ORIENTABLE SMALL COVERS OVER THE PRODUCT OF 2-CUBE WITH $n$-GON

Yanchang Chen* and Yanying Wang†

Abstract
We calculate the number of D-J equivalence classes and equivariant homeomorphism classes of all orientable small covers over the product of 2-cube with $n$-gon.

Keywords: Small cover; D-J equivalence; Equivariant homeomorphism

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1. Introduction

As defined by Davis and Januszkiewicz [5], a small cover is a smooth closed manifold $M^n$ with a locally standard $(\mathbb{Z}_2)^n$–action such that its orbit space is a simple convex polytope. For instance, the real projective space $\mathbb{R}P^n$ with a natural $(\mathbb{Z}_2)^n$–action is a small cover over an $n$-simplex. This gives a direct connection between equivariant topology and combinatorics, making research on the topology of small covers possible through the combinatorial structure of quotient spaces.

Lü and Masuda [7] showed that the equivariant homeomorphism class of a small cover over a simple convex polytope $P^n$ agrees with the equivalence class of its corresponding $(\mathbb{Z}_2)^n$–coloring under the action of the automorphism group of the face poset of $P^n$. This finding also holds true for orientable small covers by the orientability condition in [8] (see Theorem 2.5). However, general formulas for calculating the number of equivariant homeomorphism classes of (orientable) small covers over an arbitrary simple convex polytope do not exist.

In recent years, several studies have attempted to enumerate the number of equivalence classes of all small covers over a specific polytope. Garrison and Scott [6] used a computer program to calculate the number of homeomorphism classes of all small covers over a dodecahedron. Cai, Chen and Lü [2] calculated the

*College of Mathematics and Information Science, Henan Normal University, Xinxiang 453007, Henan, P. R. China Email: cyc810707@163.com
†College of Mathematics and Information Science, Hebei Normal University, Shijiazhuang 050016, P. R. China, Email: wyanying2003@yahoo.com.cn
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number of equivariant homeomorphism classes of small covers over prisms (an n-sided prism is the product of 1-cube and n-gon). Choi [3] determined the number of equivariant homeomorphism classes of small covers over cubes. However, little is known about orientable small covers. Choi [4] calculated the number of D-J equivalence classes of orientable small covers over cubes. This paper aims to determine the number of D-J equivalence classes and equivariant homeomorphism classes of all orientable small covers over $I^2 \times P_n$ (see Theorem 3.1 and Theorem 4.1), where $I^2$ and $P_n$ denote 2-cube and n-gon, respectively.

The paper is organized as follows. In Section 2, we review the basic theory on orientable small covers and calculate the automorphism group of the face poset of $I^2 \times P_n$. In Section 3, we determine the number of D-J equivalence classes of orientable small covers over $I^2 \times P_n$. In Section 4, we obtain a formula for the number of equivariant homeomorphism classes of orientable small covers over $I^2 \times P_n$.

2. Preliminaries

A convex polytope $P^n$ of dimension $n$ is simple if every vertex of $P^n$ is the intersection of $n$ facets (i.e., faces of dimension $(n-1)$) [9]. An $n$-dimensional smooth closed manifold $M^n$ is a small cover if it admits a smooth $\mathbb{Z}_2^n$-action such that the action is locally isomorphic to a standard action of $\mathbb{Z}_2^n$ on $\mathbb{R}^n$ and the orbit space $M^n/\mathbb{Z}_2^n$ is a simple convex polytope of dimension $n$.

Let $P^n$ be a simple convex polytope of dimension $n$ and $\mathcal{F}(P^n) = \{F_1, \ldots, F_\ell\}$ be the set of facets of $P^n$. Assuming that $\pi : M^n \rightarrow P^n$ is a small cover over $P^n$, then there are $\ell$ connected submanifolds $\pi^{-1}(F_1), \ldots, \pi^{-1}(F_\ell)$. Each submanifold $\pi^{-1}(F_i)$ is fixed pointwise by a $\mathbb{Z}_2$-subgroup $\mathbb{Z}_2(F_i)$ of $\mathbb{Z}_2^n$. Obviously, the $\mathbb{Z}_2$-subgroup $\mathbb{Z}_2(F_i)$ agrees with an element $\nu_i$ in $\mathbb{Z}_2^n$ as a vector space. For each face $F$ of codimension $u$, given that $P^n$ is simple, there are $u$ facets $F_{i_1}, \ldots, F_{i_u}$ such that $F = F_{i_1} \cap \cdots \cap F_{i_u}$. Then, the corresponding submanifolds $\pi^{-1}(F_{i_1}), \ldots, \pi^{-1}(F_{i_u})$ intersect transversally in the $(n-u)$-dimensional submanifold $\pi^{-1}(F)$, and the isotropy subgroup $\mathbb{Z}_2(F)$ of $\pi^{-1}(F)$ is a subtorus of rank $u$ generated by $\mathbb{Z}_2(F_{i_1}), \ldots, \mathbb{Z}_2(F_{i_u})$ (or is determined by $\nu_{i_1}, \ldots, \nu_{i_u}$ in $\mathbb{Z}_2^n$). This gives a characteristic function $\lambda : \mathcal{F}(P^n) \rightarrow (\mathbb{Z}_2)^n$ which is defined by $\lambda(F_i) = \nu_i$ such that whenever the intersection $F_{i_1} \cap \cdots \cap F_{i_u}$ is non-empty, $\lambda(F_{i_1}), \ldots, \lambda(F_{i_u})$ are linearly independent in $(\mathbb{Z}_2)^n$. Assuming that each nonzero vector of $(\mathbb{Z}_2)^n$ is a color, then the characteristic function $\lambda$ means that each facet is colored. Hence, we also call $\lambda$ a $(\mathbb{Z}_2)^n$-coloring on $P^n$.

In fact, Davis and Januszkiewicz gave a reconstruction process of a small cover by using a $(\mathbb{Z}_2)^n$-coloring $\lambda : \mathcal{F}(P^n) \rightarrow (\mathbb{Z}_2)^n$. Let $\mathbb{Z}_2(F_i)$ be the subgroup of $(\mathbb{Z}_2)^n$ generated by $\lambda(F_i)$. Given a point $p \in P^n$, we denote the minimal face containing $p$ in its relative interior by $F(p)$. Assuming that $F(p) = F_{i_1} \cap \cdots \cap F_{i_u}$ and $\mathbb{Z}_2(F(p)) = \bigoplus_{i=1}^n \mathbb{Z}_2(F_{i_i})$, then $\mathbb{Z}_2(F(p))$ is a $u$-dimensional subgroup of $(\mathbb{Z}_2)^n$. Let $M(\lambda)$ denote $P^n \times (\mathbb{Z}_2)^n / \sim$, where $(p,g) \sim (q,h)$ if $p = q$ and $g^{-1}h \in \mathbb{Z}_2(F(p))$. The free action of $(\mathbb{Z}_2)^n$ on $P^n \times (\mathbb{Z}_2)^n$ descends to an action on $M(\lambda)$ with quotient $P^n$. Thus, $M(\lambda)$ is a small cover over $P^n$ [5].
Two small covers $M_1$ and $M_2$ over $P^n$ are called weakly equivariantly homeomorphic if there is an automorphism $\varphi : (\mathbb{Z}_2)^n \to (\mathbb{Z}_2)^n$ and a homeomorphism $f : M_1 \to M_2$ such that $f(t \cdot x) = \varphi(t) \cdot f(x)$ for every $t \in (\mathbb{Z}_2)^n$ and $x \in M_1$. If $\varphi$ is an identity, then $M_1$ and $M_2$ are equivariantly homeomorphic. Following [5], two small covers $M_1$ and $M_2$ over $P^n$ are called Davis-Januszkiewicz equivalent (or simply, D-J equivalent) if there is a weakly equivariant homeomorphism $f : M_1 \to M_2$ covering the identity on $P^n$.

By $\Lambda(P^n)$, we denote the set of all $(\mathbb{Z}_2)^n$-colorings on $P^n$. We have

2.1. Theorem. ([5]) All small covers over $P^n$ are given by $\{M(\lambda) | \lambda \in \Lambda(P^n)\}$, i.e., for each small cover $M^n$ over $P^n$, there is a $(\mathbb{Z}_2)^n$-coloring $\lambda$ with an equivariant homeomorphism $\lambda : M^n \to P^n$ covering the identity on $P^n$.

Nakayama and Nishimura [8] found an orientability condition for a small cover.

2.2. Theorem. For a basis $\{e_1, \ldots, e_n\}$ of $(\mathbb{Z}_2)^n$, a homomorphism $\varepsilon : (\mathbb{Z}_2)^n \to \mathbb{Z}_2 = \{0, 1\}$ is defined by $\varepsilon(e_i) = 1$ for $i = 1, \ldots, n$. A small cover $M(\lambda)$ over a simple convex polytope $P^n$ is orientable if and only if there exists a basis $\{e_1, \ldots, e_n\}$ of $(\mathbb{Z}_2)^n$ such that the image of $\varepsilon\lambda$ is $\{1\}$.

A $(\mathbb{Z}_2)^n$-coloring that satisfies the orientability condition in Theorem 2.2 is an orientable coloring of $P^n$. We know that there exists an orientable small cover over every simple convex 3-polytope [8]. Similarly, we know the existence of orientable small covers over $I^2 \times P_n$ by the existence of orientable colorings and determine the number of D-J equivalence classes and equivariant homeomorphism classes.

By $O(P^n)$, we denote the set of all orientable colorings on $P^n$. There is a natural action of $GL(n, \mathbb{Z}_2)$ on $O(P^n)$ defined by the correspondence $\lambda \mapsto \sigma \cdot \lambda$, and the action on $O(P^n)$ is free. We assume that $F_1, \ldots, F_n$ of $\mathcal{F}(P^n)$ meet at one vertex $p$ of $P^n$. Let $e_1, \ldots, e_n$ be the standard basis of $(\mathbb{Z}_2)^n$, and $B(P^n) = \{\lambda \in O(P^n) | \lambda(F_i) = e_i, i = 1, \ldots, n\}$. Then $B(P^n)$ is the orbit space of $O(P^n)$ under the action of $GL(n, \mathbb{Z}_2)$.

2.3. Remark. We have $B(P^n) = \{\lambda \in O(P^n) | \lambda(F_i) = e_i, i = 1, \ldots, n\}$ and for $n + 1 \leq j \leq \ell$, $\lambda(F_j) = e_{j_1} + e_{j_2} + \cdots + e_{j_{h_j} + 1}$, $1 \leq j_1 < j_2 < \cdots < j_{h_j} + 1 \leq n$. Below, we show that $\lambda(F_j) = e_{j_1} + e_{j_2} + \cdots + e_{j_{h_j} + 1}$ for $n + 1 \leq j \leq \ell$. If $\lambda \in O(P^n)$, there exists a basis $\{e_1', \ldots, e_n'\}$ of $(\mathbb{Z}_2)^n$ such that for $1 \leq i \leq \ell$, $\lambda(F_i) = e_{i_1'} + e_{i_2'} + \cdots + e_{i_{2i_{h_{i'}}} + 1'}$, $1 \leq i_1 < \cdots < i_{2i_{h_{i'}} + 1} \leq n$. Given that $\lambda(F_i) = e_i$ for $i = 1, \ldots, n$, then $e_i = e_{i_1'} + \cdots + e_{i_{2i_{h_{i'}}} + 1'}$. Thus, for $n + 1 \leq j \leq \ell$, $\lambda(F_j)$ is not of the form $e_{j_1} + \cdots + e_{j_{k}}$, $1 \leq j_1 < \cdots < j_{k} \leq n$.

Given that $B(P^n)$ is the orbit space of $O(P^n)$, then we have

2.4. Lemma. $|O(P^n)| = |B(P^n)| \times |GL(n, \mathbb{Z}_2)|$.

Note that $|GL(n, \mathbb{Z}_2)| = \prod_{k=1}^{n} (2^n - 2^{k-1})$ [1]. Two orientable small covers $M(\lambda_1)$ and $M(\lambda_2)$ over $P^n$ are D-J equivalent if and only if there is $\sigma \in GL(n, \mathbb{Z}_2)$ such that $\lambda_1 = \sigma \circ \lambda_2$. Thus the number of D-J equivalence classes of orientable small covers over $P^n$ is $|B(P^n)|$.

Let $P^n$ be a simple convex polytope of dimension $n$. All faces of $P^n$ form a poset (i.e., a partially ordered set by inclusion). An automorphism of $\mathcal{F}(P^n)$ is a
2.5. Theorem. Two orientable small covers over an n-dimensional simple convex polytope $P^n$ are equivariantly homeomorphic if and only if there is $h \in \text{Aut}(\mathcal{F}(P^n))$ such that $\lambda_1 = \lambda_2 \circ h$, where $\lambda_1$ and $\lambda_2$ are their corresponding orientable colorings on $P^n$.

Proof. Theorem 2.5 is proven true by combining Lemma 5.4 in [7] with Theorem 2.2. □

According to Theorem 2.5, the number of orbits of $O(P^n)$ under the action of $\text{Aut}(\mathcal{F}(P^n))$ is the number of equivariant homeomorphism classes of orientable small covers over $P^n$. Thus, we count the number of orbits. Burnside Lemma is very useful in enumerating the number of orbits.

Burnside Lemma Let $G$ be a finite group acting on a set $X$. Then the number of orbits $X$ under the action of $G$ equals $\frac{1}{|G|} \sum_{g \in G} |X_g|$, where $X_g = \{x \in X | gx = x\}$.

Burnside Lemma suggests that, to determine the number of the orbits of $O(P^n)$ under the action of $\text{Aut}(\mathcal{F}(P^n))$, the structure of $\text{Aut}(\mathcal{F}(P^n))$ should first be understood. We shall particularly be concerned when the simple convex polytope is $I^2 \times P_n$.

For convenience, we introduce the following marks. By $F'_1, F'_2, F'_3$, and $F'_4$ we denote four edges of the 2-cube $I^2$ in their general order (here $I^2$ is considered as a 4-gon). Similarly, by $F''_5, F''_6, \ldots, F''_{n+4}$, we denote all edges of $n$-gon $P_n$ in their general order. Set $\mathcal{F}' = \{F_i = F'_i \times P_n | 1 \leq i \leq 4\}$, and $\mathcal{F}'' = \{F_i = I^2 \times F''_i | 5 \leq i \leq n+4\}$. Then $\mathcal{F}(I^2 \times P_n) = \mathcal{F}' \cup \mathcal{F}''$.

Next, we determine the automorphism group of face poset of $I^2 \times P_n$.

2.6. Lemma. When $n=4$, the automorphism group $\text{Aut}(\mathcal{F}(I^2 \times P_n))$ is isomorphic to $(\mathbb{Z}_2)^4 \times S_4$, where $S_4$ is the symmetric group on four symbols. When $n \neq 4$, $\text{Aut}(\mathcal{F}(I^2 \times P_n))$ is isomorphic to $D_4 \times D_n$, where $D_n$ is the dihedral group of order $2n$.

Proof. When $n=4$, $I^2 \times P_4$ is a 4-cube $I^4$. Obviously, the automorphism group $\text{Aut}(\mathcal{F}(I^4))$ contains a symmetric group $S_4$ because there is exactly one automorphism for each permutation of the four pairs of opposite sides of $I^4$. All elements of $\text{Aut}(\mathcal{F}(I^4))$ can be written in a simple form as $\chi_1 \chi_2 \chi_3 \chi_4 \cdot u$, where $e_1, e_2, e_3, e_4 \in \mathbb{Z}_2$, with reflections $\chi_1, \chi_2, \chi_3, \chi_4$ and $u \in S_4$. Thus, the automorphism group $\text{Aut}(\mathcal{F}(I^4))$ is isomorphic to $(\mathbb{Z}_2)^4 \times S_4$.

When $n \neq 4$, the facets of $\mathcal{F}'$ and $\mathcal{F}''$ are mapped to $\mathcal{F}'$ and $\mathcal{F}''$, respectively, under the automorphisms of $\text{Aut}(\mathcal{F}(I^2 \times P_n))$. Given that the automorphism group $\text{Aut}(\mathcal{F}(I^2))$ is isomorphic to $D_4$ and $\text{Aut}(\mathcal{F}(P_n))$ is isomorphic to $D_n$, $\text{Aut}(\mathcal{F}(I^2 \times P_n))$ is isomorphic to $D_4 \times D_n$. □

2.7. Remark. Let $x, y, x', y'$ be the four automorphisms of $\text{Aut}(\mathcal{F}(I^2 \times P_n))$ with the following properties:
(a) $x(F_i) = F_{i+1}(1 \leq i \leq 3), x(F_4) = F_1, x(F_j) = F_j, 5 \leq j \leq n + 4;$
(b) $y(F_i) = F_{5-i}(1 \leq i \leq 4), y(F_j) = F_j, 5 \leq j \leq n + 4;$
(c) $x'(F_i) = F_i(1 \leq i \leq 4), x'(F_j) = F_{j+1}(5 \leq j \leq n + 3), x'(F_{n+4}) = F_5;$
(d) $y'(F_i) = F_i(1 \leq i \leq 4), y'(F_j) = F_{n+9-j}, 5 \leq j \leq n + 4.$

Then, when $n \neq 4$, all automorphisms of $\text{Aut}(\mathcal{F}(I^2 \times P_n))$ can be written in a simple form as follows:

\begin{equation}
 x^u y^v x'^{u'} y'^{v'}, \; u \in \mathbb{Z}_4, u' \in \mathbb{Z}_n, v, v' \in \mathbb{Z}_2
\end{equation}

with $x^4 = y^2 = x'^n = y'^2 = 1, x^u y = y x^4 - u$, and $x'^{u'} y' = y' x'^{n-u'}$.

**References**