CLASSIFICATION OF CUBIC EDGE-TRANSITIVE GRAPHS OF ORDER $14p^2$  

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Abstract

A graph is called edge-transitive if its automorphism group acts transitively on its set of edges. In this paper we classify all connected cubic edge-transitive graphs of order $14p^2$, where $p$ is a prime.

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1. Introduction

Throughout this paper, graphs are assumed to be finite, simple, undirected and connected. For a graph $X$, we denote by $V(X)$, $E(X)$, $A(X)$ and $\text{Aut}(X)$ the vertex set, the edge set, the arc set and the full automorphism group of $X$, respectively. For the group-theoretic concepts and notations not defined here we refer to [3, 4, 14, 19, 24].

Let $G$ be a finite group and $S$ a subset of $G$ such that $1 \notin S$ and $S = S^{-1}$. The Cayley graph $X = \text{Cay}(G, S)$ on $G$ with respect to $S$ is defined to have vertex set $V(X) = G$ and edge set $E(X) = \{(g, sg) | g \in G, s \in S\}$. The Cayley graph $X = \text{Cay}(G, S)$ is said to be normal if $G \triangleleft \text{Aut}(X)$. By definition, $\text{Cay}(G, S)$ is connected if and only if $S$ generates the group $G$.

An $s$-arc of a graph $X$ is an ordered $(s + 1)$-tuple $(v_0, v_1, \ldots, v_s)$ of vertices of $X$ such that $v_{i-1}$ is adjacent to $v_i$ for $1 \leq i \leq s$ and $v_{i-1} \neq v_{i+1}$ for $1 \leq i < s$. A graph $X$ is said to be $s$-arc-transitive if $\text{Aut}(X)$ acts transitively on the set of its $s$-arcs. In particular, 0-arc-transitive means vertex-transitive, and 1-arc-transitive means arc-transitive or symmetric. $X$ is said to be $s$-regular if $\text{Aut}(X)$ acts regularly on the set of its $s$-arcs. Tutte [20] showed that every finite connected cubic symmetric graph is $s$-regular for $1 \leq s \leq 5$. A subgroup of $\text{Aut}(X)$ is said to be $s$-regular if it acts regularly

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on the set of $s$-arcs of $X$. If a subgroup $G$ of $\text{Aut}(X)$ acts transitively on $V(X)$ and $E(X)$, we say that $X$ is $G$-vertex-transitive and $G$-edge-transitive, respectively. In the special case, when $G = \text{Aut}(X)$, we say that $X$ is vertex-transitive and edge-transitive, respectively.

It can be shown that a $G$-edge-transitive but not $G$-vertex-transitive graph $X$ is necessarily bipartite, where the two parts of the bipartition are orbits of $G \subseteq \text{Aut}(X)$. Moreover, if $X$ is regular then these two parts have the same cardinality. A regular $G$-edge-transitive but not $G$-vertex-transitive graph $X$ will be referred to as a $G$-semisymmetric graph. In particular if $G = \text{Aut}(X)$, $X$ is said to be semisymmetric.

The classification of cubic symmetric graphs of different orders is given in many papers. In [2, 3], the cubic $s$-regular graphs up to order $2048$ are classified. Throughout this paper, $p$ and $q$ are prime numbers. The $s$-regular cubic graphs of some orders such as $2p^2$, $4p^2$, $6p^2$, $10p^2$ were classified in [8-11]. Recently cubic $s$-regular graphs of order $2pq$ were classified in [25].

The study of semisymmetric graphs was initiated by Folkman [13]. For example, cubic semisymmetric graphs of orders $6p^2$, $8p^2$ and $2pq$ were classified in [15, 1, 7]. In this paper we classify cubic edge-transitive (symmetric or semisymmetric) graphs of order $14p^2$.

1.1. Theorem. Let $p$ be a prime and $X$ a connected cubic edge-transitive graph of order $14p^2$. Then $X$ is isomorphic either to the semisymmetric graph $S126$ or to one $s$-regular graph, where $1 \leq s \leq 3$. Furthermore,

1. $X$ is 1-regular if and only if $X$ is isomorphic to one of the graphs $F56A$, $F126$, $F350$, $F686A$, $F686C$, $F1694$, $EF14p^2$, where $p \geq 13$, or to Cay($G$, $S$), where $G = \langle a, b \mid a^2 = b^{7p^2} = 1, aba = b^{-1} > \cong D_{14p^2}$, $S = \{a, ba, b^{t+1}a\}$, $t^2 + t + 1 = 0$ (mod$7p^2$), $p \geq 13$ and $3(p - 1)$.

2. $X$ is 2-regular if and only if $X$ is isomorphic to one of the graphs $F56B$ and $F686B$.

3. $X$ is 3-regular if and only if $X$ is isomorphic to $F56C$.

2. Preliminaries

Let $X$ be a graph and $N$ a subgroup of $\text{Aut}(X)$. For $u$, $v \in V(X)$, denote by $\{u, v\}$ the edge incident to $u$ and $v$ in $X$, and by $N_X(u)$ the set of vertices adjacent to $u$ in $X$. The quotient graph $X_N$ (also denoted by $X/N$) induced by $N$ is defined as the graph such that the set $\Sigma$ of $N$-orbits in $V(X)$ is the vertex set of $X_N$, and $B, C \in \Sigma$ are adjacent if and only if there exist $u \in B$ and $v \in C$ such that $\{u, v\} \in E(X)$.

A graph $\tilde{X}$ is called a covering of a graph $X$ with projection $\varphi : \tilde{X} \to X$ if there is a surjection $\varphi : V(\tilde{X}) \to V(X)$ such that $\varphi|_{N_{\tilde{X}}(\tilde{v})} : N_{\tilde{X}}(\tilde{v}) \to N_X(v)$ is a bijection for any vertex $\tilde{v} \in V(\tilde{X})$ and $v \in \varphi^{-1}(v)$. A covering graph $\tilde{X}$ of $X$ with projection $\varphi$ is said to be regular (or a $K$-covering) if there is a semiregular subgroup $K$ of the automorphism group $\text{Aut}(\tilde{X})$ such that the graph $X$ is isomorphic to the quotient graph $\tilde{X}_K$, say by $h$, and the quotient map $\tilde{X} \to \tilde{X}_K$ is the composition $\phi h$ of $\phi$ and $h$. The fibre of an edge or a vertex is its preimage under $\phi$.

The group of automorphisms of $\tilde{X}$ mapping fibres to fibres is called the fibre-preserving subgroup of $\text{Aut}(\tilde{X})$.

Let $X$ be a graph and let $K$ be a finite group. By $a^{-1}$ we mean the reverse arc to an arc $a$. A voltage assignment (or, a $K$-voltage assignment) of $X$ is a function $\phi : A(X) \to K$ with the property that $\phi(a^{-1}) = \phi(a)^{-1}$ for each arc $a \in A(X)$. The values of $\phi$ are called voltages, and $K$ is the voltage group. The graph $X \times_\phi K$ derived from a voltage assignment $\phi : A(X) \to K$ has vertex set $V(X) \times K$ and edge set $E(X) \times K$, so that
the edge \((e, g)\) of \(X \times_A K\) joins the vertex \((u, g)\) to \((v, \phi(a)g)\) for \(a = (u, v) \in A(X)\) and \(g \in K\), where \(e = u, v\).

Clearly, the derived graph \(X \times_A K\) is a covering of \(X\); the first coordinate projection \(\phi : X \times_A K \to X\) is called the natural projection. By defining \((u, g')\) is \((u, g'g)\) for any \(g \in K\) and \((u, g') \in V(X \times_A K)\), \(K\) becomes a subgroup of \(\text{Aut}(X \times_A K)\) which acts semiregularly on \(V(X \times_A K)\). Therefore, \(X \times_A K\) can be viewed as a \(K\)-covering. For each \(u \in V(X)\) and \(u, v \in E(X)\), the vertex set \(\{(u, g) \mid g \in K\}\) is the fibre of \(u\) and the edge set \(\{(u, g)(v, \phi(a)g) \mid g \in K\}\) is the fibre of \(u, v\), where \(a = (u, v)\). Conversely, each regular covering \(\tilde{X}\) of \(X\) with a covering transformation group \(K\) can be derived from a \(K\)-voltage assignment.

Let \(\tilde{X}\) be a \(K\)-covering of \(X\) with a projection \(\phi\). If \(\alpha \in \text{Aut}(X)\) and \(\tilde{\alpha} \in \text{Aut}(\tilde{X})\) satisfy \(\tilde{\alpha} \phi = \phi \alpha\), we call \(\tilde{\alpha}\) a lift of \(\alpha\), and \(\alpha\) the projection of \(\tilde{\alpha}\). Concepts such as a lift of a subgroup of \(\text{Aut}(X)\) and the projection of a subgroup of \(\text{Aut}(X)\) are self-explanatory. The lifts and the projections of such subgroups are of course subgroups in \(\text{Aut}(\tilde{X})\) and \(\text{Aut}(X)\), respectively. In particular, if the covering graph \(\tilde{X}\) is connected, then the covering transformation group \(K\) is the lift of the trivial group, that is \(K = \{\tilde{\alpha} \in \text{Aut}(\tilde{X}) : \phi = \tilde{\alpha} \phi\}\).

Clearly, if \(\tilde{\alpha}\) is a lift of \(\alpha\), then \(K\tilde{\alpha}\) are all the lifts of \(\alpha\). The projection \(\phi\) is called vertex-transitive (edge-transitive) if some vertex-transitive (edge-transitive) subgroup of \(\text{Aut}(X)\) lifts along \(\phi\), and semisymmetric if it is edge- but not vertex-transitive.

The next proposition is a special case of [22, Proposition 2.5].

**2.1. Proposition.** Let \(X\) be a \(G\)-semisymmetric cubic graph with bipartition sets \(U(X)\) and \(W(X)\), where \(G \leq A := \text{Aut}(X)\). Moreover, suppose that \(N\) is a normal subgroup of \(G\). Then,

1. If \(N\) is intransitive on bipartition sets, then \(N\) acts semiregularly on both \(U(X)\) and \(W(X)\), and \(X\) is a regular \(N\)-covering of the \(G/N\)-semisymmetric cubic graph \(X_N\).
2. If \(3\) does not divide \(|A/N|\), then \(N\) is semisymmetric on \(X\).

**2.2. Proposition.** [17, Proposition 2.4] The vertex stabilizers of a connected \(G\)-semisymmetric cubic graph \(X\) have order \(2^\ell \cdot 3^r\), where \(0 \leq r \leq 7\). Moreover, if \(u\) and \(v\) are two adjacent vertices, then the edge stabilizer \(G_u \cap G_v\) is a common Sylow 2-subgroup of \(G_u\) and \(G_v\).

**2.3. Proposition.** [19, pp.230] Let \(G\) be a finite group and let \(p\) be a prime. If \(G\) has an abelian Sylow \(p\)-subgroup, then \(p\) does not divide \(|G \cap Z(G)|\).

**2.4. Proposition.** [24, Proposition 4.4] Every transitive abelian group \(G\) on a set \(\Omega\) is regular, and the centralizer of \(G\) in the symmetric group on \(\Omega\) is \(G\).

**2.5. Proposition.** [12, Theorem 9] Let \(X\) be a connected abelian group of prime valency and let \(G\) be an \(s\)-regular subgroup of \(\text{Aut}(X)\) for some \(s \geq 1\). If a normal subgroup \(N\) of \(G\) has more than two orbits, then it is semiregular and \(G/N\) is an \(s\)-regular subgroup of \(\text{Aut}(X_N)\), where \(X_N\) is the quotient graph of \(X\) corresponding to the orbits of \(N\). Furthermore, \(X\) is a regular \(N\)-covering of \(X_N\).

The next proposition is a special case of [23, Theorem 1.1].

**2.6. Proposition.** Let \(X\) be a connected edge-transitive \(Z_u\)-cover of the Heawood graph \(F_{14}\). Then \(n = 3^a \cdot p_1^{\alpha_1} \cdots p_t^{\alpha_t}\), \(k \geq 3\), \(i \geq 2\), and \(\alpha_i = 1\) \((\mod 3)\), and \(X\) is symmetric and isomorphic to a normal Cayley graph \(\text{Cay}(G, S)\) for some group \(G\) with respect to a generating set \(S\). Furthermore, if \(7\)
is coprime to \( n \), then \( G = \langle a, b \mid a^2 = b^{7n} = 1, aba = b^{-1} \rangle \cong D_{14n} \), \( S = \{a, ba, b^{i+1}a\} \), \( t^2 + t + 1 = 0 \pmod{7n} \), and \( X \) is 1-regular.

\[ \Box \]

3. Main results

Let \( p \) be a prime and let \( X \) be a cubic edge-transitive graph of order \( 14p^2 \). By [21], every cubic edge and vertex-transitive graph is arc-transitive and consequently, \( X \) is either symmetric or semisymmetric.

For a prime \( p \geq 13 \), denote by \( EF14p^2 \) the \( Z_p \times Z_p \)-covering of the Heawood graph \( F14 \) with voltage assignment \((2, 0), (-1, 1), (1, -1), (1, 1), (-1, -1), (1, 1), (0, 0), (2, 0)\).

By [2, 3], we have the following lemma.

**3.1. Lemma.** Let \( p \) be a prime and \( X \) a connected cubic symmetric graph of order \( 14p^2 \), where \( p < 13 \). Then \( X \) is isomorphic to one of the 1-regular graphs \( F56A, F126, F535, F686A, F686C \) and \( F1694 \), or to the 2-regular graphs \( F56B \) or \( F686B \), or to the 3-regular graph \( F696C \).

**Proof.** By Tutte [20], \( X \) is at most 5-regular and hence \(|A| = 2^s \cdot 3 \cdot 7 \cdot p^2 \) for some \( s \), where \( 1 \leq s \leq 5 \). Let \( Q = O_p(A) \) be the maximal normal \( p \)-subgroup of \( A \). We show that \(|Q| = p^2 \) as follows.

Let \( N \) be a minimal normal subgroup of \( A \). Thus \( N \cong L \times \cdots \times L = L^k \), where \( L \) is a simple group. If \( N \) is unsolvable then by [4], \( L \cong PSL(2, 7) \) or \( PSL(2, 13) \) of orders \( 2^3 \cdot 3 \cdot 7 \) and \( 2^2 \cdot 3 \cdot 7 \cdot 13 \), respectively. Since \( 3^2 \mid |A| \), we have \( k = 1 \) and so \( N \cong PSL(2, 7) \) or \( PSL(2, 13) \). Thus \( N \) has more than two orbits and then by Proposition 2.5, \( N \) is semiregular. Therefore, \(|N| \mid 14p^2 \), and this is impossible. Hence \( N \) is solvable and so elementary abelian.

Suppose first that \( Q = 1 \). Thus \( N \) is an elementary abelian \( q \)-group, for \( q = 2, 3 \) or 7 and so \( N \) has more than two orbits on \( X \). By Proposition 2.5, \( N \) is semiregular and hence \(|N| \mid 14p^2 \). It follows that \(|N| = 2 \) or 7. If \(|N| = 2 \), by Proposition 2.5 \( X_N \) is a cubic symmetric graph of odd order \( 7p^2 \), a contradiction.

Suppose that \(|N| = 7 \). By Proposition 2.5, \( X_N \) is a cubic \( A/N \)-symmetric graph of order \( 2p^2 \). Let \( T/N \) be a minimal normal subgroup of \( A/N \). By a similar argument as above, \( T/N \) is elementary abelian and hence \(|T/N| = 2 \) or \( p \). If \(|T/N| = 2 \), then \(|T| = 14 \) and \( X_T \) is a cubic symmetric graph of odd order \( p^2 \), a contradiction. So, \(|T/N| = p \) and also \(|T| = 7p \). Since \( p \geq 13 \), the Sylow \( p \)-subgroup of \( T \) is characteristic and so normal in \( A \), a contrary to the our assumption that \( Q = 1 \).

We now suppose that \(|Q| = p \). Let \( P \) be a Sylow \( p \)-subgroup of \( A \) and \( C = C_A(Q) \) the centralizer of \( Q \) in \( A \). Clearly, \( Q < P \) and also \( P \leq C \) because \( P \) is abelian. Thus \( p^2 \mid |C| \). If \( p^3 \mid |C'| \) (\( C' \) is the derived subgroup of \( C \)) then \( Q \leq C' \) and hence \( p \mid |C' \cap Q| \), forcing that \( p \mid |C' \cap Z(C)| \) because \( Q \leq Z(C) \). This contradicts Proposition 2.3. Consequently, \( p^2 \mid |C'| \) and so \( C' \) has more than two orbits on \( X \). By Proposition 2.5, \( C' \) is semiregular on \( X \) and hence \(|C'| \mid 14p^2 \).

Let \( K/C' \) be a Sylow \( p \)-subgroup of \( C/C' \). Since \( C/C' \) is abelian, \( K/C' \) is characteristic and hence normal in \( A/C' \), implying that \( K \nmid A \). Note that \( p^2 \mid |K| \) and \(|K| \mid 14p^2 \). If \(|K| = 14p^2 \) then \( K \) has a normal subgroup of order \( 7p^2 \), say \( H \). Since \( p \geq 13 \), the Sylow \( p \)-subgroup of \( H \) is characteristic and consequently normal in \( K \) and also normal
in $A$. Also, if $|K| < 14p^2$, $K$ has a characteristic Sylow $p$-subgroup of order $p^2$ which is normal in $A$. However, this is contrary to our assumption $|Q| = p$. Therefore, $|Q| = p^2$.

Clearly, $Q \cong Z_{p^2}$ or $Z_p \times Z_p$. Then by Proposition 2.5, $X$ is a regular $Q$-covering of the symmetric graph $X_Q$ of order 14. By [3] the only cubic symmetric graph of order 14 is the Heawood graph $F_{14}$. Suppose that $Q \cong Z_{p^2}$. Since $p \geq 13$, it is coprime to $p^2$ and hence by Proposition 2.6, $X$ is isomorphic to a 1-regular graph $Cay(G, S)$, where $G = \langle a, b | a^2 = b^{7p^2} = 1, aba = b^{-1} \rangle \cong D_{14p^2}$, $S = \{a, ba, b^{13}a\}$, $t^2 + t + 1 = 0$ (mod $7p^2$), $p \geq 13$ and $3(p - 1)$.

Now, suppose that $Q \cong Z_p \times Z_p$. Then by [18, Table 2], $X$ is isomorphic to $EF14p^2$, where $p \geq 13$. Hence the result follows. □

3.3. Lemma. Let $p$ be a prime. Then, $S_{126}$ is the only cubic semisymmetric graph of order $14p^2$. 

Proof. Let $X$ be a cubic semisymmetric graph of order $14p^2$. If $p < 11$, then by [4] there is only one cubic semisymmetric graph $S_{126}$ of order $14p^2$, in which $p = 3$. Hence we can assume that $p \geq 11$. Set $A := \text{Aut}(X)$. By Proposition 2.2, $|A_o| = 2^{r} \cdot 3$, where $0 \leq r \leq 7$ and hence $|A| = 2^{r} \cdot 3 \cdot 7 \cdot p^2$. Let $Q = O_p(A)$ be the maximal normal $p$-subgroup of $A$. We show that $|Q| = p^2$ as follows.

Let $N$ be a minimal normal subgroup of $A$. Thus $N \cong L^k$, where $L$ is a simple group. Let $N$ be unsolvable. By [5], $L$ is isomorphic to $PSL(2, 7)$ or $PSL(2, 13)$ of orders $2^3 \cdot 7$ and $2^2 \cdot 3 \cdot 7 \cdot 13$, respectively. Note that $3^2 \nmid |A|$, forcing $k = 1$. Also, 3 does not divide $|A/N|$, and hence by Proposition 2.1 $N$ is semisymmetric on $X$. Consequently, $7p^2 \nmid |N|$, a contradiction because $p \geq 11$. Therefore, $N$ is solvable and so elementary abelian. It follows that $N$ acts intransitively on the bipartition sets of $X$, and by Proposition 2.1 it is semiregular on each partition. Hence $|N| \nmid 7p^2$.

Suppose first that $Q = 1$. This implies that $N \cong Z_7$. Consequently, by Proposition 2.1, $X_N$ is a cubic $A/N$-semisymmetric group of order $2p^2$. Let $T/N$ be a minimal normal subgroup of $A/N$. If $T/N$ is unsolvable then by a similar argument as above, $T/N$ is isomorphic to one of the two simple groups in the previous paragraph, implying that $7p^2 \nmid |T|$ and this is impossible. Hence, $T/N$ is solvable and so elementary abelian. If $T/N$ acts transitively on one partition of $X_N$, by Proposition 2.4 $|T/N| = p^2$ and hence $|T| = 7p^2$. Since $p \geq 11$, the Sylow $p$-subgroup of $T$ is characteristic and consequently normal in $A$. It contradicts our assumption $Q = 1$. Therefore, $T/N$ acts intransitively on the bipartition sets of $X_N$ and by Proposition 2.1, it is semiregular on each partition, which force $|T/N| \neq p^2$. Hence $|T/N| = p$ and so $|T| = 7p$. Again, $A$ has a normal $p$-subgroup, a contradiction.

We now suppose that $|Q| = p$. Let $C = C_A(Q)$ be the centralizer of $Q$ in $A$ and $C'$ the derived subgroup of $C$. By the same argument as in the previous lemma, $p^2 \nmid |C'|$ and so $C'$ acts intransitively on the bipartition sets of $X$. Then by Proposition 2.1, it is semiregular and hence $|C'| \nmid 7p^2$.

Let $K/C'$ be a Sylow $p$-subgroup of $C/C'$. Since $C/C'$ is abelian, $K/C'$ is characteristic and hence normal in $A/C'$, implying that $K \triangleleft A$. Note that $p^3 \nmid |K|$ and $|K| \nmid 7p^2$. Then, $K$ has a characteristic Sylow $p$-subgroup of order $p^2$ which is normal in $A$, contrary to our assumption $|Q| = p$.

Therefore, $|Q| = p^2$. Clearly, $Q \cong Z_{p^2}$ or $Z_p \times Z_p$. By Proposition 2.1, the semisymmetric graph $X$ is a regular $Q$-covering of a $A/Q$-semisymmetric graph $X_Q$ of order 14 which is the Heawood graph $F_{14}$ under a projection, say $\phi$. Since $Q \triangleleft A$, the group $A$ is projected along $\phi$ and consequently, $\phi$ is a semisymmetric $Q$-covering projection and also, $X$ is a semisymmetric $Q$-covering of the Heawood graph. But by Proposition 2.6,
there is no semisymmetric $\mathbb{Z}_p$-covering of the Heawood graph and also by [16, Theorem 7.1], there is no semisymmetric $\mathbb{Z}_p \times \mathbb{Z}_p$-covering projection of the Heawood graph, a contradiction. Hence the result follows. □

Now, the proof of Theorem 1.1 follows by Lemmas 3.1, 3.2 and 3.3.

References