NEW INTEGRAL INEQUALITIES
VIA \((\alpha, m)\)-CONVEXITY AND QUASI-CONVEXITY

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Abstract

In this paper, we establish some new integral inequalities involving Beta function via \((\alpha, m)\)-convexity and quasi-convexity, respectively. Our results in special cases recapture known results.

Keywords: Hermite’s inequality, Euler Beta function, Hölder’s inequality, \((\alpha, m)\)-convexity, quasi-convexity


1. Introduction

Let \(I\) be an interval in \(\mathbb{R}\). Then \(f : I \to \mathbb{R}\) is said to be convex (see [17, P.1]) if

\[
f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)
\]

holds for all \(x, y \in I\) and \(t \in [0,1]\).

In [27], Toader defined \(m\)-convexity as follows:

1.1. Definition. The function \(f : [0, b] \to \mathbb{R}, \ b > 0\) is said to be \(m\)-convex, where \(m \in [0,1]\), if

\[
f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y)
\]

holds for all \(x, y \in [0, b]\) and \(t \in [0,1]\). We say that \(f\) is \(m\)-concave if \(-f\) is \(m\)-convex.

In [18], Miheșan defined \((\alpha, m)\)-convexity as follows:

1.2. Definition. The function \(f : [0, b] \to \mathbb{R}, \ b > 0\), is said to be \((\alpha, m)\)-convex, where \((\alpha, m) \in [0,1]^2\), if

\[
f(tx + m(1-t)y) \leq t^\alpha f(x) + m(1-t^\alpha)f(y)
\]

holds for all \(x, y \in [0, b]\) and \(t \in [0,1]\).
Denote by $K_m^+(b)$ the class of all $(\alpha, m)$-convex functions on $[0, b]$ for which $f(0) \leq 0$. It can be easily seen that for $(\alpha, m) = (1, m)$, $(\alpha, m)$-convexity reduces to $m$-convexity and for $(\alpha, m) = (1, 1)$, $(\alpha, m)$-convexity reduces to the concept of usual convexity defined on $[0, b]$, $b > 0$. For recent results and generalizations concerning $m$-convex and $(\alpha, m)$-convex functions see [4, 6, 10, 19, 21, 26].

We recall that the notion of quasi-convex functions generalizes the notion of convex functions. More precisely, a function $f : [a, b] \to \mathbb{R}$ is said to be quasi-convex on $[a, b]$ if
\[
 f(\lambda x + (1 - \lambda)y) \leq \max\{f(x), f(y)\}
\]
holds for any $x, y \in [a, b]$ and $\lambda \in [0, 1]$. Clearly, any convex function is a quasi-convex function. Furthermore, there exist quasi-convex functions which are not convex (see [14]).

One of the most famous inequalities for convex functions is Hadamard’s inequality. This double inequality is stated as follows: Let $f$ be a convex function on some nonempty interval $[a, b]$ of real line $\mathbb{R}$, where $a \neq b$. Then
\[
 f \left( \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2}.
\]
Hadamard’s inequality for convex functions has received renewed attention in recent years and a remarkable variety of refinements and generalizations have been found (see, for example, [1]-[19], [22]-[26], [28]). In [4], Bakula et al. establish several Hadamard type inequalities for differentiable $m$-convex and $(\alpha, m)$-convex functions.

Recently, Ion [14] established two estimates on the Hermite-Hadamard inequality for functions whose first derivatives in absolute value are quasi-convex. Namely, he obtained the following results:

1.3. Theorem. Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I$, $a, b \in I$ with $a < b$. If $|f'|$ is quasi-convex on $[a, b]$, then the following inequality holds:
\[
 \frac{|f(a) + f(b)|}{2} - \frac{1}{b - a} \int_a^b f'(u)du \leq \frac{b - a}{4} \left\{ \max \left\{ |f'(a)|, |f'(b)| \right\} \right\}.
\]

1.4. Theorem. Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I$, $a, b \in I$ with $a < b$ and let $p > 1$. If $|f'|^{1/p}$ is quasi-convex on $[a, b]$, then the following inequality holds:
\[
 \frac{|f(a) + f(b)|}{2} - \frac{1}{b - a} \int_a^b f'(u)du \leq \frac{b - a}{2(p + 1)^{1/p}} \left( \max \left\{ |f'(a)|^{1/p}, |f'(b)|^{1/p} \right\} \right)^{p+1}.
\]

In [2], Alomari et al. obtained the following result.

1.5. Theorem. Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I$, $a, b \in I$ with $a < b$ and let $q \geq 1$. If $|f'|^q$ is quasi-convex on $[a, b]$, then the following inequality holds:
\[
 \frac{|f(a) + f(b)|}{2} - \frac{1}{b - a} \int_a^b f'(u)du \leq \frac{b - a}{4} \left( \max \left\{ |f'(a)|^q, |f'(b)|^q \right\} \right)^{1/q}.
\]

In [20], Özdemir et al. used the following lemma in order to establish several integral inequalities via some kinds of convexity.

1.6. Lemma. Let $f : [a, b] \subset [0, \infty) \to \mathbb{R}$ be continuous on $[a, b]$ such that $f \in L([a, b])$, $a < b$. Then the equality
\[
 \int_a^b (x - a)^p(b - x)^qf(x)dx = (b - a)^{p+q+1}\int_0^1 (1 - t)^p t^q f(ta + (1 - t)b)dt
\]
holds for some fixed $p, q > 0$.

Especially, Özdemir et al. [20] discussed the following new results connecting with $m$-convex function and quasi-convex function, respectively:
1.7. **Theorem.** Let \( f : [a, b] \to \mathbb{R} \) be continuous on \([a, b]\) such that \( f \in L([a, b]) \), \( 0 \leq a < b < \infty \). If \( f \) is \( m-\)convex on \([a, b]\), for some fixed \( m \in (0,1] \) and \( p, q > 0 \), then
\[
\int_a^b (x-a)^p(b-x)^q f(x) dx \\
\leq (b-a)^{p+q+1} \min \left\{ \beta(q + 2, p + 1) f(a) + m \beta(q + 1, p + 2) f \left( \frac{a}{m} \right) \right\},
\]
(1.3) \( \beta(q + 1, p + 2) f(b) + m \beta(q + 2, p + 1) f \left( \frac{a}{m} \right) \},
\)
where \( \beta(x, y) \) is the Euler Beta function.

1.8. **Theorem.** Let \( f : [a, b] \to \mathbb{R} \) be continuous on \([a, b]\) such that \( f \in L([a, b]) \), \( 0 \leq a < b < \infty \). If \( f \) is quasi-\( m \)-convex on \([a, b]\), then for some fixed \( p, q > 0 \), we have
\[
\int_a^b (x-a)^p(b-x)^q f(x) dx \leq (b-a)^{p+q+1} \max \{ f(a), f(b) \} \beta(p + 1, q + 1).
\]
(1.4) \( \int_a^b (x-a)^p(b-x)^q f(x) dx \leq (b-a)^{p+q+1} \max \{ f(a), f(b) \} \beta(p + 1, q + 1).
\)

The aim of this paper is to establish some new integral inequalities like those given in Theorems 1.7 and 1.8 for \((\alpha, m)\)-convex functions (Section 2) and quasi-\( m \)-convex functions (Section 3), respectively. Our results in special cases recapture Theorems 1.7 and 1.8, respectively. That is, this study is a continuation and generalization of [20].

2. **New integral inequalities for \((\alpha, m)\)-convex functions**

2.1. **Theorem.** Let \( f : [a, b] \to \mathbb{R} \) be continuous on \([a, b]\) such that \( f \in L([a, b]) \), \( 0 \leq a < b < \infty \). If \( f \) is \((\alpha, m)\)-convex on \([a, b]\), for some fixed \((\alpha, m) \in (0,1]^2 \) and \( p, q > 0 \), then
\[
\int_a^b (x-a)^p(b-x)^q f(x) dx \\
\leq (b-a)^{p+q+1} \min \left\{ \beta(q + \alpha + 1, p + 1) f(a) + m \beta(q + 1, p + 1) - \beta(q + \alpha + 1, p + 1) \right\} f \left( \frac{a}{m} \right),
\]
(2.1) \( \beta(q + 1, p + \alpha + 1) f(b) + m \beta(p + 1, q + 1) - \beta(q + 1, p + \alpha + 1) f \left( \frac{a}{m} \right) \},
\)
where \( \beta(x, y) \) is the Euler Beta function.

**Proof.** Since \( f \) is \((\alpha, m)\)-convex on \([a, b]\), we know that for every \( t \in [0,1] \)
\[
f(ta + (1-t)b) = f \left( ta + m(1-t) \frac{b}{m} \right) \leq t^\alpha f(a) + m (1-t^\alpha) f \left( \frac{b}{m} \right),
\]
(2.2) \( f(ta + (1-t)b) = f \left( ta + m(1-t) \frac{b}{m} \right) \leq t^\alpha f(a) + m (1-t^\alpha) f \left( \frac{b}{m} \right),
\)
Using Lemma 1.6, with \( x = ta + (1-t)b \), then we have
\[
\int_a^b (x-a)^p(b-x)^q f(x) dx \\
\leq (b-a)^{p+q+1} \int_0^1 (1-t)^p (1-t)^q f \left( \frac{b}{m} \right) f \left( \frac{a}{m} \right) \left( t^\alpha f(a) + m (1-t^\alpha) f \left( \frac{b}{m} \right) \right) dt
\]
\[
= (b-a)^{p+q+1} \left[ f(a) \int_0^1 (1-t)^p (1-t)^q f \left( \frac{b}{m} \right) dt + m f \left( \frac{b}{m} \right) \int_0^1 (1-t)^p (1-t^\alpha) f \left( \frac{b}{m} \right) dt \right].
\]
Now, we will make use of the Beta function which is defined for \( x, y > 0 \) as
\[
\beta(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt.
\]
It is known that
\[
\int_0^1 t^{q+\alpha} (1-t)^p dt = \beta(q + \alpha + 1, p + 1),
\]
\[
\int_0^1 (1-t)^p t^q (1-t^\alpha) \, dt = \int_0^1 t^q (1-t)^p \, dt - \int_0^1 t^{q+\alpha} (1-t)^p \, dt = \beta(q+1,p+1) - \beta(q+\alpha+1,p+1)].
\]

Combining all obtained equalities we get
\[
\int_a^b (x-a)^p (b-x)^q f(x) \, dx 
\leq (b-a)^{p+q+1} \left\{ \beta(q+1,p+1)f(a) + m[\beta(q+1,p+1) - \beta(q+\alpha+1,p+1)]f \left( \frac{b}{m} \right) \right\}.
\]

(2.3)

If we choose \( x = tb + (1-t)a \), analogously we obtain
\[
\int_a^b (x-a)^p (b-x)^q f(x) \, dx 
\leq (b-a)^{p+q+1} \left\{ \beta(q+1,p+\alpha+1)f(b) + m[\beta(q+1,p+1) - \beta(q+1,p+\alpha+1)]f \left( \frac{a}{m} \right) \right\}.
\]

(2.4)

Thus, by (2.3) and (2.4) we obtain (2.1), which completes the proof. \( \square \)

2.2. **Remark.** As a special case of Theorem 2.1 for \( \alpha = 1 \), that is for \( f \) be \( m-\)convex on \([a,b] \), we recapture Theorem 1.7 due to the fact that
\[
\beta(q+1,p+1) - \beta(q+1,p+1) = \beta(q+1,p+1) - \frac{q+1}{p+q+2} \beta(q+1,p+1) = \frac{p+1}{p+q+2} \beta(q+1,p+1) = \beta(q+1,p+2)
\]
and
\[
\beta(q+1,p+1) - \beta(q+1,p+\alpha+1) = \beta(q+2,p+1).
\]

2.3. **Corollary.** In Theorem 2.1, if \( p = q \), then (2.1) reduces to
\[
\int_a^b (x-a)^p (b-x)^p f(x) \, dx 
\leq (b-a)^{2p+1} \min\left\{ \beta(p+\alpha+1,p+1)f(a) + m[\beta(p+1,p+1) - \beta(p+\alpha+1,p+1)]f \left( \frac{b}{m} \right), \right. \\
\left. \beta(p+1,p+\alpha+1)f(b) + m[\beta(p+1,p+1) - \beta(p+1,p+\alpha+1)]f \left( \frac{a}{m} \right) \right\}.
\]

2.4. **Theorem.** Let \( f : [a,b] \to \mathbb{R} \) be continuous on \([a,b] \) such that \( f \in L ([a,b]) \), \( 0 \leq a < b < \infty \) and let \( k > 1 \). If \( |f|^{\frac{k}{k+1}} \) is \((\alpha,m)-\)convex on \([a,b] \), for some fixed \((\alpha,m) \in (0,1]^2\) and \( p,q > 0 \), then
\[
\int_a^b (x-a)^p (b-x)^q f(x) \, dx 
\leq \frac{(b-a)^{p+q+1}}{(\alpha+1)^{\frac{k-1}{k+1}}} [\beta(kp+1,kq+1)]^{\frac{k-1}{k}} \min\left\{ \left[ |f(a)|^{\frac{k}{k+1}} + \alpha m \left| f \left( \frac{b}{m} \right) \right|^{\frac{k}{k+1}} \right]^{\frac{k-1}{k}}, \right. \\
\left. \left[ |f(b)|^{\frac{k}{k+1}} + \alpha m \left| f \left( \frac{a}{m} \right) \right|^{\frac{k}{k+1}} \right]^{\frac{k-1}{k}} \right\}.
\]

(2.5)
Proof. Since $|f|^\frac{k}{k+1}$ is $(\alpha, m)$--convex on $[a, b]$ we know that for every $t \in [0, 1]$

$$|f(ta + (1 - t)b)|^{\frac{k}{k+1}} = \left| f \left( ta + m(1 - t) \frac{b}{m} \right) \right|^{\frac{k}{k+1}} \leq t^\alpha |f(a)|^{\frac{k}{k+1}} + m (1 - t^\alpha) \left| f \left( \frac{b}{m} \right) \right|^{\frac{k}{k+1}}.$$ 

Using Lemma 1.6, with $x = ta + (1 - t)b$, then we have

$$\int_a^b (x - a)^p (b - x)^q f(x) dx \leq (b - a)^{p+1+1} \left[ \int_0^1 (1 - t)^{kp} t^{kq} dt \right]^{\frac{1}{k+1}} \left[ \int_0^1 |f(ta + (1 - t)b)|^{\frac{k}{k+1}} dt \right]^{\frac{k+1}{k+1}}$$

$$\leq (b - a)^{p+1+1} [\beta(kq + 1, kp + 1)]^{\frac{1}{k+1}} \left[ \int_0^1 t^\alpha |f(a)|^{\frac{k}{k+1}} dt + m \int_0^1 (1 - t^\alpha) |f \left( \frac{b}{m} \right)|^{\frac{k}{k+1}} dt \right]^{\frac{k+1}{k+1}}$$

$$= (b - a)^{p+1+1} [\beta(kq + 1, kp + 1)]^{\frac{1}{k+1}} \left[ \frac{1}{\alpha + 1} |f(a)|^{\frac{k}{k+1}} + m \frac{\alpha}{\alpha + 1} |f \left( \frac{b}{m} \right)|^{\frac{k}{k+1}} \right]^{\frac{k+1}{k+1}}.$$

If we choose $x = tb + (1 - t)a$, analogously we obtain

$$\int_a^b (x - a)^p (b - x)^q f(x) dx \leq (b - a)^{p+1+1} [\beta(kp + 1, kp + 1)]^{\frac{1}{k+1}} \left[ \frac{1}{\alpha + 1} |f(b)|^{\frac{k}{k+1}} + m \frac{\alpha}{\alpha + 1} |f \left( \frac{a}{m} \right)|^{\frac{k}{k+1}} \right]^{\frac{k+1}{k+1}},$$

which completes the proof. \qed

2.5. Corollary. In Theorem 2.4, if $p = q$, then (2.5) reduces to

$$\int_a^b (x - a)^p (b - x)^q f(x) dx \leq \frac{(b - a)^{2p+1}}{(\alpha + 1)^{\frac{1}{k+1}}} [\beta(kp + 1, kp + 1)]^{\frac{1}{k+1}} \min \left\{ \left[ |f(a)|^{\frac{k}{k+1}} + am |f \left( \frac{b}{m} \right)|^{\frac{k}{k+1}} \right]^{\frac{k+1}{k+1}}, \right.$$

$$\left. \left[ |f(b)|^{\frac{k}{k+1}} + am |f \left( \frac{a}{m} \right)|^{\frac{k}{k+1}} \right]^{\frac{k+1}{k+1}} \right\}.$$

2.6. Corollary. In Theorem 2.4, if $\alpha = 1$, i.e., if $|f|^\frac{k}{k+1}$ is $m$--convex on $[a, b]$, then (2.5) reduces to

$$\int_a^b (x - a)^p (b - x)^q f(x) dx \leq \frac{(b - a)^{p+1+1}}{2^{\frac{k+1}{k}} [\beta(kp + 1, kp + 1)]^{\frac{1}{k+1}}} \min \left\{ \left[ |f(a)|^{\frac{k}{k+1}} + m |f \left( \frac{b}{m} \right)|^{\frac{k}{k+1}} \right]^{\frac{k+1}{k}}, \right.$$
2.7. Remark. As a special case of Corollary 2.6 for \( m = 1 \), that is for \( |f|^{\frac{k}{m}} \) be convex on \([a, b]\), we get
\[
\int_a^b (x - a)^p (b - x)^q f(x)dx \leq \left(\frac{b - a}{2}\right)^{p+q+1} |\beta(kp + 1, kq + 1)|^{\frac{1}{r}} \left[|f(a)|^{\frac{1}{r}} + |f(b)|^{\frac{1}{r}}\right]^{\frac{p+q+1}{r}}.
\]

2.8. Theorem. Let \( f : [a, b] \to \mathbb{R} \) be continuous on \([a, b]\) such that \( f \in L([a, b]), 0 \leq a < b < \infty \) and let \( l \geq 1 \). If \( |f|^{l} \) is \((\alpha, m)\)–convex on \([a, b]\), for some fixed \((\alpha, m) \in (0, 1]^2\) and \( p, q > 0 \), then
\[
\int_a^b (x - a)^p (b - x)^q f(x)dx 
\leq (b - a)^{p+q+1} |\beta(p + 1, q + 1)|^{\frac{1}{r}} \times \min \left\{ \left[\beta(q + \alpha + 1, p + 1)|f(a)|^l + m|\beta(q + 1, p + 1) - \beta(q + \alpha + 1, p + 1)| f \left(\frac{b}{m}\right)\right]^{\frac{1}{r}}, \right\}
\]
(2.6)
\[
\left[\beta(q + 1, p + \alpha + 1)|f(b)|^l + m|\beta(q + 1, p + 1) - \beta(q + 1, p + \alpha + 1)| f \left(\frac{a}{m}\right)\right]^{\frac{1}{r}}.
\]

Proof. Since \( |f|^l \) is \((\alpha, m)\)–convex on \([a, b]\), we know that for every \( t \in [0, 1] \)
\[
|f(ta + (1 - t)b)|^l = \left| f \left(\frac{ta + m(1 - t) \frac{b}{m}}{a}\right)\right| \leq t^\alpha |f(a)|^l + m(1 - t^\alpha) \left| f \left(\frac{b}{m}\right)\right|^l.
\]
Using Lemma 1.6, with \( x = ta + (1 - t)b \), then we have
\[
\int_a^b (x - a)^p (b - x)^q f(x)dx = (b - a)^{p+q+1} \int_0^1 [(1 - t)^p t^q]^{\frac{1}{r}} |(1 - t)^p t^q|^{\frac{1}{r}} f(ta + (1 - t)b)dt
\]
\[
\leq (b - a)^{p+q+1} \left[ \int_0^1 (1 - t)^p t^q dt \right]^{\frac{1}{r}} \left[ \int_0^1 (1 - t)^p t^q |f(ta + (1 - t)b)|^l dt \right]^{\frac{1}{r}}
\]
\[
\leq (b - a)^{p+q+1} |\beta(q + 1, p + 1)|^{\frac{1}{r}} \times \left[\beta(q + \alpha + 1, p + 1)|f(a)|^l + m|\beta(q + 1, p + 1) - \beta(q + \alpha + 1, p + 1)| f \left(\frac{b}{m}\right)\right]^{\frac{1}{r}},
\]
\[
\times \left[\beta(q + 1, p + \alpha + 1)|f(b)|^l + m|\beta(q + 1, p + 1) - \beta(q + 1, p + \alpha + 1)| f \left(\frac{a}{m}\right)\right]^{\frac{1}{r}}.
\]
If we choose \( x = tb + (1 - t)a \), analogously we obtain
\[
\int_a^b (x - a)^p (b - x)^q f(x)dx \leq (b - a)^{p+q+1} |\beta(p + 1, q + 1)|^{\frac{1}{r}} \times \left[\beta(q + 1, p + \alpha + 1)|f(a)|^l + m|\beta(q + 1, p + 1) - \beta(q + 1, p + \alpha + 1)| f \left(\frac{b}{m}\right)\right]^{\frac{1}{r}},
\]
which completes the proof. \( \Box \)
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2.9. Corollary. In Theorem 2.8, if \(p = q\), then (2.6) reduces to
\[
\int_a^b (x-a)^p (b-x)^q f(x) dx \\
\leq (b-a)^{p+1} \beta(p+1,p+1) \frac{|f(a)|}{|f(b)|}\min \left\{ \left[ \beta(p+1,p+1) |f(a)|^q + m|\beta(p+1,p+1) - \beta(p+1,p+1)| \right] \left[ \frac{b}{m} \right] \right\}^{\frac{1}{q}}
\]

2.10. Corollary. In Theorem 2.8, if \(\alpha = 1\), i.e., if \(|f|^q\) is \(m\)-convex on \([a,b]\), then (2.6) reduces to
\[
\int_a^b (x-a)^p (b-x)^q f(x) dx \\
\leq (b-a)^{p+q+1} \beta(p+1,q+1) \frac{|f(a)|}{|f(b)|}\min \left\{ \left[ \beta(q+2,p+1) |f(a)|^q + m\beta(q+2,p+1) \right] \left[ \frac{b}{m} \right] \right\}^{\frac{1}{q}}
\]

2.11. Remark. As a special case of Corollary 2.10 for \(m = 1\), that is for \(|f|^q\) be convex on \([a,b]\), we get
\[
\int_a^b (x-a)^p (b-x)^q f(x) dx \\
\leq (b-a)^{p+q+1} \beta(p+1,q+1) \frac{|f(a)|}{|f(b)|}\min \left\{ \left[ \beta(q+2,p+1) |f(a)|^q + \beta(q+2,p+1) \right] \left[ \frac{b}{m} \right] \right\}^{\frac{1}{q}}
\]

3. New integral inequalities for quasi-convex functions

3.1. Theorem. Let \(f : [a,b] \to \mathbb{R}\) be continuous on \([a,b]\) such that \(f \in L([a,b])\), \(0 \leq a < b \leq \infty\) and let \(k > 1\). If \(|f|^{\frac{k}{k-1}}\) is quasi-convex on \([a,b]\), for some fixed \(p,q > 0\), then
\[
\int_a^b (x-a)^p (b-x)^q f(x) dx \leq (b-a)^{p+q+1} \beta(kp,q,kq+1) \frac{|f(a)|}{|f(b)|}\left( \max \left\{ \left[ f(a) \right]^{\frac{k}{k-1}}, \left[ f(b) \right]^{\frac{k}{k-1}} \right\} \right)^{\frac{1}{k-1}}
\]

Proof. By Lemma 1.6, Hölder’s inequality, the definition of Beta function and the fact that \(|f|^{\frac{k}{k-1}}\) is quasi-convex on \([a,b]\), we have
\[
\int_a^b (x-a)^p (b-x)^q f(x) dx \\
\leq (b-a)^{p+q+1} \left[ \int_0^1 (1-t)^kpq dt \right] \frac{1}{k} \left[ \int_0^1 f\left( ta + (1-t)b \right) \left[ \frac{b}{m} \right] \right]^{\frac{k-1}{k}}
\]
\[
\leq (b-a)^{p+q+1} \beta(kp,q,kq+1) \frac{1}{k} \left[ \int_0^1 \max \left\{ \left[ f(a) \right]^{\frac{k}{k-1}}, \left[ f(b) \right]^{\frac{k}{k-1}} \right\} dt \right]^{\frac{k-1}{k}}
\]
\[
= (b-a)^{p+q+1} \beta(kp,q,kq+1) \frac{1}{k} \left[ \max \left\{ \left[ f(a) \right]^{\frac{k}{k-1}}, \left[ f(b) \right]^{\frac{k}{k-1}} \right\} \right]^{\frac{k-1}{k}}
\]
which completes the proof.
3.2. Corollary. Let $f$ be as in Theorem 3.1. Additionally, if
(1) $f$ is increasing, then we have
$$\int_a^b (x-a)^p(b-x)^q f(x)dx \leq (b-a)^{p+q+1} \left[\beta(kp+1, kp+1)\right]^\frac{1}{p} f(b).$$
(2) $f$ is decreasing, then we have
$$\int_a^b (x-a)^p(b-x)^q f(x)dx \leq (b-a)^{p+q+1} \left[\beta(kp+1, kp+1)\right]^\frac{1}{p} f(a).$$

3.3. Theorem. Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ such that $f \in L([a, b])$, $0 \leq a < b < \infty$ and let $l \geq 1$. If $|f|^l$ is quasi-convex on $[a, b]$, for some fixed $p, q > 0$, then
$$\int_a^b (x-a)^p(b-x)^q f(x)dx \leq (b-a)^{p+q+1} \beta(p+1, q+1) \left(\max\left\{|f(a)|^l, |f(b)|^l\right\}\right)^\frac{1}{l},$$
where $\beta(x, y)$ is the Euler Beta function.

Proof. By Lemma 1.6, Hölder’s inequality, the definition of Beta function and the fact that $|f|^l$ is quasi-convex on $[a, b]$, we have
$$\int_a^b (x-a)^p(b-x)^q f(x)dx = (b-a)^{p+q+1} \int_0^1 [(1-t)^p t^q]^{\frac{1}{l}} \left[(1-t)^p t^q\right]^{\frac{1}{l}} f(ta + (1-t)b)dt$$
$$\leq (b-a)^{p+q+1} \left[\int_0^1 (1-t)^p t^q dt\right]^{\frac{1}{l}} \left[\int_0^1 (1-t)^p t^q \left|f(ta + (1-t)b)\right|^l dt\right]^{\frac{1}{l}}$$
$$\leq (b-a)^{p+q+1} \left[\beta(q+1, p+1)\right]^{\frac{1}{l}} \left[\max\left\{|f(a)|^l, |f(b)|^l\right\}\beta(q+1, p+1)\right]^{\frac{1}{l}}$$
$$= (b-a)^{p+q+1} \beta(p+1, q+1) \left(\max\left\{|f(a)|^l, |f(b)|^l\right\}\right)^\frac{1}{l},$$
which completes the proof. \qed

3.4. Corollary. Let $f$ be as in Theorem 3.3. Additionally, if
(1) $f$ is increasing, then we have
$$\int_a^b (x-a)^p(b-x)^q f(x)dx \leq (b-a)^{p+q+1} \beta(p+1, q+1) f(b).$$
(2) $f$ is decreasing, then we have
$$\int_a^b (x-a)^p(b-x)^q f(x)dx \leq (b-a)^{p+q+1} \beta(p+1, q+1) f(a).$$

References
New integral inequalities via \((\alpha, m)\)-convexity and quasi-convexity


