φ given by*

$$\varphi(z) = 1 + c_{p+n}z^n + c_{n+1}z^{n+1} + \ldots, \tag{2.1}$$

is analytic in U*. Then*

$$\varphi(z) + \frac{z\varphi'(z)}{\gamma} \prec h(z) \quad (\operatorname{Re}\gamma \geq 0,\ \gamma \neq 0),$$

implies

$$\varphi(z) \prec \psi(z) = \frac{\gamma}{n} z^{-\frac{\gamma}{n}} \int_0^z t^{\frac{\gamma}{n}-1} h(t)\,\mathrm{d}\,t \prec h(z), \tag{2.2}$$

and ψ is the best dominant of (2.2).

2.2. Lemma. [25] *Let Φ be analytic in* U*, with*

$$\Phi(0) = 1 \quad \textit{and} \quad \operatorname{Re}\Phi(z) > \frac{1}{2},\ z \in \mathrm{U}.$$

Then, for any function F analytic in U*, the set $(\Phi * F)(\mathrm{U})$ is contained in the convex hull of $F(\mathrm{U})$, i.e. $(\Phi * F)(\mathrm{U}) \subset \operatorname{co} F(\mathrm{U})$.*

For real or complex numbers a, b and c, the *Gauss hypergeometric function* is defined by

$$\begin{aligned} {}_2F_1(a,b,c;z) &= 1 + \frac{a \cdot b}{c}\frac{z}{1!} + \frac{a(a+1)\cdot b(b+1)}{c(c+1)}\frac{z^2}{2!} + \cdots \\ &= \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k}\frac{z^k}{k!}, \quad a, b \in \mathbb{C},\ c \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}, \end{aligned} \tag{2.3}$$

where $(d)_k = d(d+1)\ldots(d+k-1)$ and $(d)_0 = 1$. The series (2.3) converges absolutely for $z \in \mathrm{U}$, hence it represents an analytic function in U (see [26, Chapter 14]).

Each of the following identities are fairly well-known:

2.3. Lemma. [26, Chapter 14] *For all real or complex numbers a, b and c, with $c \neq 0, -1, -2, \ldots$, the following equalities hold:*

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,\mathrm{d}\,t = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\ {}_2F_1(a,b,c;z) \tag{2.4}$$

where $\operatorname{Re} c > \operatorname{Re} b > 0$,

$${}_2F_1(a,b,c;z) = (1-z)^{-a}\ {}_2F_1\left(a, c-b, c; \frac{z}{z-1}\right), \tag{2.5}$$

and

$${}_2F_1(a,b,c;z) ={}_2 F_1(b,a,c;z). \tag{2.6}$$

3. Main Results

Unless otherwise mentioned, we assume throughout this paper that

$$\alpha > 0,\ -1 \leq B < A \leq 1,\ \lambda \geq 0,\ p \in \mathbb{N},\ m \in \mathbb{N}_0.$$

3.1. Theorem. *Let $g \in A_n(p)$ be a given function, and suppose that the function $f \in A_n(p)$ satisfies the subordination condition*

$$\frac{(1-\alpha)\left(D^m_{\lambda,p}h)(z)\right)' + \alpha\left(D^{m+1}_{\lambda,p}h(z)\right)'}{pz^{p-1}} \prec \frac{1+Az}{1+Bz},$$

where $h = f * g$, and $\lambda > 0$. Then

$$\frac{\left(D^m_{\lambda,p}h(z)\right)'}{pz^{p-1}} \prec Q(z) \prec \frac{1+Az}{1+Bz}, \tag{3.1}$$

where

$$Q(z) = \begin{cases} \frac{A}{B} + (1-\frac{A}{B})(1+Bz)^{-1}\ {}_2F_1\left(1, 1, \frac{p}{\alpha\lambda n}+1; \frac{Bz}{1+Bz}\right), & \text{if } B \neq 0, \\ 1 + \frac{p}{\alpha\lambda n + p}Az, & \text{if } B = 0, \end{cases} \tag{3.2}$$

is the best dominant of (3.1)*. Furthermore,*

$$\operatorname{Re}\frac{\left(D^m_{\lambda,p}h(z)\right)'}{pz^{p-1}} > \eta,\ z \in \mathrm{U} \quad (0 \le \eta < 1), \tag{3.3}$$

where

$$\eta = \begin{cases} \frac{A}{B} + (1-\frac{A}{B})(1-B)^{-1}\ {}_2F_1\left(1, 1, \frac{p}{\alpha\lambda n}+1; \frac{B}{B-1}\right), & \text{if } B \neq 0, \\ 1 - \frac{p}{\alpha\lambda n + p}A, & \text{if } B = 0. \end{cases} \tag{3.4}$$

The inequality (3.3) *is the best possible.*

Proof. If we let

$$\varphi(z) = \frac{\left(D^m_{\lambda,p}h(z)\right)'}{pz^{p-1}},\ z \in \mathrm{U}, \tag{3.5}$$

then φ is of the form (2.1), and it is analytic in U. Applying the identity (1.3) in (3.5) and differentiating the resulting equation with respect to z, we get

$$\frac{(1-\alpha)\left(D^m_{\lambda,p}h(z)\right)' + \alpha\left(D^{m+1}_{\lambda,p}h(z)\right)'}{pz^{p-1}} = \varphi(z) + \frac{\alpha\lambda z\varphi'(z)}{p} \prec \frac{1+Az}{1+Bz}.$$

Using Lemma 2.1 for $\gamma = \dfrac{p}{\alpha\lambda}$, we deduce that

$$\begin{aligned} &\frac{\left(D^m_{\lambda,p}h(z)\right)'}{pz^{p-1}} \\ &\qquad \prec Q(z) \\ &\qquad = \frac{p}{\alpha\lambda n} z^{-\frac{p}{\alpha\lambda n}} \int_0^z t^{\frac{p}{\alpha\lambda n}-1}\frac{1+At}{1+Bt}\,\mathrm{d}\,t \\ &\qquad = \begin{cases} \frac{A}{B} + (1-\frac{A}{B})(1+Bz)^{-1}\ {}_2F_1\left(1, 1, \frac{p}{\alpha\lambda n}+1; \frac{Bz}{1+Bz}\right), & \text{if } B \neq 0, \\ 1 + \frac{p}{\alpha\lambda n + p}Az, & \text{if } B = 0, \end{cases} \end{aligned}$$

where we have also made a change of variables followed by the use of the identities (2.4), (2.5), and (2.6). Next we will show that

$$\inf\{\operatorname{Re}Q(z) : |z| < 1\} = Q(-1).$$

Indeed, for $|z| \le r < 1$ we have

$$\operatorname{Re}\frac{1+Az}{1+Bz} \ge \frac{1-Ar}{1-Br}.$$

Setting

$$G(z,s)=\frac{1+Azs}{1+Bzs}$$

and

$$\mathrm{d}\,\nu(s)=\frac{p}{\alpha\lambda n}s^{\frac{p}{\alpha\lambda n}-1}\,\mathrm{d}\,s,\ 0\le s\le 1,$$

which is a positive measure on the closed interval $[0,1]$, we have $Q(z)=\int\limits_0^1 G(s,z)\,\mathrm{d}\,\nu(s)$, and thus

$$\operatorname{Re}Q(z)\geq\int_0^1\frac{1-Asr}{1-Bsr}\,\mathrm{d}\,\nu(s)=Q(-r),\ |z|\le r<1.$$

Letting $r\to 1^-$ in the above inequality, we obtain the assertion (3.2).

Finally, the estimate (3.3) is the best possible as the function Q is the best dominant of (3.1), which completes the proof of the theorem. □

Taking $g(z)=z^p+\dfrac{z^{p+n}}{1-z}$ in Theorem 3.1, we have the following result:

3.2. Corollary. *If the function $f\in A_n(p)$ satisfy the subordination condition*

$$\frac{(1-\alpha)\left(D_{\lambda,p}^m f(z)\right)'+\alpha\left(D_{\lambda,p}^{m+1}f(z)\right)'}{pz^{p-1}}\prec\frac{1+Az}{1+Bz},$$

with $\lambda>0$, then

$$\frac{\left(D_{\lambda,p}^m f(z)\right)'}{pz^{p-1}}\prec Q(z)\prec\frac{1+Az}{1+Bz},$$

where Q is given by (3.2)*, and it is the best dominant. Furthermore,*

$$\operatorname{Re}\frac{\left(D_{\lambda,p}^m f(z)\right)'}{pz^{p-1}}>\eta,\ z\in\mathrm{U},\tag{3.6}$$

where η is given by (3.4)*, and the inequality* (3.6) *is the best possible.* □

For $m=0$ and $g=g^*$ given by (1.5), using the identity

$$z\left(\mathrm{H}_{l,s}^p[\alpha_1]f(z)\right)'=\alpha_1\,\mathrm{H}_{l,s}^p[\alpha_1+1]f(z)+(p-\alpha_1)\,\mathrm{H}_{l,s}^p[\alpha_1]f(z),$$

Theorem 3.1 reduces to the next result:

3.3. Corollary. *Let $\lambda>0$, let $\alpha_i\in\mathbb{C}$, $i=\overline{1,l}$, and $\beta_j\in\mathbb{C}\setminus\{0,-1,-2,\dots\}$, $j=\overline{1,s}$, with $l\le s+1$, $l,s\in\mathbb{N}_0$, and suppose that the function $f\in A_n(p)$ satisfy the subordination condition*

$$\frac{\left(1-\frac{\lambda\alpha\alpha_1}{p}\right)\left(\mathrm{H}_{l,s}^p[\alpha_1]f(z)\right)'+\frac{\lambda\alpha\alpha_1}{p}\left(\mathrm{H}_{l,s}^p[\alpha_1+1]f(z)\right)'}{pz^{p-1}}\prec\frac{1+Az}{1+Bz}.$$

Then

$$\frac{\left(\mathrm{H}_{l,s}^p[\alpha_1]f(z)\right)'}{pz^{p-1}}\prec Q(z)\prec\frac{1+Az}{1+Bz},$$

where Q is given by (3.2)*, and it is the best dominant. Furthermore,*

$$\operatorname{Re}\frac{\left(\mathrm{H}_{l,s}^p[\alpha_1]f(z)\right)'}{pz^{p-1}}>\eta,\ z\in\mathrm{U},\tag{3.7}$$

where η *is given by* (3.4)*, and the inequality* (3.7) *is the best possible.* □

Taking in Theorem 3.1 the parameter $m = 0$ and $g = g_*$ of the form (1.4), and using the identity (see [7])

$$\lambda z\,(\mathrm{I}_p(s,\lambda,l)f(z))' = (p+l)\,\mathrm{I}_p(s+1,\lambda,l)f(z) - [p(1-\lambda)+l]\,\mathrm{I}_p(s,\lambda,l)f(z),\ \lambda \geq 0,$$

we deduce the following result:

3.4. Corollary. *Let* $\lambda > 0$*,* $p \in \mathbb{N}$*, and* $l, s \in \mathbb{N}_0$*, and suppose that the function* $f \in A_n(p)$ *satisfy the subordination condition*

$$\frac{\left[1-\alpha\left(1+\frac{l}{p}\right)\right]\left(\mathrm{I}_p(s,\lambda,l)f(z)\right)' + \alpha\left(1+\frac{l}{p}\right)\left(\mathrm{I}_p(s+1,\lambda,l)f(z)\right)'}{pz^{p-1}} \prec \frac{1+Az}{1+Bz}.$$

Then

$$\frac{(\mathrm{I}_p(s,\lambda,l)f(z))'}{pz^{p-1}} \prec Q(z) \prec \frac{1+Az}{1+Bz},$$

where Q *is given by* (3.2)*, and it is the best dominant. Furthermore,*

$$\text{(3.8)} \qquad \operatorname{Re}\frac{(\mathrm{I}_p(s,\lambda,l)f(z))'}{pz^{p-1}} > \eta,\ z \in \mathrm{U},$$

where η *is given by* (3.4)*, and the inequality* (3.8) *is the best possible.* □

3.5. Theorem. *Let* $g \in A_n(p)$ *be a given function, and suppose that* $f \in T^m_{p,n}(\lambda;\eta)$*,* $(0 \leq \eta < p)$*. Then*

$$\operatorname{Re}\frac{(1-\alpha)\left(D^m_{\lambda,p}h(z)\right)' + \alpha\left(D^{m+1}_{\lambda,p}h(z)\right)'}{z^{p-1}} > \eta,\ |z| < R,$$

where $h = f * g$*, and*

$$\text{(3.9)} \qquad R = \left[\frac{\sqrt{(\alpha\lambda n)^2+p^2}-\alpha\lambda n}{p}\right]^{\dfrac{1}{n}}.$$

The result is the best possible.

Proof. Since $f \in T^m_{p,n}(\lambda;\eta)$, we write

$$\text{(3.10)} \qquad \frac{\left(D^m_{\lambda,p}h(z)\right)'}{z^{p-1}} = \eta + (p-\eta)u(z).$$

Then the function u is of the form (2.1), analytic in U, and has a positive real part in U. Substituting the relation (1.3) in (3.10), and differentiating the resulting equation with respect to z, we have

$$\text{(3.11)} \qquad \frac{1}{p-\eta}\left[\frac{(1-\alpha)\left(D^m_{\lambda,p}h(z)\right)' + \alpha\left(D^{m+1}_{\lambda,p}h(z)\right)'}{z^{p-1}} - \eta\right] = u(z) + \frac{\alpha\lambda}{p}zu'(z).$$

Applying the following well-known estimate [17]

$$\frac{|zu'(z)|}{\operatorname{Re}u(z)} \leq \frac{2nr^n}{1-r^{2n}},\ |z| = r < 1,$$

in (3.11), we get

$$\operatorname{Re}\frac{1}{p-\eta}\left[\frac{(1-\alpha)\left(D_{\lambda,p}^{m}h(z)\right)'+\alpha\left(D_{\lambda,p}^{m+1}h(z)\right)'}{z^{p-1}}-\eta\right]$$
$$\geq \operatorname{Re}u(z)\left(1-\frac{2\lambda\alpha n r^{n}}{p\left(1-r^{2n}\right)}\right),\ |z|=r<1.$$

It is easy to see that the right-hand side of the inequality (**??**) is positive whenever $r<R$, where R is given by (3.9).

In order to show that the bound R is the best possible, we consider the function $f\in A_n(p)$ such that, for the given function $g\in A_n(p)$ we have

$$\frac{\left(D_{\lambda,p}^{m}h(z)\right)'}{z^{p-1}}=\eta+(p-\eta)\frac{1+z^{n}}{1-z^{n}},\ z\in \mathrm{U}\quad (0\leq\eta<p).$$

Noting that

$$\frac{1}{p-\eta}\left[\frac{(1-\alpha)\left(D_{\lambda,p}^{m}h(z)\right)'+\alpha\left(D_{\lambda,p}^{m+1}h(z)\right)'}{z^{p-1}}-\eta\right]=\frac{p(1-z^{2n})-2\alpha\lambda n z^{n}}{p\left(1-z^{n}\right)^{2}}=0,$$

for $z=R\exp\left(\dfrac{i\pi}{n}\right)$, the proof of the Theorem 3.5 is complete. □

Putting $\alpha=1$ in Theorem 3.5, we obtain the following result:

3.6. Corollary. *Let $g\in A_n(p)$ be a given function, and suppose that $f\in T_{p,n}^{m}(\lambda;\eta)$, $(0\leq\eta<p)$. Then $f\in T_{p,n}^{m+1}(\lambda;\eta)$ for $|z|<R^{*}$, where*

$$R^{*}=\left[\frac{\sqrt{(\lambda n)^{2}+p^{2}}-\lambda n}{p}\right]^{\frac{1}{n}}.$$

The result is the best possible. □

Now we define the integral operator $F_{\delta,p}:A_n(p)\to A_n(p)$ by

$$F_{\delta,p}(f)(z)=\frac{\delta+p}{z^{\delta}}\int_{0}^{z}t^{\delta-1}f(t)\,\mathrm{d}t,\ z\in \mathrm{U}\quad (\delta>-p).$$

3.7. Theorem. *Let $g\in A_n(p)$ be a given function, and suppose that $f\in T_{p,n}^{m}(\lambda;A,B)$. Then*

$$\frac{\left(D_{\lambda,p}^{m}F_{\delta,p}h(z)\right)'}{pz^{p-1}}\prec\Theta(z)\prec\frac{1+Az}{1+Bz}, \tag{3.12}$$

where $h=f*g$, and the function Θ is given by

$$\Theta(z)=\begin{cases}\frac{A}{B}+(1-\frac{A}{B})(1+Bz)^{-1}\,{}_2F_1\left(1,1,\frac{\delta+p}{n}+1;\frac{Bz}{1+Bz}\right), & \text{if } B\neq 0,\\ 1+\frac{\delta+p}{p+n+\delta}Az, & \text{if } B=0,\end{cases}$$

and it is the best dominant of (3.12). Furthermore,

$$\operatorname{Re}\frac{\left(D_{\lambda,p}^{m}F_{\delta,p}h(z)\right)'}{pz^{p-1}}>\vartheta,\ z\in \mathrm{U}\quad (\delta>-p),$$

where

$$\vartheta=\begin{cases}\frac{A}{B}+(1-\frac{A}{B})(1-B)^{-1}\,{}_2F_1\left(1,1,\frac{\delta+p}{n}+1;\frac{B}{B-1}\right), & \text{if } B\neq 0,\\ 1-\frac{\delta+p}{p+n+\delta}A, & \text{if } B=0.\end{cases}$$

The result is the best possible.

Proof. Letting

$$\varphi(z) = \frac{\left(D^m_{\lambda,p} F_{\delta,p} h(z)\right)'}{pz^{p-1}}, \ z \in \mathrm{U}, \tag{3.13}$$

then φ is of the form (2.1), and analytic in U. Using the operator identity

$$z\left(D^m_{\lambda,p} F_{\delta,p} h(z)\right)' = (p+\delta) D^m_{\lambda,p} h(z) - \delta D^m_{\lambda,p} F_{\delta,p} h(z)$$

in (3.13), and differentiating the resulting equation with respect to z, we have

$$\frac{\left(D^m_{\lambda,p} h(z)\right)'}{pz^{p-1}} = \varphi(z) + \frac{z\varphi'(z)}{p+\delta} \prec \frac{1+Az}{1+Bz}.$$

Now the remaining part of Theorem 3.7 follows by employing the techniques that were used in the proof of Theorem 3.1. □

It is easy to see that,

$$\frac{\left(D^m_{\lambda,p} F_{\delta,p} h(z)\right)'}{pz^{p-1}} = \frac{p+\delta}{pz^{p+\delta}} \int_0^z t^\delta \left(D^m_{\lambda,p} h(t)\right)' \mathrm{d}\, t, \ z \in \mathrm{U},$$

whenever $f \in A_n(p)$ with $\delta > -p$. In view of the above identity, Theorem 3.7 for the special case $A = 1 - 2\eta$ $(0 \leq \eta < 1)$ and $B = -1$ yields the next result:

3.8. Corollary. *Let $g \in A_n(p)$ be a given function, and suppose that $f \in A_n(p)$ satisfies the inequality*

$$\mathrm{Re}\, \frac{\left(D^m_{\lambda,p} h(z)\right)'}{pz^{p-1}} > \eta, \ z \in \mathrm{U} \quad (0 \leq \eta < 1),$$

where $h = f * g$, and $\lambda > 0$. Then

$$\mathrm{Re}\left[\frac{p+\delta}{pz^{p+\delta}} \int_0^z t^\delta \left(D^m_{\lambda,p} h(t)\right)' \mathrm{d}\, t\right] > \eta + (1-\eta)\left[{}_2F_1\left(1, 1, \frac{p+\delta}{n} + 1; \frac{1}{2}\right) - 1\right], \ z \in \mathrm{U}\ (\delta > -p).$$

The result is the best possible. □

3.9. Theorem. *Let $g \in A_n(p)$ be a given function, and suppose that the function $H \in A_n(p)$ satisfies the inequality*

$$\mathrm{Re}\, \frac{D^m_{\lambda,p} H(z)}{z^p} > 0, \ z \in \mathrm{U}.$$

If the function $f \in A_n(p)$ satisfies the inequality

$$\left| \frac{D^m_{\lambda,p} h(z)}{D^m_{p,\lambda} H(z)} - 1 \right| < 1, \ z \in \mathrm{U},$$

where $h = f * g$, then

$$\mathrm{Re}\, \frac{z\left(D^m_{\lambda,p} h(z)\right)'}{D^m_{\lambda,p} h(z)} > 0, \ |z| < R_0,$$

where

$$R_0 = \frac{\sqrt{9n^2 + 4p(p+n)} - 3n}{2(p+n)}. \tag{3.14}$$

Proof. Letting

$$\varphi(z)=\frac{D^m_{\lambda,p}h(z)}{D^m_{\lambda,p}H(z)}-1=e_nz^n+e_{n+1}z^{n+1}+\cdots,z\in\mathrm{U}, \tag{3.15}$$

then φ is analytic in U, with $\varphi(0)=0$ and $|\varphi(z)|<1$ for all $z\in\mathrm{U}$.

Defining the function ψ by

$$\psi(z)=\begin{cases}\dfrac{\varphi(z)}{z^n}, & z\in\mathrm{U}\setminus\{0\},\\ \dfrac{\varphi^{(n)}(0)}{n!}, & z=0,\end{cases}$$

then ψ is analytic in $\mathrm{U}\setminus\{0\}$ and continuous in U, hence it is analytic in the whole unit disc U. If $r\in(0,1)$ is an arbitrary number, since $|\varphi(z)|<1$ for all $z\in\mathrm{U}$, we deduce

$$|\psi(z)|\le\max_{|z|=r}\left|\frac{\varphi(z)}{z^n}\right|\le\max_{|z|=r}\frac{|\varphi(z)|}{|z|^n}<\frac{1}{r^n},\ |z|\le r<1.$$

By letting $r\to1^-$ in the above inequality, we get $|\psi(z)|<1$ for all $z\in\mathrm{U}$, i.e. $\varphi(z)=z^n\psi(z)$, where the function ψ is analytic in U, and $|\psi(z)|<1$, $z\in\mathrm{U}$.

Therefore, (3.15) leads to

$$D^m_{\lambda,p}h(z)=D^m_{\lambda,p}H(z)\left[1+z^n\psi(z)\right],$$

and differentiating logarithmically with respect to z the above relation, we obtain

$$\frac{z\left(D^m_{\lambda,p}h(z)\right)'}{D^m_{\lambda,p}h(z)}=\frac{z\left(D^m_{\lambda,p}H(z)\right)'}{D^m_{\lambda,p}H(z)}+\frac{z^n\left[n\psi(z)+z\psi'(z)\right]}{1+z^n\psi(z)}. \tag{3.16}$$

Letting $\chi(z)=\dfrac{D^m_{\lambda,p}H(z)}{z^p}$, we see that the function χ is of the form (2.1), analytic in U with $\operatorname{Re}\chi(z)>0$ for all $z\in\mathrm{U}$, and

$$\frac{z\left(D^m_{\lambda,p}H(z)\right)'}{D^m_{\lambda,p}H(z)}=\frac{z\chi'(z)}{\chi(z)}+p,$$

so we find from (3.16) that

$$\operatorname{Re}\frac{z\left(D^m_{\lambda,p}h(z)\right)'}{D^m_{\lambda,p}h(z)}\ge p-\left|\frac{z\chi'(z)}{\chi(z)}\right|-\left|\frac{z^n\left[n\psi(z)+z\psi'(z)\right]}{1+z^n\psi(z)}\right|,\ z\in\mathrm{U}. \tag{3.17}$$

Using the following known estimates [21] (see also [17])

$$\left|\frac{\chi'(z)}{\chi(z)}\right|\le\frac{2nr^{n-1}}{1-r^{2n}}\quad\text{and}\quad\left|\frac{n\psi(z)+z\psi'(z)}{1+z^n\psi(z)}\right|\le\frac{n}{1-r^n},\ |z|=r<1,$$

from (3.17) we deduce that

$$\operatorname{Re}\frac{z\left(D^m_{\lambda,p}h(z)\right)'}{D^m_{\lambda,p}h(z)}\ge\frac{p-3nr^n-(p+n)r^{2n}}{1-r^{2n}},\ |z|=r<1,$$

and the right-hand side fraction is positive provided that $r<R_0$, where R_0 is given by (3.14). □

3.10. Theorem. *Let $g\in A_n(p)$ be a given function, and suppose that the function $H\in A_n(p)$ satisfies the inequality*

$$\operatorname{Re}\frac{H(z)}{z^p}>\frac{1}{2},\ z\in\mathrm{U}. \tag{3.18}$$

If $f\in T^m_{p,n}(\lambda;A,B)$, then

$$f*H\in T^m_{p,n}(\lambda;A,B).$$

Proof. A simple calculation shows that

$$\frac{\left(D^m_{\lambda,p}(f*g*H)(z)\right)'}{pz^{p-1}} = \frac{\left(D^m_{\lambda,p}(f*g)(z)\right)'}{pz^{p-1}} * \frac{H(z)}{z^p}.$$

Using the assumption (3.18) and the fact that the function $\dfrac{1+Az}{1+Bz}$ is convex in U, from Lemma 2.2 follows

$$\frac{\left(D^m_{\lambda,p}(f*g*H)(z)\right)'}{pz^{p-1}} \prec \frac{1+Az}{1+Bz},$$

that is $f*H \in T^m_{p,n}(\lambda; A, B)$. □

3.11. Remark. Specializing in the above results the parameters λ and m, and the function g, we obtain new results corresponding to the operators defined in the introduction.

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