A NEW SUBCLASS OF HARMONIC MAPPINGS WITH POSITIVE REAL PART

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Abstract

Complex-valued harmonic functions that are univalent and sense preserving in the unit disk $U$ can be written in the form $f = h + \bar{g}$, where $h$ and $g$ are analytic in $U$. In this paper, we introduce a class $HP(\beta, \alpha)$, $(\alpha \geq 0, 0 \leq \beta < 1)$ of all functions $f = h + \bar{g}$ for which $\Re \{\alpha z (h'(z) + g'(\bar{z})) + h(z) + g(\bar{z})\} > \beta$, $f(0) = 1$. We give sufficient coefficient conditions for normalized harmonic functions to be in $HP(\beta, \alpha)$. These conditions are also shown to be necessary when the coefficients are negative. This leads to distortion bounds and extreme points.

Key Words: Harmonic mappings, extreme points, distortion bounds.

Mathematics Subject Classification: 30 C 45

1. Introduction

A continuous function $f = u + iv$ is a complex-valued harmonic function in a domain $D \subset \mathbb{C}$ if both $u$ and $v$ are real harmonic in $D$. In any simply connected domain we can write $f = h + \bar{g}$, where $h$ and $g$ are analytic in $D$. See Clunie and Sheil-Small [1].

There has been interest [2] in studying the class $P_H$ of all the functions of the form $f = h + \bar{g}$ that are harmonic in $U = \{z : |z| < 1\}$ and such that for $z \in U$, $\Re f(z) > 0$, where

$$h(z) = 1 + \sum_{n=1}^{\infty} a_n z^n \text{ and } g(z) = \sum_{n=1}^{\infty} b_n z^n \tag{1}$$

are analytic in $U$.

The class $P_H(\beta)$ of all functions of the form (1) with $\Re f(z) > \beta$, $0 \leq \beta < 1$ and $f(0) = 1$ is studied in [4]. Obviously, $P_H(0) = P_H$ and $P_H(\beta) \subset P_H$.

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We denote by $HP(\beta, \alpha)$ the class of all functions of the form (1) that satisfy the condition

\[ \Re\{az(h(z) + g'(z)) + h(z) + g(z)\} > \beta, \ \alpha \geq 0, \ 0 \leq \beta < 1. \tag{2} \]

Clearly, $HP(0, 0) = P_H$ and $HP(\beta, 0) = P_H(\beta)$. Moreover, if $0 \leq \beta_1 \leq \beta_2 < 1$, then $HP(\beta_2, \alpha) \subset HP(\beta_1, \alpha)$ and if $0 \leq \alpha_1 \leq \alpha_2$, then $HP(\beta, \alpha_2) \subset HP(\beta, \alpha_1)$.

We further denote by $HR(\beta, \alpha)$ the subclass of $HP(\beta, \alpha)$ such that the functions $h$ and $g$ in $f = h + g$ are of the form

\[ h(z) = 1 - \sum_{n=1}^{\infty} a_n z^n \]  

and

\[ g(z) = -\sum_{n=1}^{\infty} b_n z^n \]  

with $a_n \geq 0$ and $b_n \geq 0$ for all $n \geq 1$.

2. Main Result

2.1. Theorem: Let $f = h + g$ be given by (1). Furthermore, let

\[ \sum_{n=1}^{\infty} \frac{\alpha n + 1}{1 - \beta} (|a_n| + |b_n|) \leq 1 \]  \tag{4}

where $\alpha \geq 0$ and $0 \leq \beta < 1$. Then $f \in HP(\beta, \alpha)$.

Proof. We show that the inequality (4) is a sufficient condition for $f$ to be in $HP(\beta, \alpha)$. According to the condition (2) we only need to show that if (4) holds then

\[ |1 - \beta + \alpha z(h'(z) + g'(z)) + h(z) + g(z)| - |1 + \beta - \alpha z(h'(z) + g'(z)) - h(z) - g(z)| > 0. \tag{5} \]

Substituting $h(z)$ and $g(z)$ in (5) yields by (4),

\[ |1 - \beta + \alpha z(h'(z) + g'(z)) + h(z) + g(z)| - |1 + \beta - \alpha z(h'(z) + g'(z)) - h(z) - g(z)| = \]

\[ = 2 - \beta + \sum_{n=1}^{\infty} (\alpha n + 1)(a_n + b_n)z^n - \beta - \sum_{n=1}^{\infty} (\alpha n + 1)(a_n + b_n)z^n \]

\[ \geq 2(1 - \beta) - 2 \sum_{n=1}^{\infty} (\alpha n + 1)(|a_n| + |b_n|)|z^n| \]

\[ > 2(1 - \beta) \left\{ 1 - \sum_{n=1}^{\infty} \frac{\alpha n + 1}{1 - \beta} (|a_n| + |b_n|) \right\} \geq 0. \tag*{\square} \]

The harmonic mappings

\[ f(z) = 1 + \sum_{n=1}^{\infty} \frac{1 - \beta}{\alpha n + 1} (x_n z^n + y_n z^n), \tag{6} \]
A new subclass of Harmonic Mappings

where
\[ \sum_{n=1}^{\infty} (|x_n| + |y_n|) = 1, \]
show that the coefficient bound given by (4) is sharp.

The functions of the form (6) are in \( H\hat{P}(\beta, \alpha) \) because
\[ \sum_{n=1}^{\infty} \frac{\alpha n + 1}{1 - \beta} (|a_n| + |b_n|) = \sum_{n=1}^{\infty} (|x_n| + |y_n|) = 1. \]

The restriction imposed in Theorem 2.1 on the moduli of the coefficients of \( f = h + \tilde{g} \) enables us to conclude for arbitrary rotation of the coefficients of \( f \) that the resulting functions would still be harmonic and \( f \in H\hat{P}(\beta, \alpha) \). Our next theorem establishes that such coefficient bounds cannot be improved.

2.2. Theorem: Let \( f = h + \tilde{g} \) be given by (3). Then \( f \in H\hat{R}(\beta, \alpha) \) if and only if
\[ \sum_{n=1}^{\infty} \frac{\alpha n + 1}{1 - \beta} (|a_n| + |b_n|) \leq 1, \]
where \( \alpha \geq 0 \) and \( 0 \leq \beta < 1 \).

Proof. The if part follows from Theorem 2.1 upon noting that if \( f = h + \tilde{g} \in H\hat{P}(\beta, \alpha) \) are of the form (3) then \( f \in H\hat{R}(\beta, \alpha) \).

Suppose that \( f \in H\hat{R}(\beta, \alpha) \). Then we find from (2) that
\[ \Re \left\{ 1 - \sum_{n=1}^{\infty} (\alpha n + 1)(a_n + b_n)z^n \right\} > \beta, \ z \in U, \ \alpha \geq 0, \ 0 \leq \beta < 1. \]
If we choose \( z \) to be real and let \( z \to 1^- \), we get
\[ 1 - \sum_{n=1}^{\infty} (\alpha n + 1)(a_n + b_n) \geq \beta \]
or equivalently,
\[ \sum_{n=1}^{\infty} (\alpha n + 1)(a_n + b_n) \leq 1 - \beta, \]
which is precisely the assertion (7) of Theorem 2.2.

2.3. Theorem: If \( f \in H\hat{R}(\beta, \alpha) \), then
\[ |f(z)| \leq 1 + \frac{1 - \beta}{1 + \alpha} r, \ |z| < 1 \]
and
\[ |f(z)| \geq 1 - \frac{1 - \beta}{1 + \alpha} r, \ |z| < 1. \]
Proof. Let \( f \in HR(\beta, \alpha) \). Taking the absolute value of \( f \) we obtain

\[
|f(z)| \leq 1 + \sum_{n=1}^{\infty} (a_n + b_n)|z|^n \\
\leq 1 + \sum_{n=1}^{\infty} (a_n + b_n)r \\
\leq 1 + \frac{1 - \beta}{1 + \alpha} \sum_{n=1}^{\infty} \frac{\alpha + 1}{1 - \beta} (a_n + b_n)r \\
\leq 1 + \frac{1 - \beta}{1 + \alpha}r,
\]

and

\[
|f(z)| \geq 1 - \sum_{n=1}^{\infty} (a_n + b_n)|z|^n \\
\geq 1 - \sum_{n=1}^{\infty} (a_n + b_n)r \\
\geq 1 - \frac{1 - \beta}{1 + \alpha} \sum_{n=1}^{\infty} \frac{\alpha + 1}{1 - \beta} (a_n + b_n)r \\
\geq 1 - \frac{1 - \beta}{1 + \alpha}r.
\]

The bounds given in Theorem 2.3 for the functions \( f = h + \bar{g} \) of the form (3) also hold for functions of the form (1) if the coefficient condition (4) is satisfied. The functions

\[
f(z) = 1 - \frac{1 - \beta}{1 + \alpha} z \quad \text{and} \quad f(z) = 1 - \frac{1 - \beta}{1 + \alpha} \bar{z}
\]

for \( 0 \leq \beta < 1 \) and \( \alpha \geq 0 \) show that the bounds given in Theorem 2.3 are sharp.

The following covering result follows from the second inequality in Theorem 2.3.

2.4. Corollary. If \( f \in HR(\beta, \alpha) \), then

\[
\left\{ w : |w| < \frac{\alpha + \beta}{1 + \alpha} \right\} \subset f(U).
\]

As \( HR(\beta, \alpha) \) is a convex family, \( HR(\beta, \alpha) \) has a non-empty set of extreme points.

2.5. Theorem: Set

\[
h_n(z) = 1 - \frac{1 - \beta}{\alpha n + 1} z^n \quad \text{and} \quad g_n(z) = 1 - \frac{1 - \beta}{\alpha n + 1} z^n, \quad \text{for } n = 1, 2, \ldots
\]

Then \( f \in HR(\beta, \alpha) \) if and only if it can be expressed in the form

\[
f(z) = \sum_{n=1}^{\infty} (\lambda_n h_n + \gamma_n g_n), \quad (8)
\]
where \( \lambda_n \geq 0, \gamma_n \geq 0 \) and \( \sum_{n=1}^{\infty} (\lambda_n + \gamma_n) = 1 \).

In particular, the extreme points of \( HR(\beta, \alpha) \) are \( \{h_n\} \) and \( \{g_n\} \).

**Proof.** For functions \( f \) of the form (8) we have

\[
f(z) = \sum_{n=1}^{\infty} (\lambda_n h_n + \gamma_n g_n) = 1 - \sum_{n=1}^{\infty} \frac{1 - \beta}{\alpha n + 1} (\lambda_n z^n + \gamma_n \bar{z}^n).
\]

Then

\[
\sum_{n=1}^{\infty} \frac{\alpha n + 1}{1 - \beta} \left[ \frac{1 - \beta}{\alpha n + 1} (\lambda_n + \gamma_n) \right] = \sum_{n=1}^{\infty} (\lambda_n + \gamma_n) = 1
\]

and so \( f \in HR(\beta, \alpha) \).

Conversely, suppose that \( f \in HR(\beta, \alpha) \). Set

\[
\lambda_n = \frac{\alpha n + 1}{1 - \beta} a_n \quad \text{and} \quad \gamma_n = \frac{\alpha n + 1}{1 - \beta} b_n, \quad \text{for} \quad n = 1, 2, \ldots.
\]

Then by Theorem 2.2, \( 0 \leq \lambda_n \leq 1 \) and \( 0 \leq \gamma_n \leq 1 \), \( (n = 1, 2, \ldots) \). Consequently, we obtain

\[
f(z) = \sum_{n=1}^{\infty} (\lambda_n h_n + \gamma_n g_n),
\]

as required. \( \square \)

Following Ruscheweyh [3], we call the set

\[
N_\delta(f) = \left\{ F : F(z) = 1 - \sum_{n=1}^{\infty} (|A_n|z^n + |B_n|z^n) \quad \text{and} \quad \sum_{n=1}^{\infty} n(|a_n - A_n| + |b_n - B_n|) \leq \delta \right\}.
\]

the \( \delta \)-neighborhood of \( f \in P_H \). In particular, for the constant function \( I(z) = 1 \), we immediately have

\[
N_\delta(I) = \left\{ f : f(z) = 1 - \sum_{n=1}^{\infty} (|a_n|z^n + |b_n|z^n) \quad \text{and} \quad \sum_{n=1}^{\infty} n(|a_n| + |b_n|) \leq \delta \right\}.
\]

**2.6. Theorem:** Let \( \delta = (1 - \beta)/\alpha \). Then \( HR(\beta, \alpha) \subset N_\delta(I) \).

**Proof.** Let \( f \) belong to \( HR(\beta, \alpha) \). We have

\[
\sum_{n=1}^{\infty} n(a_n + b_n) = \frac{1}{\alpha} \sum_{n=1}^{\infty} \alpha n(a_n + b_n)
\]

\[
\leq \frac{1}{\alpha} \sum_{n=1}^{\infty} (\alpha n + 1)(a_n + b_n)
\]

\[
\leq \frac{1}{\alpha} (1 - \beta) = \delta.
\]

Hence \( f(z) \in N_\delta(I) \). \( \square \)
References


