Hacettepe Journal of Mathematics and Statistics
Volume 43 (3) (2014), 383 – 389

Coefficient bounds for certain classes of bi-univalent functions

B.A. Frasin *

Abstract

In this paper, we introduce two new subclasses of the function class Σ of bi-univalent functions defined in the open unit disk. Furthermore, we find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in these new subclasses.

Keywords: Analytic and univalent functions, bi-univalent functions, starlike and convex functions, coefficients bounds.

2000 AMS Classification: 30C45

1. Introduction and definitions

Let $\mathcal{A}$ denote the class of functions of the form :

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disc $\mathcal{U} = \{z : |z| < 1\}$. Further, by $\mathcal{S}$ we shall denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathcal{U}$. A function $f(z)$ belonging to $\mathcal{S}$ is said to be starlike of order α if it satisfies

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \qquad (z \in \mathcal{U}) \tag{1.2}$$

for some $\alpha (0 \le \alpha < 1)$. We denote by $\mathcal{S}^*(\alpha)$ the subclass of $\mathcal{S}$ consisting of functions which are starlike of order α in $\mathcal{U}$. Also, a function $f(z)$ belonging to $\mathcal{S}$ is said to be convex of order α if it satisfies

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \qquad (z \in \mathcal{U}) \tag{1.3}$$

for some $\alpha (0 \le \alpha < 1)$. We denote by $\mathcal{K}(\alpha)$ the subclass of $\mathcal{S}$ consisting of functions which are convex of order α in $\mathcal{U}$.

*Faculty of Science, Department of Mathematics, Al al-Bayt University, P.O. Box: 130095 Mafraq, Jordan Email:bafrasin@yahoo.com

Gao and Zhou [5] showed some mapping properties of the following subclass of $\mathcal{A}$:

$$\mathcal{R}(\alpha,\beta)=\{f\in\mathcal{A}:\Re((f'(z)+\beta zf''(z))>\alpha,\quad \beta>0,\ 0\le\alpha<1;\ z\in\mathcal{U}\}.$$

Yang and Liu [12, Theorem 3.1, p.9], proved that the class $\mathcal{R}(\alpha,\beta)\subset\mathcal{S}$ iff

$2(1-\alpha)\sum\limits_{m=1}^{\infty}\frac{(-1)^{m-1}}{\beta m+1}\le 1.$

It is well known that every function $f\in\mathcal{S}$ has an inverse f^{-1}, defined by

$$f^{-1}(f(z))=z\qquad(z\in\mathcal{U})$$

and

$$f^{-1}(f(w))=w\qquad(|w|<r_0(f);\ r_0(f)\ge\frac{1}{4})$$

where

$$f^{-1}(w)=w-a_2w^2+(2a_2^2-a_3)w^3-(5a_2^3-5a_2a_3+a_4)w^4+\cdots.$$

A function is said to be bi-univalent in $\mathcal{U}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathcal{U}$.

Let Σ denote the class of bi-univalent functions in $\mathcal{U}$ given by (1.1). Example of functions in the class Σ are

$$\frac{z}{1-z},\qquad \log\frac{1}{1-z},\qquad \log\sqrt{\frac{1+z}{1-z}}.$$

However, the familiar Koebe function is not a member of Σ. Other common examples of functions in $\mathcal{U}$ such as

$$\frac{2z-z^2}{2}\ \text{and}\ \frac{z}{1-z^2}$$

are also not members of Σ.

Lewin [6] investigated the bi-univalent function class Σ and showed that $|a_2|<1.51$. Subsequently, Brannan and Clunie [1] conjectured that $|a_2|<\sqrt{2}$. Netanyahu [7], on the other hand, showed that $\max\limits_{f\in\Sigma}|a_2|=4/3$.

The coefficient estimate problem for each of the Taylor–Maclaurin coefficients $|a_n|$ $(n\ge 3; n\in\mathbb{N})$ is presumably still an open problem.

Brannan and Taha [2] (see also [10]) introduced certain subclasses of the bi-univalent function class Σ similar to the familiar subclasses $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$(see [3]). Thus, following Brannan and Taha [2] (see also [10]), a function $f\in\mathcal{A}$ is in the class $\mathcal{S}^*_\Sigma[\alpha]$ of strongly bi-starlike functions of order $\alpha(0<\alpha\le 1)$ if each of the following conditions are satisfied:

$$f\in\Sigma\ \text{and}\ \left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right|<\frac{\alpha\pi}{2}\qquad(0<\alpha\le 1,\ z\in\mathcal{U})$$

and

$$\left|\arg\left(\frac{zg'(w)}{g(w)}\right)\right|<\frac{\alpha\pi}{2}\qquad(0<\alpha\le 1,\ w\in\mathcal{U}),$$

where g is the extension of f^{-1} to $\mathcal{U}$. The classes $\mathcal{S}^*_\Sigma(\alpha)$ and $\mathcal{K}_\Sigma(\alpha)$ of bi-starlike functions of order α and bi-convex functions of order α, corresponding (respectively) to the function classes defined by (1.2) and (1.3), were also introduced analogously. For each of the function classes $\mathcal{S}^*_\Sigma(\alpha)$ and $\mathcal{K}_\Sigma(\alpha)$, they found non-sharp estimates on the first two Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$ (for details, see [7,8]).

The object of the present paper is to introduce two new subclasses of the function class Σ and find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in these new subclasses of the function class Σ employing the techniques used earlier by Srivastava *et al.* [9] (see also, [4] and [11]).

In order to derive our main results, we have to recall here the following lemma [8].

1.1. Lemma. *If $h \in \mathcal{P}$ then $|c_k| \leq 2$ for each k,*

where $\mathcal{P}$ is the family of all functions h analytic in $\mathcal{U}$ for which $\Re h(z) > 0$
$h(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots$ for $z \in \mathcal{U}$.

2. Coefficient bounds for the function class $\mathcal{H}_\Sigma(\alpha, \beta)$

2.1. Definition. A function $f(z)$ given by (1.1) is said to be in the class $\mathcal{H}_\Sigma(\alpha, \beta)$ if the following conditions are satisfied:

$$(2.1) \qquad f \in \Sigma \text{ and } \left|\arg\left(f'(z) + \beta z f''(z)\right)\right| < \frac{\alpha\pi}{2} \qquad (z \in \mathcal{U})$$

and

$$(2.2) \qquad \left|\arg\left(g'(w) + \beta w g''(w)\right)\right| < \frac{\alpha\pi}{2} \qquad (w \in \mathcal{U}),$$

where $\beta > 0, 0 < \alpha < 1$, $2(1-\alpha)\sum\limits_{m=1}^{\infty}\frac{(-1)^{m-1}}{\beta m+1} \leq 1$, and the function g is given by

$$(2.3) \qquad g(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots .$$

We begin by finding the estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the class $\mathcal{H}_\Sigma(\alpha, \beta)$.

2.2. Theorem. *Let $f(z)$ given by (1.1) be in the class $\mathcal{H}_\Sigma(\alpha, \beta)$ where $\beta > 0, 0 < \alpha < 1$, and $2(1-\alpha)\sum\limits_{m=1}^{\infty}\frac{(-1)^{m-1}}{\beta m+1} \leq 1$. Then*

$$(2.4) \qquad |a_2| \leq \frac{2\alpha}{\sqrt{2(\alpha+2) + 4\beta(\alpha+\beta+2-\alpha\beta)}}$$

and

$$(2.5) \qquad |a_3| \leq \frac{\alpha^2}{(1+\beta)^2} + \frac{2\alpha}{3(1+2\beta)}.$$

Proof. It follows from (2.1) and (2.2) that

(2.6) $\quad f'(z) + \beta z f''(z) = [p(z)]^{\alpha}$

and

(2.7) $\quad g'(w) + \beta w g''(w) = [q(w)]^{\alpha}$

where $p(z)$ and $q(w)$ in $\mathcal{P}$ and have the forms

(2.8) $\quad p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots$

and

(2.9) $\quad q(w) = 1 + q_1 w + q_2 w^2 + q_3 w^3 + \cdots .$

Now, equating the coefficients in (2.6) and (2.7), we get

$$2(1+\beta)a_2 = \alpha p_1, \tag{2.10}$$

$$3(1+2\beta)a_3 = \alpha p_2 + \frac{\alpha(\alpha-1)}{2}p_1^2, \tag{2.11}$$

$$-2(1+\beta)a_2 = \alpha q_1 \tag{2.12}$$

and

$$3(1+2\beta)(2a_2^2 - a_3) = \alpha q_2 + \frac{\alpha(\alpha-1)}{2}q_1^2. \tag{2.13}$$

From (2.10) and (2.12), we get

$$p_1 = -q_1 \tag{2.14}$$

and

$$8(1+\beta)^2 a_2^2 = \alpha^2(p_1^2 + q_1^2). \tag{2.15}$$

Now from (2.11), (2.13) and (2.15), we obtain

$$\begin{aligned} 6(1+2\beta)a_2^2 &= \alpha(p_2+q_2) + \frac{\alpha(\alpha-1)}{2}(p_1^2+q_1^2) \\ &= \alpha(p_2+q_2) + \frac{4(\alpha-1)(1+\beta)^2}{\alpha}a_2^2. \end{aligned}$$

Therefore, we have

$$a_2^2 = \frac{\alpha^2(p_2+q_2)}{2(\alpha+2)+4\beta(\alpha+\beta+2-\alpha\beta)}.$$

Applying Lemma 1.1 for the coefficients p_2 and q_2, we immediately have

$$|a_2| \leq \frac{2\alpha}{\sqrt{2(\alpha+2)+4\beta(\alpha+\beta+2-\alpha\beta)}}.$$

This gives the bound on $|a_2|$ as asserted in (2.4).

Next, in order to find the bound on $|a_3|$, by subtracting (2.13) from (2.11), we get

$$6(1+2\beta)a_3 - 6(1+2\beta)a_2^2 = \alpha p_2 + \frac{\alpha(\alpha-1)}{2}p_1^2 - \left(\alpha q_2 + \frac{\alpha(\alpha-1)}{2}q_1^2\right). \tag{2.16}$$

Upon substituting the value of a_2^2 from (2.15) and observing that $p_1^2 = q_1^2$, it follows that

$$a_3 = \frac{\alpha^2 p_1^2}{4(1+\beta)^2} + \frac{\alpha(p_2 - q_2)}{6(1+2\beta)}.$$

Applying Lemma 1.1 once again for the coefficients p_1, p_2, q_1 and q_2, we readily get

$$|a_3| \leq \frac{\alpha^2}{(1+\beta)^2} + \frac{2\alpha}{3(1+2\beta)}.$$

This completes the proof of Theorem 2.2. □

Putting $\beta = 1$ in Theorem 2.2, we have

2.3. Corollary. *Let $f(z)$ given by (1.1) be in the class $\mathcal{H}_\Sigma(\alpha, 1)$ where $0 < \alpha < 1$, and $2(1-\alpha) \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m+1} \leq 1$. Then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{2(\alpha+2)+12}} \tag{2.17}$$

and

$$|a_3| \leq \frac{9\alpha^2 + 8\alpha}{36}. \tag{2.18}$$

3. Coefficient bounds for the function class $\mathcal{H}_\Sigma(\gamma, \beta)$

3.1. Definition. A function $f(z)$ given by (1.1) is said to be in the class $\mathcal{H}_\Sigma(\gamma, \beta)$ if the following conditions are satisfied:

$$f \in \Sigma \text{ and } \Re\left(f'(z) + \beta z f''(z)\right) > \gamma \qquad (z \in \mathcal{U}) \tag{3.1}$$

and

$$\Re\left(g'(w) + \beta w g''(w)\right) > \gamma \qquad (w \in \mathcal{U}), \tag{3.2}$$

where $\beta > 0, 0 \leq \gamma < 1$, $2(1-\gamma) \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{\beta m+1} \leq 1$, and the function g is given by (2.3).

3.2. Theorem. *Let $f(z)$ given by (1.1) be in the class $\mathcal{H}_\Sigma(\gamma, \beta)$, where $\beta > 0, 0 \leq \gamma < 1$, and $2(1-\gamma) \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{\beta m+1} \leq 1$. Then*

$$|a_2| \leq \sqrt{\frac{2(1-\gamma)}{3(1+2\beta)}} \tag{3.3}$$

and

$$|a_3| \leq \frac{(1-\gamma)^2}{(1+\beta)^2} + \frac{2(1-\gamma)}{3(1+2\beta)}. \tag{3.4}$$

Proof. It follows from (3.1) and (3.2) that there exist p and $q \in \mathcal{P}$ such that

$$f'(z) + \beta z f''(z) = \gamma + (1-\gamma)p(z) \tag{3.5}$$

and

$$g'(w) + \beta w g''(w) = \gamma + (1-\gamma)q(w) \tag{3.6}$$

where $p(z)$ and $q(w)$ have the forms (2.8) and (2.9), respectively. Equating coefficients in (3.5) and (3.6) yields

$$2(1+\beta)a_2 = (1-\gamma)p_1, \tag{3.7}$$

$$3(1+2\beta)a_3 = (1-\gamma)p_2, \tag{3.8}$$

$$-2(1+\beta)a_2 = (1-\gamma)q_1 \tag{3.9}$$

and

$$3(1+2\beta)(2a_2^2 - a_3) = (1-\gamma)q_2 \tag{3.10}$$

From (3.7) and (3.9), we get

$$p_1 = -q_1 \tag{3.11}$$

and

$$8(1+\beta)^2 a_2^2 = (1-\gamma)^2(p_1^2 + q_1^2). \tag{3.12}$$

Also, from (3.8) and (3.10), we find that

$$6(1+2\beta)a_2^2 = (1-\gamma)(p_2 + q_2).$$

Thus, we have

$$\left|a_2^2\right| \leq \frac{(1-\gamma)}{6(1+2\beta)}(|p_2| + |q_2|) = \frac{2(1-\gamma)}{3(1+2\beta)}$$

which is the bound on $\left|a_2^2\right|$ as given in (3.3).

Next, in order to find the bound on $|a_3|$, by subtracting (3.10) from (3.8), we get

$$6(1+2\beta)a_3 - 6(1+2\beta)a_2^2 = (1-\gamma)(p_2 - q_2)$$

or, equivalently,

$$a_3 = a_2^2 + \frac{(1-\gamma)(p_2 - q_2)}{6(1+2\beta)}.$$

Upon substituting the value of a_2^2 from (3.12), we obtain

$$a_3 = \frac{(1-\gamma)^2(p_1^2 + q_1^2)}{8(1+\beta)^2} + \frac{(1-\gamma)(p_2 - q_2)}{6(1+2\beta)}.$$

Applying Lemma 1.1 for the coefficients p_1, p_2, q_1 and q_2, we readily get

$$|a_3| \leq \frac{(1-\gamma)^2}{(1+\beta)^2} + \frac{2(1-\gamma)}{3(1+2\beta)}$$

which is the bound on $|a_3|$ as asserted in (3.4). □

Putting $\beta = 1$ in Theorem 3.2, we have

3.3. Corollary. *Let* $f(z)$ *given by (1.1) be in the class* $\mathcal{H}_\Sigma(\gamma, 1)$, *where* $0 \leq \gamma < 1$, *and* $2(1-\gamma)\sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m+1} \leq 1$.

$$|a_2| \leq \frac{1}{3}\sqrt{2(1-\gamma)} \tag{3.13}$$

and

$$|a_3| \leq \frac{(1-\gamma)(9(1-\gamma)+8)}{36}. \tag{3.14}$$

Acknowledgements. The author would like to thank the referees for their helpful comments and suggestions.

References

[1] Brannan, D. A. and Clunie, J. G. (Eds.), *Aspects of Contemporary Complex Analysis* Proceedings of the NATO Advanced Study Institute held at the University of Durham, Durham; July 1-20, 1979, (Academic Press, New York and London, 1980).

[2] Brannan, D. A. and Taha, T. S. *On some classes of bi-univalent functions*, in: S.M. Mazhar, A. Hamoui, N.S. Faour (Eds.), Mathematical Analysis and Its Applications, Kuwait; February 18-21, 1985, in: KFAS Proceedings Series, vol. 3, Pergamon Press (Elsevier Science Limited), Oxford, 1988, pp. 53-60; see also Studia Univ. Babeş-Bolyai Math. 31 (2) (1986) 70-77.

[3] Brannan, D. A., Clunie, J. and Kirwan, W. E. *Coefficient estimates for a class of starlike functions*, Canad. J. Math., **22**, 476-485, 1970.

[4] Frasin, B. A. and Aouf, M. K *New subclasses of bi-univalent functions*, Appl. Math. Letters, **24**, no. 9, 1569-1573, 2011.

[5] Gao, C. Y. and Zhou, S. Q. *Certain subclass of starlike functions*, Applied Mathematics and Computation, vol. **187**, no. 1, pp. 176–182, 2007.

[6] Lewin, M *On a coefficient problem for bi-univalent functions*, Proc. Amer. Math. Soc. **18**, 63-68, 1967.

[7] Netanyahu, E. *The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in* $|z| < 1$, Arch. Rational Mech. Anal., **32**, 100-112, 1969.

[8] Pommerenke, C. *Univalent functions*, (Vandenhoeck and Rupercht, Göttingen, 1975).

[9] Srivastava, H. M., Mishra, A. K. and Gochhayat, P. *Certain subclasses of analytic and bi-univalent functions*, Appl. Math. Lett., **23**, 1188-1192, 2010.

[10] Taha, T. S. *Topics in Univalent Function Theory*, Ph.D. Thesis, University of London, 1981.

[11] Xu, Q, Gui, Y. and Srivastava, H. M. *Coefficient estimates for a certain subclass of analytic and bi-univalent functions*, Appl. Math. Lett., **25**, 990-994, 2012.

[12] Yang, D. G. and Liu, J. L *A class of analytic functions with missing coefficients,* Abstract and Applied Analysis, Volume **2011**, Article ID 456729, 2011.