FUNDAMENTAL GROUPS OF QUASI GRAPHS OF GROUPS

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Abstract
A graph is called a quasi graph if the possibility of an edge of the graph being equal to its inverse is not excluded. Quasi HNN groups are a new generalizations of HNN groups. In this paper we introduce the concepts of a quasi graph of groups and its fundamental group, and show that the fundamental group of a quasi graph of groups is a quasi HNN group. The embedding theorem for the fundamental group of a quasi graph of groups is formulated and proved. Furthermore, we find the structures of groups induced by the vertices of a quasi graph of groups.

Keywords: Quasi graphs, Quasi graphs of groups, Quasi HNN groups, Fundamental groups of quasi graphs of groups.

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1. Introduction
In [6, p. 37], Serre and Bass introduced the concepts of graphs of groups and their fundamental groups under the condition that the edge of a graph does not equal its inverse. Then they showed that the fundamental group of a graph of groups is an HNN group of base the tree product of all of the vertex groups (see [2, p. 139]). For further information on fundamental groups of graphs of groups we refer the readers to H. Bass [1, p. 6], G. Baumslag [2, p. 131] or D. E. Cohen [3, p. 198]. In this paper we generalize the above concepts in the case where an edge of the graph can be equal to its inverse.

This paper is divided into 5 sections. In section 2, we introduce the concepts of quasi graphs of groups and formulate the fundamental groups of the quasi graph of groups. In section 3, we form and prove the embedding theorem for fundamental groups of quasi graphs of groups. In section 4, we form the structures of the fundamental groups of quasi graphs of groups relative to maximal subtrees of the graphs. In section 5, we find the structures of groups induced by the vertices of quasi graphs of groups.

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2. Quasi graph of groups

A quasi graph $X$ consists of two disjoint sets, $V(X)$ (the set of vertices of $X$) and $E(X)$ (the set of edges of $X$), with $V(X)$ non-empty, together with three functions $\partial_0 : E(X) \to V(X)$, $\partial_1 : E(X) \to V(X)$, and $\eta : E(X) \to E(X)$ satisfying the conditions that $\partial_0 \eta = \partial_1$, $\partial_1 \eta = \partial_0$, and $\eta$ is an involution fixing some elements of $E(X)$.

For simplicity, if $e \in E(X)$ we write $\partial_0 (e) = o(e)$, $\partial_1 (e) = t(e)$, and $\eta (e) = \bar{e}$. This implies that $o(\bar{e}) = t(e)$, $t(\bar{e}) = o(e)$ and $\bar{e} = e$. For a graph $X$, let

$$E_1 (X) = \{ e \in E(X) : e \neq \bar{e} \} \text{ and } E_2 (X) = \{ f \in E(X) : f = \bar{f} \}.$$ 

It is clear that $E_1 (X) \cap E_2 (X) = \emptyset$, and $E_1 (X) \cup E_2 (X) = E(X)$. We note that this concept of quasi graph is more general than given in [6, p. 13] if $E_2 (X) \neq \emptyset$.

An orientation of a graph $X$ is a set of edges of $X$ consisting of exactly one member of each of $\{ e, \bar{e} \}$ for which $e \neq \bar{e}$, together every edge of $E_2 (X)$. There are obvious definitions of subgraph, path, and tree. For more details we refer the readers to [1, 2, 3] or [6].

The following definition of a quasi graph of groups is more general that given in [6, p. 37].

2.1. Definition. A quasi graph of groups is a pair $\Phi = (Z, \Gamma)$ satisfying the following conditions:

1. $Z$ is a connected quasi graph;
2. $\Gamma$ is a mapping from $Z$ into the class of all groups;
3. The image of $x \in Z$ under $\Gamma$ is denoted by $\Gamma_x$;
4. $\Gamma_e = \Gamma_x$ for every $e \in E(Z)$;
5. Each $e \in E(Z)$ is equipped with a monomorphism $\lambda_e : \Gamma_e \to \Gamma_{1(e)}$;
6. If $e \in E_2 (Z)$, i.e. $\bar{e} = e$, then there is an automorphism $\mu_e : \Gamma_e \to \Gamma_e$ such that $\mu_e^2 : \Gamma_e \to \Gamma_e$ is an inner automorphism determined by $a_e \in \Gamma_e$ and $a_e$ is fixed by $\mu_e$. That is, $\mu_e (a_e) = a_e$, and $\mu_e^2 (a) = a_e a e^{-1}$ for every $a \in \Gamma_e$.

2.2. Notation. For simplicity we adopt the following notation.

(i) For each $e \in E(Z)$, let $\lambda_e (\Gamma_e) = \Gamma^e$;
(ii) For each $a \in \Gamma_e$, let $\lambda_e (a) = a^e$.

It is clear that $\Gamma^e \leq \Gamma_{1(e)}$ and $\Gamma^e \leq \Gamma_{\omega(e)}$.

2.3. Proposition. Let $\Phi (Z, \Gamma)$ be a quasi graph of groups, and $e$ an edge of $Z$. Then.

(i) The mapping $\phi_e : \Gamma^e \to \Gamma^e$ given by $\phi_e (a^e) = a^e$, $a \in \Gamma_e$ is an isomorphism and $\phi_e \phi_e^{-1}$ if $e \in E_1 (Z)$.
(ii) if $e \in E_2 (Z)$, then the mapping $\alpha_e : \Gamma^e \to \Gamma^e$ given by $\alpha_e (a^e) = (\mu_e (a))^e$ is an automorphism, $\alpha_e^2$ is an inner automorphism determined by $a_e^e$, and $a_e^e$ is fixed by $\alpha_e$.

Proof. (i) Clear.

(ii) It is clear that $\alpha_e$ is an automorphism.

We have $\alpha_e (a^e) = (\mu_e (a_e))^e = a_e^e$, because $\mu_e (a_e) = a_e$. This implies that $a_e^e$ is fixed by $\alpha_e$. Let $a \in \Gamma_e$. Then:
\[\alpha^2_e(a^e) = \alpha_e(\alpha_e(a^e)) \]
\[= \alpha_e((\mu_e(a))^e) \]
\[= (\mu_e(\mu_e(a))^e \]
\[= (\mu_e^2(a))^e, \text{ because } \mu_e^2 \text{ is an inner automorphism determined by } a_e \in \Gamma_e \]
\[= (aa^{-1}_e)^e \]
\[= a_ee^a(a_e)^{-1}. \]

This implies that \(\alpha^2_e\) is an inner automorphism determined by \(a_e^e\).

This completes the proof. \(\square\)

### 3. The fundamental group of a quasi graph of groups

In this section we generalise the concept of the fundamental group of a graph of groups from the case where an edge of the graph cannot equal its inverse [6, p. 41] to the case where an edge of the graph can equal its inverse.

First we introduce the following notation needed to define the concept of the fundamental group of a quasi graph of groups \(\Phi(Z, \Gamma)\).

1. For any vertex \(v \in Z\), let \(\langle \text{gen } \Gamma_v \mid \text{rel } \Gamma_v \rangle\) stand for any presentation of \(\Gamma_v\), where \(\text{gen } \Gamma_v\) is the set of generating symbols and \(\text{rel } \Gamma_v\) is the set of relations of the presentation of \(\Gamma_v\), such that \(\text{gen } \Gamma_v \cap \text{gen } \Gamma_v = \emptyset\) for \(u \neq v, u, v \in V(Z)\).

If \(v \in V(Z)\) and \(g \in \Gamma_v\), let \(w(g)\) stand for a word of the generating symbols \(\text{gen } \Gamma_v\) of \(\Gamma_v\) of value \(g\).

2. For any edge \(e \in Z\), let \(e \cdot \Gamma^e \cdot e^{-1} = \Gamma^e\) stand for the set of relations \(ew(g)e^{-1} = w(\phi_e(g))\), \((w_\alpha e g)\) if \(e \in E_2(Z)\), \(g \in \Gamma^e\), where \(w(g)\) and \(w(\phi_e(g))\), \((w_\alpha e g)\) if \(e \in E_2(Z)\) \(\) are words in the sets of generating symbols of \(\Gamma_{i(e)}, \Gamma_{o(e)}\) of value \(g\) and \(\phi_e(g)\), \((w_\alpha e g)\) if \(e \in E_2(Z)\), respectively.

3. For any edge \(e \in Z\), let \(\Gamma^e = \Gamma^e\) be the relations \(w(g) = w(\phi_e(g))\), \((w_\alpha e g)\) if \(e \in E_2(Z)\), defined similarly as in (2).

4. For each edge \(e \in E(Z)\), define \(\delta_e = \begin{cases} 1 & \text{if } e \in E_1(Z) \\ a_ee^a & \text{if } e \in E_2(Z) \end{cases}\). It is clear that \(\delta_e \in \Gamma^e\), \(\delta_e = \delta_e\), and, if \(e \in E_2(Z)\), then \(\alpha_e(\delta_e) = \delta_e\), and \(\alpha^2_e(a) = \delta_a a \delta_a^{-1}\) for every \(a \in \Gamma^e\).

5. For any edge \(e \in Z\), let \(ee = \delta_e\) stand for the relation \(ee = w(\delta_e)\), where \(w(\delta_e)\) is a word in the sets of generating symbols of \(\Gamma_{i(e)}\) of value \(\delta_e\).

#### 3.1. Definition. The fundamental group of a quasi graph of groups \(\Phi(Z, \Gamma)\) is denoted by \(\pi\Phi(Z, \Gamma)\), and defined to be the group with presentation

\[\pi\Phi(Z, \Gamma) = \langle \text{gen } \Gamma_v, e \mid \text{rel } \Gamma_v, e \cdot \Gamma^e \cdot e^{-1} = \Gamma^e, ee = \delta_e \rangle, \]

where \(v \in V(Z)\) and \(e \in E(Z)\).

#### 3.2. Proposition. Let \(\Phi(Z, \Gamma)\) be a quasi graph of groups, and \(A\) an orientation of \(Z\). Let \(\pi\Phi(Z, A, \Gamma)\) be the group with presentation

\[\pi\Phi(Z, A, \Gamma) = \langle \text{gen } \Gamma_v, e, f \mid \text{rel } \Gamma_v, e \cdot \Gamma^e \cdot e^{-1} = \Gamma^e, f \cdot \Gamma^f \cdot f^{-1} = \Gamma^f, f^2 = \delta_f \rangle, \]

where, \(v \in V(Z)\), \(e\) and \(f\) are edges of \(A\) such that \(e \in E_1(Z)\) and \(f \in E_2(Z)\). Then \(\pi\Phi(Z, \Gamma)\) and \(\pi\Phi(Z, A, \Gamma)\) are isomorphic.
Proof. We use the Tietze transformations $T_1$–$T_4$ of [5, p. 50].

Case 1 Let $e \in E_1(Z)$. It is clear that the relation $\bar{e} = e^{-1}$ is a consequence of the relation $e \bar{e} = 1$. Then by $T_1$ we add $\bar{e} = e^{-1}$ to the relations of $\pi \Phi(Z, \Gamma)$, and by $T_2$ we delete $e \bar{e} = 1$ from the relations of $\pi \Phi(Z, \Gamma)$. Since $e$ is in $A$, $e \neq \bar{e}$, therefore $\bar{e}$ is not in $A$. Then by $T_4$ we delete $e$ from the set of generating symbols of $\pi \Phi(Z, \Gamma)$ and substitute $\bar{e}$ by $e^{-1}$ in the relations $\bar{e}(g)\bar{e}^{-1} = w(\phi_e(g))$, $g \in \Gamma^e$, (or $\bar{e} \cdot \Gamma^e \cdot \bar{e}^{-1} = \Gamma^e$). We get the relation $e^{-1}w(g)e = w(\phi_e^{-1}(g))$. This relation is a consequence of the relation $\bar{e}(g)\bar{e}^{-1} = w(\phi_e^{-1}(g))$ because $\phi_e = \phi_e^{-1}$.

Similar to the above, we delete the relation $e^{-1}w(g)e = w(\phi_e^{-1}(g))$. Then we obtain the group $\pi \Phi(Z, A, \Gamma)$.

Case 2. Let $f \in E_2(Z)$. This case is similar to Case 1. This completes the proof. □

The following corollary shows that $\pi \Phi(Z, \Gamma)$ is independent of the choice of orientation.

3.3. Corollary. Let $\Phi(Z, \Gamma)$ be a quasi graph of groups, and let $A_1$, $A_2$ be two orientations of $Z$. Then $\pi \Phi(Z, A_1, \Gamma)$ and $\pi \Phi(Z, A_2, \Gamma)$ are isomorphic. □

3.4. Corollary. Let $\Phi(Z, \Gamma)$ be a quasi graph of groups such that $Z$ is a tree. Then $\pi \Phi(Z, \Gamma)$ has the presentation

$$\langle \text{gen } (\Gamma_v) | \text{rel } (\Gamma_v), \Gamma^m = \Gamma^n \rangle,$$

where $v \in V(Z)$ and $m \in E(T)$. □

3.5. Examples. Let $\Phi(Z, \Gamma)$ be a quasi graph of groups. Then.

1. If $\Gamma_v$ is trivial for all $v, v \in V(Z)$, then $\pi \Phi(Z, \Gamma) = F \ast C$, where $F$ is the free group generated by $E_1(Z)$, and $C$ is the free product of the groups of order 2 generated by $f, f \in E_2(Z)$.

2. If $\Gamma^e$ is trivial for any edge $x$ of $Z$, then $\pi \Phi(Z, A, \Gamma) = G \ast F \ast C$, where $G$ is the free product of the groups $G_v, v \in V(Z)$, and $F$ and $C$ are as above.

3. Let $E(Z) = E_1(Z) = \{e\}$. Then.

i. If $o(e) = t(e) = v$, then

$$\pi \Phi(Z, \Gamma) = \langle \text{gen } (\Gamma_v), e | \text{rel } (\Gamma_v), e \cdot \Gamma^e \cdot e^{-1} = \Gamma^e \rangle$$

is an HNN group with base $\Gamma_v$, associated pairs $(\Gamma^e, \Gamma^e)$ of subgroups of $\Gamma_v$, and stable letter $e$.

ii. If $o(e) = u, \ t(e) = v, u \neq v$, then

$$\pi \Phi(Z, \Gamma) = \langle \text{gen } (\Gamma_u), \text{gen } (\Gamma_v), e | \text{rel } (\Gamma_v), \text{rel } (\Gamma_v), \Gamma^e = \Gamma^e, e \cdot \Gamma^e \cdot e^{-1} = \Gamma^e \rangle$$

is an HNN group with base the group with presentation,

$$G = \langle \text{gen } (\Gamma_u), \text{gen } (\Gamma_v), e | \text{rel } (\Gamma_v), \text{rel } (\Gamma_v), \Gamma^e = \Gamma^e \rangle,$$

associated pairs $(\Gamma^e, \Gamma^e)$ of subgroups of $G$, and of stable letter $e$.

Now we formulate and prove the embedding theorem for fundamental groups of quasi graphs of groups.

In [4], Khanfar and Mahmood introduced a new class of groups, called quasi HNN groups, defined as follows:

Let $G$ be a group, $I$ and $J$ two disjoint indexed sets.

Let $\{A_i : i \in I\}$, $\{B_i : i \in I\}$ and $\{C_j : j \in J\}$ be families of subgroups of $G$. For each $i \in I$, let $\phi_i : A_i \rightarrow B_i$ be an onto isomorphism, and for each $j \in J$ let $\alpha_j : C_j \rightarrow C_j$ be an automorphism such that $\alpha_j^2$ is the inner automorphism determined by $c_j \in C_j$ and $c_j$ is fixed by $\alpha_j$. That is, $\alpha_j(c_j) = c_j$ and $\alpha_j^2(c) = c_jcc_j^{-1}$ for all $c \in C_j$. 

The group $G^*$ with presentation

$$\langle \text{gen}(G), t_e, t_v | \text{rel}(G), t_vAt_v^{-1} = B_i, t_jC_jt_j^{-1} = C_j, t_j^2 = c_j, i \in I, j \in J \rangle$$

is called a quasi HNN group of base $G$ and associated pairs $(A_i, B_i)$ and $(C_j, C_j)$, $i \in I$, $j \in J$ of subgroups of $G$, where the generators and relations of $G^*$ are defined similarly as in the group $\pi \Phi(Z, \Gamma)$.

The embedding theorem for quasi HNN groups can be stated as follows. For the proof see [4, Theorem 1, p. 18].

3.6. Lemma. $G$ is embedded in $G^*$. \hfill $\Box$

3.7. Proposition. $\pi \Phi(Z, \Gamma)$ is a quasi HNN group having as base a free product of groups, and for every vertex $v$ of $Z$, $\Gamma_v$ is embedded in $\pi \Phi(Z, \Gamma)$.

Proof. Let $F$ be the free product of the groups $\langle \text{gen}(\Gamma_v) | \text{rel}(\Gamma_v) \rangle$, $v \in V(Z)$. Then $\pi \Phi(Z, \Gamma)$ is the quasi HNN group with base $F$, associated pairs $(\Gamma^e, \Gamma^o)$ and $(\Gamma^f, \Gamma^f)$ of subgroups of $F$, where the index sets are $I = E_1(Z)$, $J = E_2(Z)$, and the isomorphism $\phi_e : \Gamma^e \rightarrow \Gamma^o$ and the automorphism $\alpha_e : \Gamma^o \rightarrow \Gamma^e$ are those of Proposition 2.3.

Then the embedding theorem for quasi HNN groups implies that for every vertex $v$ of $Z$, $\Gamma_v$ is embedded in $\pi \Phi(Z, \Gamma)$. This completes the proof. \hfill $\Box$

3.8. Proposition. For each edge $e$ of $E(Z)$, each vertex $v$ of $V(Z)$, and each $g \in \Gamma_v$, let $t_e$ be the value of $e$ in the group $\pi \Phi(Z, \Gamma)$. Then:

1. $\pi \Phi(Z, \Gamma)$ is generated by the elements $t_e$ and $g$.
2. $t_e \notin \Gamma_v$.
3. $t_e = t_e^{-1}$ if $e \in E_1(Z)$.
4. $t_egt_e^{-1} = \phi_e(g)$ if $e \in E_1(Z)$, (or $t_egt_e^{-1} = \alpha_e(g)$ if $e \in E_2(Z)$), and $g \in \Gamma_v$.
5. $t_et_e = \delta_e$.

Proof. The proof follows easily from the relations of $\pi \Phi(Z, \Gamma)$. \hfill $\Box$

3.9. Remark. No confusion will be caused by the notations $t_e$ and $t(\epsilon)$.

3.10. Definition. By a word $w$ in $\pi \Phi(Z, \Gamma)$ we mean an expression of the form $w = x_{g_0}y_1x_{g_1} \cdots y_1x_{g_n}, n \geq 0$, where $y_1, \ldots, y_n$ is path in $Z$ such that

(a) $g_0 \in \Gamma_{o(y_1)}$.

(b) $g_i \in \Gamma_{t(y_i)}$ and $x_{g_i}$ is a word in the generating symbols $\text{gen}(\Gamma_{t(y_i)})$ of $\Gamma_{t(y_i)}$ of value $g_i$ for $i = 1, \ldots, n$.

Let $w$ be a word as defined above. We have the following concepts.

1. $w$ is called trivial if $w = 1$.
2. Define $o(w) = o(y_1)$ and $t(w) = t(y_n)$.
   i. If $o(w) = t(w) = v$, then $w$ is called a closed word in $\pi \Phi(Z, \Gamma)$ of type $v$.
   ii. $n$ is called the length of $w$, and is denoted by $|w| = n$.
3. The value $t_w$ of $w$ is defined to be the element $t_w = g_0t_{g_1}g_1 \cdots t_{g_n}g_n$ of $\pi \Phi(Z, \Gamma)$.
4. If $w' = x_{g_0}y_1x_{g_1} \cdots y_1x_{g_n}x_{h_0}z_1x_{h_1} \cdots z_1x_{h_m}$ is a word in $\pi \Phi(Z, \Gamma)$ such that $t(w) = o(w')$, then $w \cdot w'$ is defined to be the word $ww' = x_{g_0}y_1x_{g_1} \cdots y_1x_{g_n}x_{h_0}z_1x_{h_1} \cdots z_1x_{h_m}$.

   It is clear that $ww'$ is word in $\pi \Phi(Z, \Gamma)$. Also, $o(ww') = o(w), t(ww') = t(w'), |ww'| = |w| + |w'|$, and $t_{ww'} = t_{w}t_{w'}$.
5. $w$ is called a reduced word in $\pi \Phi(Z, \Gamma)$ if either $n = 0$ and $g_0 \neq 1$, or else $n > 0$, and $w$ contains no subword of the form $y_iw_{g_i}y_i$ if $g_i \in \Gamma^{w_i}$, and $y_i \in E(Z)$, or contains no subword of the form $y_iw_{g_i}y_i$ if $g_i \in \Gamma^{w_i}$ and $y_i \in E_2(Z)$.
3.11. Proposition.

(1) For every nontrivial word $w$ in $\pi \Phi(Z, \Gamma)$, there exists a reduced word $w_0$ in $\pi \Phi(Z, \Gamma)$ such that $o(w) = o(w_0)$, $t(w) = t(w_0)$ and $t_w = t_{w_0}$.

(2) If $w$ is a reduced word in $\pi \Phi(Z, \Gamma)$, then $t_w \neq 1$.

Proof. (1) It is clear that replacing each subword of $w$ of the form $y_i x_i \bar{y}_i$ if $g_i \in \Gamma^{y_i}$ and $y_i \in E_1(Z)$ by $x_{\phi_{y_i}(g_i)}$ or $y_i x_i y_i$ if $g_i \in \Gamma^{y_i}$ and $y_i \in E_2(Z)$ by $x_{\phi_{y_i}(g)}$ yields a reduced word $w_0$ of $\pi \Phi(Z, \Gamma)$ of the required properties.

(2) The definition of a reduced word in $\pi \Phi(Z, \Gamma)$ implies that $w \neq 1$. Then $t_w \neq 1$.

This completes the proof. \qed

3.12. Theorem. If $w = x_{y_0} y_1 x_{y_1} \cdots y_n x_{y_n}$ and $w'$ are two reduced words in $\pi \Phi(Z, \Gamma)$ such that $o(w) = o(w')$, $t(w) = t(w')$, and $t_w = t_{w'}$, then $w' = x_{h_0} y_1 x_{h_1} \cdots y_n x_{h_n}$, where $h_i = a_i g_i \phi_{y_{i+1}}(a_{i+1}^{-1})$, $a_i \in \Gamma^{y_i}$, and $y_{i+1} \in E_1(Z)$, or $(h_i = a_i g_i \phi_{y_{i+1}}(a_{i+1}^{-1}), a_i \in \Gamma^{y_i}$, and $y_{i+1} \in E_2(Z)$, for $i = 1, \ldots, n - 1$ with the convention that $a_0 = 1$ and $h_n = a_n g_n$, $a_n \in \Gamma^{y_n}$.

Proof. Let $w' = x_{h_0} z_1 x_{h_1} \cdots z_{m-1} x_{h_{m-1}} z_m x_{h_m}$, and $\bar{w} = x_{h_0} \bar{y}_n x_{h_{m-1}} \bar{y}_{n-1} \cdots x_{k_1} \bar{y}_1 x_{k_0}$, where $k_i = 1, \ldots, n$ and $k_0 = g_0^{-1}$. It is clear that $\bar{w}$ is a reduced word in $\pi \Phi(Z, \Gamma)$ such that $o(w) = t(\bar{w})$, $t(w) = o(\bar{w})$, and $t_{\bar{w}} = t_{w'}^{-1}$. Let $w_0$ be the word $w_0 = g_n^{-1} \delta_n^{-1} g_{n-1}^{-1} \delta_{n-1}^{-1} \cdots g_1^{-1} \delta_1^{-1} g_0^{-1}$.

Then $t_{w_0} = 1$, the identity element in $\pi \Phi(Z, v_0, \Gamma)$. Therefore Proposition 3.11(2) implies that

$$w_0 = x_{h_0} z_1 x_{h_1} \cdots z_{m-1} x_{h_{m-1}} z_m x_{h_m} x_{k_n} \bar{y}_n x_{k_{n-1}} \cdots x_{k_1} \bar{y}_1 x_{k_0}$$

is not reduced. Since $w$ and $\bar{w}$ are reduced, therefore $y_n = z_m$ (or $y_n = z_m$ if $\bar{y}_n = y_n$), and $h_m g_n^{-1} \delta_n^{-1} \in \Gamma^{y_n}$. Since $\delta_n^{-1} \in \Gamma^{y_n}$, therefore $h_m g_n^{-1} \delta_n^{-1} \in \Gamma^{y_n}$. Then $h_m = a_n g_n$.

We substitute $z_m \cdot a_n \cdot \bar{y}_n$ by $\phi_{y_n}(a_n)$ (or $\alpha_{y_n}(a_n)$) in $w$ to get the new word

$$w_1 = x_{h_0} z_1 x_{h_1} \cdots z_{m-1} x_{h_{m-1}} z_m x_{h_m} x_{k_n} \bar{y}_n x_{k_{n-1}} \cdots x_{k_1} \bar{y}_1 x_{k_0},$$

where $k = h_{m-1} \phi_{y_n}(a_n) g_{n-1}^{-1} \delta_{n-1}^{-1}$. Then $w_1$ is not reduced, and similar to the above we have $z_{m-1} = \bar{y}_{n-1}$ (or $z_{m-1} = y_{n-1}$ if $\bar{y}_{n-1} = y_{n-1}$), and $h_{m-1} \phi_{y_n}(a_n) g_{n-1}^{-1} \delta_{n-1}^{-1} \in \Gamma^{y_{n-1}}$. Since $\delta_{n-1}^{-1} \in \Gamma^{y_{n-1}}$, therefore $h_{m-1} \phi_{y_n}(a_n) g_{n-1}^{-1} \delta_{n-1}^{-1} \in \Gamma^{y_{n-1}}$, and $h_{m-1} = a_{n-1} g_{n-1} \phi_{y_n}(a_{n-1})^{-1}$.

Continuing the above processes yields

$$z_1 = y_1 (or \ z_1 = \bar{y}_1 \ if \ \bar{y}_1 = y_1), \ and \ h_1 \phi_{y_2}(a_2) \delta_2^{-1} \delta_1^{-1} \in a_1 \in \Gamma^{y_1}.$$ 

Then $h_1 = a_1 g_1 \phi_{y_2}(a_2)^{-1}$. Finally we get $h_0 \phi_{y_1}(a_1)^{-1} \delta_0^{-1} = 1$. This implies that $h_0 = g_0 \phi_{y_1}(a_0^{-1})$. Consequently, $w' = h_0 y_1 x_{h_1} \cdots y_n x_{h_n}$, which completes the proof. \qed

The proof of the following corollary follows immediately from Theorem 3.12.

3.13. Corollary. If $w = x_{y_0} y_1 x_{y_1} \cdots y_n x_{y_n}$ and $w' = x_{h_0} z_1 x_{h_1} \cdots z_{m-1} x_{h_{m-1}} z_m x_{h_m}$ are two reduced words in $\pi \Phi(Z, \Gamma)$ such that $o(w) = o(w')$, $t(w) = t(w')$, and $t_w = t_{w'}$, then $n = m, z_i = y_i$ (or $z_i = \bar{y}_i$ if $y_i \in E_2(Z)$), and there exist unique elements $a_i \in \Gamma^{y_i}$, $i = 1, \ldots, m - 1$, with the convention that $a_0 = 1$, such that $h_i = a_i g_i \phi_{y_{i+1}}(a_{i+1}^{-1})$ (or $h_i = a_i g_i \alpha_{y_{i+1}}(a_{i+1}^{-1})$), $i = 1, \ldots, m - 1$ and $h_m = a_m g_m$. \qed
4. The fundamental group of a quasi graph of groups relative to maximal subtrees

Throughout this section, $\Phi(Z, \Gamma)$ is a quasi graph of groups, and $T$ a maximal subtree of $Z$.

4.1. Definition. The fundamental group of $\Phi(Z, \Gamma)$ relative to $T$ is denoted by $\pi\Phi(Z, T, \Gamma)$, and defined to be the group

$$\pi\Phi(Z, T, \Gamma) = \pi\Phi(Z, \Gamma)/(\text{relations, } e = 1 \text{ for all } e \in E(T)).$$

4.2. Proposition. $\pi\Phi(Z, T, \Gamma)$ has the presentation

$$\langle \text{gen (} \Gamma_v \text{), } e, f \mid \text{rel (} \Gamma_v \text{), } e \cdot \Gamma^v \cdot e^{-1} = \Gamma^e, \ f \cdot \Gamma^f \cdot f^{-1} = \Gamma^f, \ e\bar{e} = 1, \ f^2 = \delta_f, \ e = 1 \text{ if } e \in E(T) \rangle,$$

where $v \in V(Z)$, $e \in E_1(Z)$, and $f \in E_2(Z)$.

Proof. Let $N$ be the normal closure of the set of edges $e \in E(T)$ in $\Phi(Z, \Gamma)$. Then it is clear that $\pi\Phi(Z, T, \Gamma) = \pi\Phi(Z, \Gamma)/N$, the quotient group of $\pi\Phi(Z, \Gamma)$ modulo $N$. Then by [5, Theorem 2.1, p. 71], $\pi\Phi(Z, T, \Gamma)$ has the required presentation. This completes the proof. □

4.3. Proposition. Let $A$ be an orientation of $Z$, and $\pi\Phi(Z, T, A, \Gamma)$ the group with presentation

$$\langle \text{gen (} \Gamma_v \text{), } e, f \mid \text{rel (} \Gamma_v \text{), } e \cdot \Gamma^v \cdot e^{-1} = \Gamma^e, \ f \cdot \Gamma^f \cdot f^{-1} = \Gamma^f, \ f^2 = \delta_f \rangle,$$

where $v \in V(Z)$, $e$ and $f$ are edges of $A - E(T)$ such that $e \in E_1(Z)$, $f \in E_2(Z)$, and $m \in E(T)$. Then $\pi\Phi(Z, T, \Gamma)$ is isomorphic to $\pi\Phi(Z, T, A, \Gamma)$.

Proof. Similar to the proof of Proposition 3.2, it is clear that $\pi\Phi(Z, T, \Gamma)$ has the given presentation. □

4.4. Corollary. Let $A_1$ and $A_2$ be two orientations of $Z$, and $T$ a maximal subtree of $Z$. Then $\pi\Phi(Z, T, A_1, \Gamma)$ and $\pi\Phi(Z, T, A_2, \Gamma)$ are isomorphic. □

4.5. Examples. (1) If $\Gamma_v$ is trivial for all $v, v \in V(Z)$, then

$$\pi\Phi(Z, T, \Gamma) = F \ast C,$$

where $F$ is the free group generated by $E_1(Z)$, and $C$ the free product of the groups of order 2 generated by $f, f \in E_2(Z)$.

(2) If $\Gamma^x$ is trivial for all edges $x$ of $Z$, then

$$\pi\Phi(Z, T, \Gamma) = G \ast F,$$

where $F$ is as above, and $G$ is the free product of the groups $\Gamma_v$.

(3) If $E(Z) = E_1(Z) = \{e\}$, then

(i) If $o(e) = t(e) = v$, then

$$\pi\Phi(Z, T, \Gamma) = \langle \text{gen (} \Gamma_v \text{), } e \mid \text{rel (} \Gamma_v \text{), } e \cdot \Gamma^e \cdot e^{-1} = \Gamma^e \rangle$$

is the HNN group with base $\Gamma_v$, associated pairs ($\Gamma^e$, $\Gamma^f$) of subgroups of $\Gamma_v$, and of stable letter $e$. 
Proof. We apply Theorem 4.7 by taking
\[
\phi(Z, T, \Gamma) = \langle \text{gen}(\Gamma_u), \text{gen}(\Gamma_v), e | \text{rel}(\Gamma_u), \text{rel}(\Gamma_v), \Gamma^e = e^{-1}\Gamma^e \rangle
\]
is the HNN group with base \(G\), where \(G\) is the group with presentation
\[
G = \langle \text{gen}(\Gamma_u), \text{gen}(\Gamma_v), e | \text{rel}(\Gamma_u), \text{rel}(\Gamma_v), \Gamma^e = \Gamma^e \rangle,
\]
associated pairs \((\Gamma^e, \Gamma^e)\) of subgroups of \(G\), and of stable letter \(e\).

4.6. Proposition. \(\phi(Z, T, \Gamma)\) is a quasi HNN group with base a tree product of groups. Moreover, for every vertex \(v\) of \(Z\), \(\Gamma_v\) is embedded in \(\phi(Z, T, \Gamma)\).

Proof. Let \(G = \langle \text{gen}(\Gamma_v), e, f | \text{rel}(\Gamma_v), \Gamma^m = \Gamma^m \rangle\). Then \(G\) is the tree product of the groups \(\Gamma_v\), with amalgamation subgroups \(\Gamma^m\). Then \(\phi(Z, T, \Gamma)\) is the quasi HNN group with base \(G\), associated pairs \((\Gamma^e, \Gamma^e)\) and \((\Gamma^f, \Gamma^f)\) of subgroups of \(G\), where \(v \in V(Z)\), \(e\) and \(f\) are edges of \(A - E(T)\), \(e \in E_1(Z)\), \(f \in E_2(Z)\), and \(m \in E(T)\), and \(A\) is an orientation of \(Z\). Moreover, by Lemma 3.6, \(\Gamma_v\) is embedded in \(\phi(Z, T, \Gamma)\). This completes the proof.

For the rest of this section we take the value \(t_e\) of each edge \(e\) of \(T\) to be \(t_e = 1\).

4.7. Theorem. For every element \(g\) of \(\phi(Z, T, \Gamma)\) and any two vertices \(u\) and \(v\) of \(T\), there exists a reduced word \(w\) in \(\phi(Z, T, \Gamma)\) such that \(t_w = g\), \(o(w) = u\) and \(t(w) = v\).

Proof. We consider two cases.

Case 1. \(g = 1\). Then, \(T\) being a subtree of \(Z\) implies that there exists a unique reduced path \(y_1, \ldots, y_n\) in \(T\) joining \(u\) and \(v\). Let \(w = y_1y_2\ldots y_n1\). Then \(w\) is a reduced word in \(\phi(Z, T, \Gamma)\) on which \(t_w = 1\), \(o(w) = u\) and \(t(w) = v\). Hence, \(w\) is the required word.

Case 2. \(g \neq 1\). Now, Proposition 3.8 implies that \(g\) can be written as a product of elements of the groups \(\Gamma_v\), and of elements \(t_e\) for every vertex \(v\) of \(V(Z)\) and every edge \(e\) of \(E(Z)\). Thus, \(g = g_{y_1}t_{y_1}g_{y_2}t_{y_2}\cdots t_{y_n}g_{y_n}\), where \(g_{y_i} \in \Gamma_{u_{y_i}}\) for some vertices \(u_0, u_1, \ldots, u_n\) of \(V(Z)\), and edges \(y_1, \ldots, y_n\) of \(E(Z)\). By taking the unique reduced paths in \(T\) between \(u\) and the vertices \(u_0, u_1, \ldots, u_n\), between \(u_n\) and \(v\), and the identity element, we obtain the word
\[
w' = h_0z_1h_1\cdots z_nh_m
\]
in \(\phi(Z, T, \Gamma)\) satisfying \(o(w') = u\) and \(t(w') = v\). Since \(t_y = 1\) for each edge \(y\) of \(T\), therefore \(t_{w'} = g\). Then Proposition 3.11 implies that there exists a reduced word \(w_0\) in \(\phi(Z, \Gamma)\) such that \(o(w') = o(w_0) = u\), \(t(w') = t(w_0) = v\), and \(t_{w'} = t_{w_0} = g\).

This completes the proof.

4.8. Corollary. Every element of \(\phi(Z, T, \Gamma)\) is the value of a closed and reduced word in \(\phi(Z, T, \Gamma)\) of type \(v_0\) for an arbitrary vertex \(v_0\) of \(V(Z)\). Moreover, if \(w\) is a nontrivial closed and reduced word in \(\phi(Z, \Gamma)\), then the value \(t_w\) of \(w\) is not the identity element in \(\phi(Z, T, \Gamma)\).

Proof. We apply Theorem 4.7 by taking \(u = v = v_0\), then applying Proposition 3.11 yields the proof.
5. Groups induced by vertices of a quasi graph of groups

Throughout this section, $\Phi(Z, \Gamma)$ is a quasi graph of groups, and $T$ a maximal subtree of $Z$.

5.1. Definition. By a path $p$ of length $n \geq 0$ in $\pi \Phi(Z, \Gamma)$ we mean an expression of the form

$$p = (a_0, x_1, a_1, \ldots, x_m, a_m),$$

where $x_1, \ldots, x_m$ is a path in $Z$ such that $a_0 \in \Gamma_{o(x_1)}$ and $a_i \in \Gamma_{t(x_i)}$ for $i = 1, \ldots, m$.

For the path $p$ defined above, we have the following concepts:

1. $p$ is called trivial if $p = (1)$.
2. $o(p) = o(x_1)$, and $t(p) = t(x_m)$.
3. If $o(p) = t(p) = v$, then $p$ is called a closed path in $\pi \Phi(Z, \Gamma)$ of type $v$.
4. $m$ is called the length of $p$, and is denoted by $|p| = m$.
5. $p$ is called reduced if either $m = 0$ and $a_0 \neq 1$, or else $m > 0$ and $p$ contains no subpath of the form $(x_i, a_i, \bar{x}_i)$ if $a_i \in \Gamma^{\pm 1}$ and $x_i \in E(Z)$, or $(x_i, a_i, x_i)$ if $a_i \in \Gamma^{\pm 1}$ and $x_i \in E_2(Z)$.
6. $(x_i, a_i, \bar{x}_i)$ if $a_i \in \Gamma^{\pm 1}$, or $(x_i, a_i, x_i)$ if $a_i \in \Gamma^{\pm 1}$, is called a reversal in $p$.

5.2. Definition. Let $p = (h_0, x_1, h_1, \ldots, x_m, h_m)$ and $q = (g_0, y_1, g_1, \ldots, y_n, g_n)$ be two paths in $\pi \Phi(Z, \Gamma)$ such that $t(p) = o(q)$.

1. The juxtaposition of $p$ and $q$ is defined to be the sequence

$$r = (h_0, x_1, h_1, \ldots, x_m, h_m, g_0, y_1, g_1, \ldots, y_n, g_n).$$

It is clear that $r$ is a path in $\pi \Phi(Z, \Gamma)$. Moreover, $o(r) = o(p)$, $t(r) = t(q)$, and $|r| = |p| + |q|$.

2. By the edge reductions on $p$ we mean the performance of the following operations on $p$.
   
   (a) Replacing $(x_i, a_i, \bar{x}_i), a_i \in \Gamma^{\pm 1}$, by $\phi_{x_i}(a_i)$ if $x_i \in E_1(Z)$.
   
   (b) Replacing $(x_i, a_i, x_i), a_i \in \Gamma^{\pm 1}$, by $x_i \delta_a$ if $x_i \in E_2(Z)$, where

   $$\delta_a = \begin{cases} 1 & \text{if } e \in E_1(Z), \\ a^e & \text{if } e \in E_2(Z). \end{cases}$$

   It is clear that the edge-reductions on the path $p$ yield a reduced path $p'$ of $\pi \Phi(Z, \Gamma)$ such that $o(p) = o(p')$, and $t(p) = t(p')$.

5.3. Proposition. Let $v_o$ be a fixed vertex of $Z$, $P_n$ the set of all closed and reduced paths in $\pi \Phi(Z, \Gamma)$ of type $v_o$ and of length $n$, $n = 0$. Let $\pi \Phi(Z, v_o, \Gamma) = \bigcup_{n \geq 0} P_n$. Then $\pi \Phi(Z, v_o, \Gamma)$ forms a group under the binary operation defined as juxtaposition plus edge-reductions on paths.

Proof. It is clear that juxtaposition plus edge-reductions on closed and reduced paths of $\pi \Phi(Z, \Gamma)$ of type $v_o$ yields a closed and reduced path in $\pi \Phi(Z, \Gamma)$ of type $v_o$. More precisely, if

$$p = (a_0, x_1, a_1, \ldots, x_m, a_m) \in P_m \text{ and } q = (b_0, y_1, b_1, \ldots, y_n, b_n) \in P_n,$$

then the juxtaposition plus edge-reductions of $p$ and $q$ is denoted by $pq$ and is given by

$$pq = (a_0, x_1, a_1, \ldots, x_m, a_m, L_{m-r}, y_{r+1}, \ldots, y_n, b_n),$$

where $y_1 = \bar{x}_{m-j+1}$ and $L_j = a_{m-j} \phi_{x_{m-j+1}}(L_{j-1}) b_j \in \Gamma^{e_{m-j}}$ if $y_j \in E_1(Z)$, or $y_j = x_{m-j+1}$ and $L_j = a_{m-j} \alpha_{x_{m-j+1}}(L_{j-1}) b_j \in \Gamma^{e_{m-j}}$ if $y_j \in E_2(Z)$, for $j = 1, \ldots, r$ and
with the convention that $L_0 = a_m b_n \in \Gamma^m$, and $r$ is the largest value of $k \geq 0$ for which none of the paths

$$(a_{m-1}, x_m, L_m, y_1, b_1), \ (a_{m-2}, x_{m-1}, L_{m-1}, y_2, b_2), \ldots ,$$

$$(a_{m-k}, x_{m-k+1}, L_{m-k+1}, y_k, b_k)$$

is reduced, where $L_m, L_{m-1}, \ldots , L_{m-k+1}$ are obtained in a similar way to $L_{m-r}$ above. This condition guarantees that $pq \in P^{m+n-2r}$.

Of course, $r \leq \min(m, n)$, and equality may occur. This implies that the given binary operation of paths is closed on $\pi \Phi(Z, \nu_0, \Gamma)$.

Clearly, the trivial path (1) is the identity element of $\pi \Phi(Z, \nu_0, \Gamma)$.

For the path $p = (a_0, x_1, a_1, \ldots , x_m, a_m) \in P_m$, it is clear that

$p^{-1} = (a_m^{-1} x_m^{-1}, a_{m-1}^{-1} x_{m-1}^{-1}, \ldots , a_1^{-1} x_1^{-1}, a_0^{-1})$

is a path in $\pi \Phi(Z, \Gamma)$ of length $m$, and that $p^{-1} \in P_m$ is the inverse of $p$.

For the associative law, let $p \in P_m, p' \in P_n$ and $p'' \in P_r$. We need to show that

$$(pp')p'' = p(p'p'').$$

If any of $m, n$ or $r$ is 0, this is obvious, and so we can assume that $m, n, r \geq 1$.

Let

$$p = (a_0, x_1, a_1, \ldots , x_m, a_m) \in P_m, \quad p' = (b_0, y_1, b_1, \ldots , y_n, b_n) \in P_n, \quad \text{and} \quad p'' = (c_0, z_1, c_1, \ldots , z_r, c_r) \in P_r.$$

From the above we have

$$pp' = (a_0, x_1, a_1, \ldots , x_{m-r}, L_{m-r}, y_{r+1}, b_{r+1}, \ldots , y_n, b_n) \in P^{m+n-2r}.$$

Likewise, juxtaposition plus edge-reductions on $p'$ and $p''$ gives

$$p'p'' = (b_0, y_1, b_1, \ldots , y_{n-s}, M_{n-s}, z_{s+1}, c_{s+1}, \ldots , z_r, c_r),$$

where $s$ and $M_{n-s} \in \Gamma^{y_{n-s}}$ are obtained in a similar way to $r$ and $L_{m-r}$ above.

There are three cases to consider.

Case 1 $r + s < n$. Then both sides of (9) are equal to

$$(a_0, x_1, a_1, \ldots , x_{m-r}, L_{m-r}, y_{r+1}, b_{r+1}, \ldots , y_n, M_{n-s}, z_{s+1}, c_{s+1}, \ldots , z_r, c_r).$$

Case 2 $r + s = n$. In this case, both sides of (9) are equal to the path

$$(a_0, x_1, a_1, \ldots , x_{m-r}, L_{m-r}, M_{n-s}, z_{s+1}, c_{s+1}, \ldots , z_r, c_r).$$

Case 3 $r + s > n$. Let

$$p_1 = (b_0, y_1, b_1, \ldots , y_{n-s}, b_{n-s}), \quad p_2 = (b_{n-s}, y_{n-s+1}, b_{n-s+1}, \ldots , y_r, b_r),$$

$$p_3 = (b_r, y_{r+1}, b_{r+1}, \ldots , y_n, b_n), \quad p_4 = (c_s, z_{s+1}, c_{s+1}, \ldots , z_r, c_r), \quad \text{and} \quad p_5 = (a_0, x_1, a_1, \ldots , x_m, a_m).$$

Then $p_2$ has length 1 by hypothesis in this case, and $p' = p_1 p_2 p_3$, which is clear by Case 1, as are $p = p_5 p_2^{-1} p_1^{-1}$ and $p'' = p_3^{-1} p_2^{-1} p_4$. Thus,

$$(pp')p'' = (p_5 p_3) (p_3^{-1} p_2^{-1} p_4) = p_5 (p_2^{-1} p_4),$$

and

$$p(p'p'') = (p_5 p_2^{-1} p_1^{-1})(p_1 p_4).$$

Since $p_3$ and $p_2^{-1}$ are adjacent in $p$, there are no edge-reductions in forming their product, and similarly with $p_2^{-1}$ and $p_4$ in $p''$, whence $p_5 (p_2^{-1} p_4) = (p_5 p_2^{-1}) p_4$ by Case 1, showing that (9) holds in every case. This completes the proof.
5.4. **Definition.** For each \( v \in V(Z) \) and \( g \in \Gamma_v \), let \( p_v \) be the unique reduced path in \( T \) joining \( v_o \) and \( v, \gamma_g \) the path

\[
\gamma_g = p_v gp_v = (1, y_1, 1, \ldots, 1, y_n, g, y_n, 1, \ldots, 1, y_1, 1)
\]
in \( \pi \Phi(Z, T, \Gamma) \), and \( \gamma_y \) the path

\[
\gamma_y = p_0(y)p_t(y) = (1, y_1, 1, \ldots, 1, y_n, 1, y, 1, y_n+1, 1, \ldots, 1, y_m, 1)
\]
in \( \pi \Phi(Z, T, \Gamma) \), where \( y_1, \ldots, y_n \) is the unique reduced path in \( T \) joining \( v_o \) and \( o(y) \), and \( y_n+1, \ldots, y_m \) the unique reduced path in \( T \) joining \( t(y) \) and \( v_o \).

The following lemma is essential for the proof of the main result of this section.

5.5. **Lemma.** For all \( v \in V(Z) \), \( g \in \Gamma_v \) and \( y \in E(Z) \), \( y \notin E(T) \), the group \( \pi \Phi(Z, v_o, \Gamma) \) is generated by \( \gamma_g \) and by \( \gamma_y \).

**Proof.** It is clear that \( \gamma_g \) and \( \gamma_y \) are reduced paths in \( \pi \Phi(Z, T, \Gamma) \) of type \( v_o \). Moreover, if \( g \in \Gamma_v \), then \( \gamma_g = (g) \) is a path of length 0 in \( \pi \Phi(Z, T, \Gamma) \) of type \( v_o \). Now let \( p \) be a path in \( \pi \Phi(Z, v_o, \Gamma) \), \( p \neq (1) \). Then \( p = (g_0, y_1, \ldots, y_n, g_n) \). It is clear that \( \gamma_{g_0} = (g_0) \) and \( \gamma_{g_n} = (g_n) \).

Let \( q = \gamma_{g_0} \gamma_{y_1} \gamma_{y_2} \cdots \gamma_{y_n} \gamma_{g_n} \). It is clear that \( q \) is a path in \( \pi \Phi(Z, T, \Gamma) \) of type \( v_o \).

The application of edge-reductions on \( q \) yields the path

\[
(\gamma_{g_0}) (\gamma_{y_1}) (\gamma_{y_2}) \cdots (\gamma_{y_n}) (\gamma_{g_n}) = (g_0) (g_1) (g_2) \cdots (g_n)
\]

because \( p(t(y_1)) = (1), t(y_1) = o(y_1+1) \) and \( o(y_1) = t(y_n) = v_o \).

Thus, \( p \) is the product of the paths \( \gamma_{g_0}, \gamma_{y_1}, \gamma_{y_2}, \ldots, \gamma_{y_n}, \gamma_{g_n} \). This completes the proof.

In [2, Prop. 1, p. 140], G. Baumslag proved that the groups \( \pi \Phi(Z, T, \Gamma) \) and \( \pi \Phi(Z, v_o, \Gamma) \) are isomorphic under the condition that no edge of the graph \( Z \) equals its inverse. Now in the following theorem we generalise this result to the case where an edge of the graph can equal its inverse. The proof is similar.

5.6. **Theorem.** The groups \( \pi \Phi(Z, T, \Gamma) \) and \( \pi \Phi(Z, v_o, \Gamma) \) are isomorphic.

**Proof.** Since \( T \) is a maximal subtree of \( Z \), therefore \( V(T) = V(Z) \) and \( v_o \in V(T) \). Then \( \pi \Phi(Z, T, \Gamma) \) is the group with presentation

\[
\langle \text{gen} (\Gamma_v), e, f | \text{rel} (\Gamma_v), e \cdot \Gamma^e \cdot e^{-1} = \Gamma^f, f \cdot \Gamma^f \cdot f^{-1} = \Gamma^e, e^2 = 1, f^2 = \delta_f, e = 1 \text{ if } e \in E(T) \rangle,
\]

where \( v \in V(Z), e \in E_1(Z) \) and \( f \in E_2(Z) \).

Let \( F \) be the free group having as base the generating symbols of \( \pi \Phi(Z, T, \Gamma) \). Then the base of \( F \) consists of the generating symbols \( x_a \) of \( \Gamma_v \) with value \( a \in \Gamma_v, v \in V(T) \), and the edges \( y \) of \( Z \). Let \( \Psi \) be the function from \( F \) to \( \pi \Phi(Z, v_o, \Gamma) \) given by \( \Psi(x_a) = \gamma_a \), and \( \Psi(y) = \gamma_y \) for all \( a \in \Gamma_v, v \in V(Z) \), and all \( y \in E(Z) \).

Now we show that \( \Psi \) satisfies the relations of \( \pi \Phi(Z, T, \Gamma) \).

(1) It is clear that \( \Psi \) satisfies the relations in \( \text{rel} (\Gamma_v) \).
(2) Let $e \in E(T)$. Then $\Psi(e) = \gamma_e$, and the application of the edge-reductions on $\gamma_e$ yields the trivial path $(1) = \gamma_1$. Therefore, $\Psi(e) = \Psi(1)$. Thus, $\Psi$ satisfies the relation $e = 1$ in $\pi \Phi(Z, T, \Gamma)$.

(3) Let $e \in E_1(Z)$ and $g \in \Gamma^e$. Then

$$\Psi(e)\Psi(g)\Psi(e^{-1}) = \Psi(e)\Psi(g)\Psi(e)$$

$$= \gamma_e \gamma_g \gamma_e \text{ because } e \bar{e} = 1, \text{ from which } e^{-1} = \bar{e},$$

$$= p_o(e)\bar{p}_{t(e)} p_{t(e)} g \bar{p}_{t(e)} p_o(e) \bar{p}_{t(e)} = p_o(e) e g \bar{p}_{t(e)},$$

$$= p_o(e) \phi_e(g) \bar{p}_{t(e)} = \gamma_{\phi_e(g)} = \Psi(\phi_e(g)).$$

and

$$\Psi(e)\Psi(e) = \gamma_e \gamma_e = p_o(e) \bar{p}_{t(e)} p_{t(e)} \bar{p}_{t(e)} = p_o(e) 1 \bar{p}_{t(e)} = \gamma_1 = \Psi(1).$$

Thus, $\Psi$ satisfies the relations $e \cdot \Gamma^e \cdot e^{-1} = \Gamma^e$ and $e\bar{e} = 1$.

(4) Let $f \in E_2(Z)$ and $g \in \Gamma^e$. Then $fgf^{-1} = f g \delta_f^{-1} f$, and

$$\Psi(f)\Psi(g)\Psi(f^{-1}) = \Psi(f)\Psi(g \delta_f^{-1})\Psi(f) = \gamma_f \gamma_g \delta_f\gamma_f$$

$$= p_o(f) \bar{p}_{t(f)} p_{t(f)} g \delta_f^{-1} \bar{p}_{t(f)} p_o(f) \bar{p}_{t(f)} = p_o(f) f g \delta_f^{-1} f \bar{p}_{t(f)}$$

$$= p_o(f) \alpha_f(g) \bar{p}_{t(f)} = \gamma_{\alpha_f(g)} = \Psi(\alpha_f(g)).$$

and

$$\Psi(f)\Psi(f) = p_o(f) \bar{p}_{t(f)} p_{t(f)} f \bar{p}_{t(f)} = p_o(f) f^2 \bar{p}_{t(f)} = p_o(f) \delta_f \bar{p}_{t(f)} = \gamma \delta_f$$

Then, $\Psi$ satisfies the relations $f \cdot \Gamma^f \cdot f^{-1} = \Gamma^f$ and $f^2 = \delta_f$.

This implies that $\Psi$ satisfies the relations of $\pi \Phi(Z, T, \Gamma)$. So by Dyck’s Theorem [3, Theorem 14, p. 19], there exists a unique homomorphism

$$\hat{\Psi} : \Phi(Z, T, \Gamma) \rightarrow \pi \Phi(Z, v_0, \Gamma)$$

satisfying $\hat{\Psi}(a) = \Psi(x_a) = \delta_a$ and $\hat{\Psi}(t_v) = \Psi(y) = \delta_y$ for all $a \in \Gamma_v$, $v \in V(Z)$, and all $y \in E(Z)$. Lemma 5.5 implies that $\pi \Phi(Z, v_0, \Gamma)$ is generated by $\delta_a$ and $\delta_y$ for all $v \in V(Z)$, $a \in \Gamma_v$, and for all $y \in E(Z)$, $y \notin E(T)$.

Dyck’s Theorem implies that $\hat{\Psi}$ is an epimorphism. Now we show that $\hat{\Psi}$ is injective. Let $g$ be an element of $\pi \Phi(Z, T, \Gamma)$, $g \neq 1$. We need to show that $\hat{\Psi}(g) \neq (1)$.

If $g \in \Gamma_{v_0}$, $v \in V(Z)$, then $\delta_v = (g) \neq (1)$. Let $g \notin \Gamma_{v_0}$. Then by Corollary 4.8, $g$ can be written as $g = g_0 x_{g_1} y_{g_1} \cdots y_{g_n} x_{g_n}$, where $x_{g_0} y_{g_1} x_{g_1} \cdots y_{g_n} x_{g_n}$ is a closed and reduced word in $\pi \Phi(Z, \Gamma)$ of type $v_0$.

Then, $\hat{\Psi}(g) = p = (g_0, y_{g_1}, \ldots, y_{g_n}, g_n)$ is a reduced path in $\pi \Phi(Z, T, \Gamma)$ of type $v_0$ on which $p \neq (1)$. This implies that $\hat{\Psi}$ is an isomorphism, and completes the proof. \qed

We end this section with the following corollary.

**5.7. Corollary.** Let $\Phi(Z, \Gamma)$ be a quasi graph of groups, $u_0$ and $v_0$ two vertices of $Z$, and $T_1, T_2$ two maximal subrays of $Z$. Then the groups $\pi \Phi(Z, u_0, \Gamma)$, $\pi \Phi(Z, v_0, \Gamma)$, $\pi \Phi(Z, T_1, \Gamma)$ and $\pi \Phi(Z, T_2, \Gamma)$ are isomorphic.

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References