A classification theorem on totally umbilical submanifolds in a cosymplectic manifold

Siraj Uddin* and Cenap Ozel †

Abstract

In the present paper, we study totally umbilical submanifolds of cosymplectic manifolds. We obtain a result on the classification of totally umbilical contact CR-submanifolds of a cosymplectic manifold.

Received 07/03/2012 : Accepted 04/07/2013


Keywords: Totally umbilical, contact CR-submanifold, cosymplectic manifold.

1. Introduction

The notion of CR-submanifolds of a Kaehler manifold was introduced by A. Bejancu [1]. Later on, many researchers worked on these submanifolds for different structures [5]. These submanifolds are the natural generalization of both holomorphic and totally real submanifolds of a Kaehler manifold. Totally umbilical CR-submanifolds of a Kaehler manifold have been studied by A. Bejancu [2], B.Y. Chen (see [6]), S. Deshmukh and S.I. Husain [8].

The submanifolds of a cosymplectic manifold have been studied by G.D. Ludden [10]. Recently, we have obtained some results for the existence or non-existence of warped submanifolds in a cosymplectic manifold [11]. In this paper, we classify all totally umbilical contact CR-submanifolds of a cosymplectic manifold.

*Institute of Mathematical Sciences, Faculty of Science, University of Malaya, 50603 Kuala Lumpur, Malaysia
E-mail: siraj.ch@gmail.com

†Department of Mathematics, Abant Izzet Baysal University, 14268 Bolu, Turkey
E-mail: cenap.ozel@gmail.com
2. Preliminaries

Let \( \tilde{M} \) be a \((2n+1)\)-dimensional almost contact manifold with almost contact structure \((\phi, \xi, \eta)\), that is \(\phi\) is a \((1,1)\) tensor field, \(\xi\) is a vector field and \(\eta\) is a 1-form, satisfying the following properties

\[
\phi^2 = -I + \eta \otimes \xi, \quad \phi \xi = 0, \quad \eta \circ \phi = 0, \quad \eta(\xi) = 1. \tag{2.1}
\]

In this case we call \((\tilde{M}, \phi, \xi, \eta)\) an almost contact manifold. There always exists a Riemannian metric \(g\) on an almost contact manifold \(\tilde{M}\) satisfying the following compatibility condition

\[
g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2.2}
\]
for any \(X, Y\) tangent to \(\tilde{M}\); with this metric the almost contact manifold is called an almost contact metric manifold.

An almost contact structure \((\phi, \xi, \eta)\) is said to be normal if \([\phi, \phi] + 2d\eta \otimes \xi\) vanishes identically on \(\tilde{M}\), where \([\phi, \phi](X, Y) = \phi^2[X, Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y]\) for any vector fields \(X, Y\) tangent to \(\tilde{M}\) is the Nijenhuis tensor of \(\phi\).

The fundamental 2-form \(\Phi\) on \(\tilde{M}\) is defined as \(\Phi(X, Y) = \phi^2(X, Y) + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y]\) for any vector fields \(X, Y\) tangent to \(\tilde{M}\). If \(\Phi = d\eta\), the almost contact structure is a contact structure. A normal almost contact structure with \(\Phi\) closed and \(d\eta = 0\) is called cosymplectic structure. It is well known that the cosymplectic structure is characterized by

\[
\tilde{\nabla}_X \phi = 0 \quad \text{and} \quad \tilde{\nabla}_X \eta = 0, \tag{2.3}
\]
where \(\tilde{\nabla}\) is the Levi-Civita connection of \(g\) on \(\tilde{M}\). From (2.3), it follows that \(\tilde{\nabla}_X \xi = 0\).

If we denote the curvature tensor of a cosymplectic manifold \(\tilde{M}\) by \(\tilde{R}\), then we have

\[
\tilde{R}(\phi X, \phi Y) = \tilde{R}(X, Y) \quad \text{and} \quad \tilde{R}(X, Y) \phi Z = \phi \tilde{R}(X, Y) Z. \tag{2.4}
\]

Blair and Goldberg [5] studied the cosymplectic structure on a Riemannian manifold from topological viewpoint. They have given a typical example of simply connected cosymplectic manifold which is the product of a simply connected Kähler manifold with \(\mathbb{R}\). They proved that a complete simply connected cosymplectic manifold is almost contact isometric to the product of a complete simply connected Kähler manifold with \(\mathbb{R}\). On the other hand the natural example of a compact cosymplectic manifold is given by the product of a compact Kähler manifold \((V, J, h)\) with the circle \(S^1\), where \(J\) is almost complex structure and \(h\) is almost Hermitian metric on \(V\). The cosymplectic structure \((\phi, \xi, \eta, g)\) on the product manifold \(\tilde{M} = V \times S^1\) is defined by

\[
\phi = J \circ (pr_1)_*, \quad \xi = \frac{E}{c}, \quad \eta = c(pr_2)_*(\theta), \quad g = (pr_1)_*(h) + c^2(pr_2)_*(\theta \otimes \theta),
\]
where \(\ast\) is the symbol for tangent map and \(pr_1 : \tilde{M} \to V\) and \(pr_2 : \tilde{M} \to S^1\) are the projections of \(V \times S^1\) onto \(V\) and \(S^1\) respectively, \(\theta\) is the length element of \(S^1\), \(E\) is its dual vector field and \(c\) is a non-zero real number [5]. In [7], De Leon and Marrero studied compact cosymplectic manifold with positive constant \(\phi\)-sectional curvature.

Let \(M\) be a submanifold of an almost contact metric manifold \(\tilde{M}\) with induced metric \(g\) and if \(\nabla\) and \(\nabla^\perp\) are the induced connections on the tangent bundle \(TM\) and the normal bundle \(T^\perp M\) of \(M\), respectively. Denote by \(\mathcal{F}(M)\) the algebra of smooth functions on \(M\) and by \(\Gamma(TM)\) the \(\mathcal{F}(M)\)-module of smooth sections of tangent bundle \(TM\) over \(M\), then Gauss and Weingarten formulae are given by

\[
\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{2.5}
\]
\[
\tilde{\nabla}_X N = -A_X X + \nabla^\perp_X N, \tag{2.6}
\]
for each $X, Y \in \Gamma(TM)$ and $N \in \Gamma(T^\perp M)$, where $h$ and $A_N$ are the second fundamental form and the shape operator (corresponding to the normal vector field $N$) respectively for the immersion of $M$ into $\tilde{M}$. They are related as
\[ g(h(X,Y), N) = g(A_N X, Y), \] (2.7)
where $g$ denotes the Riemannian metric on $\tilde{M}$ as well as induced on $M$. The mean curvature vector $H$ on $M$ is given by
\[ H = \frac{1}{n} \sum_{i=1}^{n} h(e_i, e_i) \] (2.8)
where $n$ is the dimension of $M$ and $\{e_1, e_2, \cdots, e_n\}$ is a local orthonormal frame of vector fields on $M$.

A submanifold $M$ of a Riemannian manifold $\tilde{M}$ is said to be totally umbilical if
\[ h(X,Y) = g(X,Y)H. \] (2.9)
If $h(X,Y) = 0$ for any $X,Y \in \Gamma(TM)$ then $M$ is said to be totally geodesic submanifold. If $H = 0$, then it is called minimal submanifold.

If $M$ is totally umbilical, then from (2.9), the equations (2.5) and (2.6) reduce to the following equations, respectively;
\[ \tilde{\nabla}_X Y = \nabla_X Y + g(X,Y)H, \] (2.10)
\[ \tilde{\nabla}_X N = -g(H,N)X + \nabla_X N. \] (2.11)

Now, for any $X \in \Gamma(TM)$, we write
\[ \phi X = PX + FX, \] (2.12)
where $PX$ is the tangential component and $FX$ is the normal component of $\phi X$.

Similarly for any $N \in \Gamma(T^\perp M)$, we write
\[ \phi N = BN + CN, \] (2.13)
where $BN$ is the tangential component and $CN$ is the normal component of $\phi N$. The covariant derivatives of the tensor fields $\phi$, $P$ and $F$ are respectively defined as
\[ (\tilde{\nabla}_X \phi)Y = \tilde{\nabla}_X \phi Y - \phi \tilde{\nabla}_X Y, \quad \forall X,Y \in \Gamma(\tilde{M}), \] (2.14)
\[ (\tilde{\nabla}_X P)Y = \nabla_X PY - P \nabla_X Y, \quad \forall X,Y \in \Gamma(TM), \] (2.15)
\[ (\tilde{\nabla}_X F)Y = \nabla_X FY - F \nabla_X Y, \quad \forall X,Y \in \Gamma(TM). \] (2.16)

3. Contact CR-submanifolds

In this section we consider the submanifold $M$ tangent to the structure vector field $\xi$ and defined as follows: A submanifold $M$ tangent to $\xi$ is called a contact CR-submanifold if it admits a pair of differentiable distributions $\mathcal{D}$ and $\mathcal{D}^\perp$ such that $\mathcal{D}$ is invariant and its orthogonal complementary distribution $\mathcal{D}^\perp$ is anti-invariant i.e., $TM = \mathcal{D} \oplus \mathcal{D}^\perp \oplus \langle \xi \rangle$ with $\phi(\mathcal{D}_x) \subseteq \mathcal{D}_x$ and $\phi(\mathcal{D}^\perp_x) \subseteq T^\perp_x M$, for every $x \in M$. Thus, a contact CR-submanifold $M$ tangent to $\xi$ is invariant if $\mathcal{D}^\perp$ is identically zero and an anti-invariant if $\mathcal{D}$ is identically zero, respectively. If neither $\mathcal{D} = \{0\}$ nor $\mathcal{D}^\perp = \{0\}$, then $M$ is proper contact CR-submanifold.

Let $M$ be a proper contact CR-submanifold of an almost contact metric manifold $\tilde{M}$, then for any $X \in \Gamma(TM)$, we have
\[ X = P_1X + P_2X + \eta(X)\xi, \] (3.1)
where $P_1$ and $P_2$ are the orthogonal projections from $TM$ to $D$ and $D^\perp$, respectively. For a contact CR-submanifold, from (2.12) and (3.1), we obtain

$$PX = \phi P_1 X \quad \text{and} \quad FX = \phi P_2 X.$$  

Let $M$ be a contact CR-submanifold of an almost contact metric manifold $\tilde{M}$. Then the normal bundle $T^\perp M$ is decomposed as

$$T^\perp M = \phi D^\perp \oplus \mu,$$

where $\mu$ is the orthogonal complement distribution of $\phi D^\perp$ in $T^\perp M$ and is a $\phi$-invariant subbundle of $T^\perp M$.

Let $M$ be a contact CR-submanifold of a cosymplectic manifold $\tilde{M}$, then for any $Z, W \in \Gamma(D^\perp)$ and $U \in \Gamma(TM)$, we have

$$g(A_{\phi W} Z, U) = g(h(Z, U), \phi W).$$

Using (2.5), we obtain

$$g(A_{\phi W} Z, U) = g(\tilde{\nabla}_U Z, \phi W) = -g(\phi \tilde{\nabla}_U Z, W).$$

By the structure equation (2.3), we get

$$g(A_{\phi W} Z, U) = -g(\tilde{\nabla}_U \phi Z, W).$$

Thus, from (2.6), we derive

$$g(A_{\phi W} Z, U) = g(A_{\phi Z} W, U) = g(h(W, U), \phi Z).$$

Again Using (2.5), we obtain

$$g(A_{\phi W} Z, U) = g(\tilde{\nabla}_W U, \phi Z) = -g(U, \tilde{\nabla}_W \phi Z).$$

Then from (2.6), we get

$$g(A_{\phi W} Z, U) = g(A_{\phi Z} W, U).$$

Hence, for a contact CR-submanifold of a cosymplectic manifold we conclude that

$$A_{\phi W} Z = A_{\phi Z} W \quad \forall Z, W \in \Gamma(D^\perp).$$

(3.3)

Now, for any $X \in \Gamma(D \oplus \langle \xi \rangle)$ and $Z, W \in \Gamma(D^\perp)$, we have

$$g([Z, W], \phi X) = g(\tilde{\nabla}_Z W - \tilde{\nabla}_W Z, \phi X)$$

$$= g(\phi \tilde{\nabla}_W Z - \phi \tilde{\nabla}_Z W, X).$$

Thus, from (2.14) and (2.3), we obtain

$$g([Z, W], \phi X) = g(\tilde{\nabla}_W \phi Z - \tilde{\nabla}_Z \phi W, X)$$

$$= g(A_{\phi W} Z - A_{\phi Z} W, X).$$

Thus, from (3.3), we obtain $g([Z, W], \phi X) = 0$. This means that $[Z, W] \in \Gamma(D^\perp)$, for any $Z, W \in \Gamma(D^\perp)$, that is, $D^\perp$ is integrable. Now for any $X, Y \in \Gamma(D \oplus \langle \xi \rangle)$, we have

$$h(X, PY) + \nabla_X PY = \tilde{\nabla}_X PY = \tilde{\nabla}_X \phi Y.$$ 

As $\tilde{M}$ is cosymplectic, then by (2.14) and the structure equation (2.3), we obtain

$$h(X, PY) + \nabla_X PY = \phi \tilde{\nabla}_X Y.$$ 

Using (2.5), (2.12) and (2.13), we derive

$$h(X, PY) + \nabla_X PY = P\nabla_X Y + F\nabla_X Y + Bh(X, Y) + Ch(X, Y).$$

Equating the normal components, we get

$$F\nabla_X Y = h(X, PY) - Ch(X, Y).$$

(3.4)
Similarly,
\[ F \nabla_Y X = h(Y, PX) - Ch(X, Y). \]  
(3.5)

Thus from (3.4) and (3.5), we obtain
\[ F[X, Y] = h(X, PY) - h(Y, PX). \]  
(3.6)

Hence, we conclude that \( F[X, Y] = 0 \) if and only if \( h(X, PY) = h(Y, PX) \), that is the distribution \( D \oplus (\xi) \) is integrable if and only if \( h(X, PY) = h(Y, PX) \), for all \( X, Y \in \Gamma(D \oplus (\xi)) \).

We give the following main result of this section.

3.1. Theorem \( \) Let \( M \) be a totally umbilical contact CR-submanifold of a cosymplectic manifold \( \tilde{M} \). Then at least one of the following statements is true

(i) \( M \) is totally geodesic,
(ii) the anti-invariant distribution \( D^\perp \) is one-dimensional, i.e., \( \text{dim} \ D^\perp = 1 \),
(iii) the mean curvature vector \( H \in \Gamma(\mu) \).

Proof. For a cosymplectic manifold, we have
\[ \nabla_Z \phi W = \phi \nabla_Z W, \]
for any \( Z, W \in \Gamma(D^\perp) \). Using (2.10) and (2.11), we derive
\[ -g(H, \phi W)Z + \nabla_Z \phi W = \phi \nabla_Z W + g(Z, W)\phi H. \]  
(3.7)

Taking the product with \( Z \in \Gamma(D^\perp) \) in (3.7), we get
\[ g(H, \phi W)||Z||^2 = g(Z, W)g(H, \phi Z). \]  
(3.8)

Interchanging \( Z \) and \( W \) in (3.8), we obtain
\[ g(H, \phi Z)||W||^2 = g(Z, W)g(H, \phi W). \]  
(3.9)

Thus, from (3.8) and (3.9), we deduce that
\[ g(H, \phi Z) = \frac{g(Z, W)^2}{||Z||^2||W||^2} g(H, \phi Z). \]

That is
\[ g(H, \phi Z)\{1 - \frac{g(Z, W)^2}{||Z||^2||W||^2}\} = 0. \]  
(3.10)

Hence, the equation (3.10) has a solution if at least one of the followings holds

(i) \( H = 0 \) or (ii) \( Z \parallel W \) or (iii) \( H \perp \phi D^\perp \).

That is either \( M \) is totally geodesic or as \( Z \) and \( W \) are parallel to each other for any \( Z, W \in \Gamma(D^\perp) \) that is these two vectors are linearly dependent and hence \( \text{dim} \ D^\perp = 1 \) or \( H \in \Gamma(\mu) \), this proves the theorem completely. \( \square \)

3.2. Example \( \) Consider a flat manifold of real dimension 6 which have a complex Kaehler structure of dimension 3, that is \( (\mathbb{C}^3, J, h) \) be a Kaehler manifold with complex structure \( J \) and Euclidean Hermitian metric \( h \). Then \( \tilde{M} = \mathbb{C}^3 \times \mathbb{R} \) is a cosymplectic manifold with the structure vector field \( \xi = \frac{\partial}{\partial t} \) dual 1-form \( \eta = dt \) and the metric \( g = h + dt^2 \). Now, consider \( M = \mathbb{R}^3 \times S^1 \), where \( S^1 \) is a unit circle being taken as totally real submanifold of \( \mathbb{C}^3 \). Then \( M \) is a contact CR-submanifold of \( \tilde{M} \) with the invariant distribution \( D = \mathbb{R}^2 \), anti-invariant distribution \( D^\perp = \Gamma(S^1) \) and the 1-dimensional distribution \( (\xi) = \mathbb{R} \).

Acknowledgement. The authors are thankful to the referee for providing constructive comments and valuable suggestions.
References