

GENERATING FUNCTIONS FOR THE BERNSTEIN TYPE POLYNOMIALS: A NEW APPROACH TO DERIVING IDENTITIES AND APPLICATIONS FOR THE POLYNOMIALS

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Abstract

The main aim of this paper is to construct generating functions for the Bernstein type polynomials. Using these generating functions, various functional equations and differential equations can be derived. New proofs both for a recursive definition of the Bernstein type basis functions and for derivatives of the n th degree Bernstein type polynomials can be given using these equations. This paper presents a novel method for deriving various new identities and properties for the Bernstein type basis functions by using not only these generating functions but also these equations. By applying the Fourier transform and the Laplace transform to the generating functions, we derive interesting series representations for the Bernstein type basis functions. Furthermore, we discuss analytic representations for the generalized Bernstein polynomials through the binomial or Newton distribution and Poisson distribution with mean and variance. By using the mean and the variance, we generalize Szasz-Mirakjan type basis functions.

Keywords: Bernstein polynomials; Generating function; Szasz-Mirakjan basis functions; Bezier curves; Binomial distribution; Poisson distribution; Fourier transform; Laplace transform; Functional equation .

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1. Introduction and main definition

In the literature in *Bezier Curves and Surfaces*, one can find systematic and extensive investigations not only of the classical Bernstein polynomials and Bezier curves, but also of their various generalizations and q -extensions. According to Goldman [7], freeform curves and surfaces are smooth shapes often describing man-made objects. The hood of a car, the hull of a ship, and the fuselage of an airplane are all examples of freeform shapes which differ from the classical surfaces. The classical surfaces are easy to describe with a few parameters. But the hood of a car or the hull of a ship is not easy to describe with a few parameters. Thus recently many scientists and engineers have developed mathematical techniques for describing freeform curves and surfaces. It is also well-known that scientists and engineers use freeform curves and surfaces to interpolate data and to approximate shape. The Bezier curves, which are polynomials curves, have many practical applications, ranging from the design of new fonts to the creation of mechanical components and assemblies for large scale industrial design and manufacture. By using the Bernstein polynomials, one can easily find an explicit polynomial representation for Bezier curves. Therefore, the Bernstein polynomials have many applications in theory of freeform curves and surfaces, in approximations of functions, in statistics, in numerical analysis, in p -adic analysis and in the solution of differential equations. It is also well-known that in Computer Aided Geometric Design polynomials are often expressed in terms of the Bernstein basis functions. The goal of this paper is to develop some of properties underlying the Bernstein polynomials using their novel generating functions.

Many of the known identities for the Bernstein basis functions are currently derived in an *ad hoc* fashion, using either the binomial theorem, the binomial distribution, tricky algebraic manipulations or blossoming. The aim of this paper is to derive functional equations and differential equations using novel generating functions for the Bernstein polynomials. By using these equations, we provide a *new* approach to derive both for standard identities and for new identities for the Bernstein type basis functions.

The organization of the paper is as follows:

In Section 2; We define generating functions for the Bernstein type basis functions. We find many functional equations and differential equations of this novel generating function. Using these equations, many properties of the Bernstein type basis functions can be determined. For instance, we give sum and alternating sum of the Bernstein type basis functions, some well-known properties of the Bernstein type basis functions, subdivision property, a recursive definition of the Bernstein type basis functions, derivatives of the n th degree Bernstein basis functions. We also prove many other properties of the Bernstein basis functions via functional equations. In Section 3; we give some application of the Fourier transform and the Laplace transform to the generating functions for the Bernstein type basis functions. We derive series representations for the Bernstein type basis functions. In Section 4; by using novel generating functions and their functional equation, we give some new identities related to the Bernstein type basis function. In Section 5; we give relations between the Bernstein basis functions, the binomial distribution and the Poisson distribution. Using the Poisson distribution, we give generating functions for the Szasz-Mirakjan type basis functions. By using Abel and Li's method [1], and applying our generating functions to Proposition 5.1, we derive identities which give pointwise orthogonality relations for the Bernstein polynomials and the Szasz-Mirakjan type basis functions.

2. New approach to deriving new proofs of the identities and properties for the Bernstein type basis functions

In this section, we provide fundamental properties of the Bernstein basis functions and their generating functions. We introduce some functional equations and differential equations of the novel generating functions for the Bernstein basis functions. We also give new proofs of some well known properties of the Bernstein basis functions by using functional equations and differential equations.

2.1. Generating Functions. Recently the Bernstein polynomials have been defined and studied in many different ways, for example, by q -series, by complex functions, by p -adic Volkenborn integrals and many algorithms. Here, by using entire function, related to nonnegative real parameters, we construct generating functions for the Bernstein type basis functions.

The Bernstein type basis functions $\mathbb{Y}_k^n(x; a, b, m)$ are defined as follows:

2.1. Definition. Let a and b be nonnegative real parameters with $a \neq b$. Let m be a positive integer and let $x \in [a, b]$. Let n be non-negative integer. The Bernstein type basis functions $\mathbb{Y}_k^n(x; a, b, m)$ can be defined by

$$(2.1) \quad \mathbb{Y}_k^n(x; a, b, m) = \binom{n}{k} \frac{(x-a)^k (b-x)^{n-k}}{(b-a)^m},$$

where

$$k = 0, 1, \dots, n,$$

and

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Remark 1. In the special case when $m = n$, Definition 2.1 immediately yields the corresponding well known results concerning the Bernstein basis functions $B_k^n(x, a, b)$ that appears, for example, in Goldman [7, p. 384, Eq.(24.6)] and cf. [3]:

$$\mathbb{Y}_k^n(x; a, b, n) = B_k^n(x; a, b) = \binom{n}{k} \left(\frac{x-a}{b-a}\right)^k \left(\frac{b-x}{b-a}\right)^{n-k},$$

where $k = 0, 1, \dots, n$ and $x \in [a, b]$ (cf., see also [5]). One can easily see that

$$(2.2) \quad B_k^n(x) = \binom{n}{k} x^k (1-x)^{n-k},$$

where $k = 0, 1, \dots, n$ and $x \in [0, 1]$ cf. [1]-[19]. In [7], Goldman gives many properties of the Bernstein polynomials $B_k^n(x, a, b)$. The functions $B_0^n(x, a, b), \dots, B_n^n(x, a, b)$ are called the Bernstein basis functions. Goldman [7, Chapter 26], shows that the Bernstein basis functions form a basis for the polynomials of degree n .

Generating functions for the Bernstein type basis functions can be defined as follows:

2.2. Definition. Let a and b be nonnegative real parameters with $a \neq b$. Let $t \in \mathbb{C}$. Let m be a positive integer and let $x \in [a, b]$. The Bernstein type basis functions can be defined by means of the following generating function

$$(2.3) \quad f_{\mathbb{Y},k}(x, t; a, b, m) := \sum_{n=0}^{\infty} \mathbb{Y}_k^n(x; a, b, m) \frac{t^n}{n!},$$

where $k = 0, 1, \dots, n$.

We construct novel generating functions for the Bernstein type basis functions explicitly by the following theorem:

2.3. Theorem. Let a and b be nonnegative real parameters with $a \neq b$. Let $t \in \mathbb{C}$. Let m be a positive integer and let $x \in [a, b]$. Then we have

$$(2.4) \quad f_{\mathbb{Y},k}(x, t; a, b, m) = \frac{t^k (x-a)^k e^{(b-x)t}}{(b-a)^m k!}.$$

Proof. By using (2.1) and (2.3), we have

$$\sum_{n=0}^{\infty} \mathbb{Y}_k^n(x; a, b, m) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \binom{n}{k} \frac{(x-a)^k (b-x)^{n-k} t^n}{(b-a)^m n!}.$$

From this equation, we obtain

$$\sum_{n=0}^{\infty} \mathbb{Y}_k^n(x; a, b, m) \frac{t^n}{n!} = \frac{(x-a)^k t^k}{k!(b-a)^m} \sum_{n=k}^{\infty} \frac{(b-x)^{n-k} t^{n-k}}{(n-k)!}.$$

The series on the right hand side is the Taylor series for $e^{(b-x)t}$. Thus we are led to the formula (2.4) asserted by Theorem 2.3. \square

Alternative form of the generating functions for the Bernstein type basis functions can be given as follows

$$(2.5) \quad \frac{t^k (x-a)^k}{(b-a)^m k!} = f_{\mathbb{Y},k}(x, t; a, b, m) e^{(x-b)t}.$$

Substituting $m = n$ in (2.1), we now give *another well-known generating function* for the Bernstein basis functions:

$$\sum_{n=0}^{\infty} \left(\sum_{k=0}^n B_k^n(x; a, b) t^k \right) \frac{z^n}{n!} = \sum_{n=0}^{\infty} \left(\sum_{k=0}^n \binom{n}{k} t^k \left(\frac{x-a}{b-a} \right)^k \left(\frac{b-x}{b-a} \right)^{n-k} \right) \frac{z^n}{n!}.$$

By using the Cauchy product in the above equation, we have

$$\sum_{n=0}^{\infty} \left(\sum_{k=0}^n B_k^n(x; a, b) t^k \right) \frac{z^n}{n!} = \sum_{n=0}^{\infty} \left(t \frac{x-a}{b-a} \right)^n \frac{z^n}{n!} \sum_{n=0}^{\infty} \left(\frac{b-x}{b-a} \right)^n \frac{z^n}{n!}.$$

From this equation, we find that

$$\sum_{n=0}^{\infty} \left(\sum_{k=0}^n B_k^n(x; a, b) t^k \right) \frac{z^n}{n!} = e^{z \left(\frac{b-x}{b-a} + t \frac{x-a}{b-a} \right)}.$$

After some elementary calculations in the above relation, we arrive at the following *generating function* for the Bernstein basis functions:

$$(2.6) \quad \sum_{k=0}^n B_k^n(x; a, b) t^k = \left(\frac{b-x}{b-a} + t \frac{x-a}{b-a} \right)^n.$$

Remark 2. If we set $a = 0$ and $b = 1$ in (2.6), then we have

$$(2.7) \quad \sum_{k=0}^n B_k^n(x) t^k = ((1-x) + tx)^n.$$

This generating function is given by Goldman [9]-[8, Chapter 5, pp. 299-306]. Goldman [9]-[8, Chapter 5, pp. 299-306] also constructs the following generating function for the Bernstein basis functions:

$$\sum_{k=0}^n B_k^n(x) e^{ky} = ((1-x) + te^y)^n.$$

Remark 3. If we set $a = 0$ and $b = 1$ in (2.4), we obtain a result given by Simsek [18], Simsek et al. [19] and Acikgoz et al. [2]:

$$\frac{(xt)^k}{k!} e^{(1-x)t} = \sum_{n=0}^{\infty} B_k^n(x) \frac{t^n}{n!},$$

so that, obviously;

$$\mathbb{Y}_k^n(x; 0, 1, n) = B_k^n(x),$$

where $B_k^n(x)$ denote the Bernstein basis functions.

2.2. Bernstein type polynomials. A Bernstein type polynomial $\mathcal{P}(x, a, b, m)$ is a polynomial represented in the Bernstein basis functions:

$$(2.8) \quad \mathcal{P}(x, a, b, m) = \sum_{k=0}^n c_k^n \mathbb{Y}_k^n(x; a, b, m).$$

Remark 4. If we set $a = 0$, $b = 1$ and $m = n$ in (2.8), then we have

$$P(x) = \sum_{k=0}^n c_k^n B_k^n(x)$$

(cf. [4]).

2.3. Bezier type curve. We define the *Bezier type curve* $B(x, a, b)$ with control points

$$P_0, \dots, P_n$$

as follows:

$$(2.9) \quad B(x, a, b; m) = \sum_{k=0}^n P_k \mathbb{Y}_k^n(x, a, b, m).$$

Remark 5. In the special case when $m = n$, Equation (2.9) yields the corresponding well known results concerning the Bezier curve $B(x, a, b)$ with control points P_0, \dots, P_n defined as follows (cf. [7]):

$$B(x, a, b) = \sum_{k=0}^n P_k B_k^n(x, a, b).$$

2.4. Some well-known properties of the Bernstein type basis functions. Below are some well-known properties of the Bernstein type basis functions:

Non-negative property:

$$(2.10) \quad \mathbb{Y}_k^n(x; a, b, m) \geq 0, \text{ for } 0 \leq a \leq x \leq b.$$

Symmetry property:

$$(2.11) \quad \mathbb{Y}_k^n(x; a, b, m) = \mathbb{Y}_{n-k}^n(b + a - x; a, b, m).$$

Corner values:

$$(2.12) \quad \mathbb{Y}_k^n(a; a, b, n) = \begin{cases} 0 & \text{if } k \neq 0, \\ 1 & \text{if } k = 0, \end{cases}$$

and

$$(2.13) \quad \mathbb{Y}_k^n(b; a, b, n) = \begin{cases} 0 & \text{if } k \neq n, \\ 1 & \text{if } k = n. \end{cases}$$

Remark 6. If we set $a = 0$, $b = 1$ and $m = n$, then (2.10)-(2.13) reduce to Goldman's results [9]-[8, Chapter 5, pp. 299-306]. In [9] and [8, Chapter 5, pp. 299-306], Goldman also gives many identities and properties for the univariate and bivariate Bernstein basis

functions, for example boundary values, maximum values, partitions of unity, representation of monomials, representation in terms of monomials, conversion to monomial form, linear independence, Descartes' law of sign, discrete convolution, unimodality, subdivision, directional derivatives, integrals, Marsden identities, De Boor-Fix formulas, and the other properties.

In the next section, by using the same method in [18], we give some functional equations. By using this equations, we find sum and alternating sum of the Bernstein basis functions.

2.5. Sum of the Bernstein type basis functions. Using the same method proposed in [18], we get the following functional equation:

$$\sum_{k=0}^{\infty} f_{\mathbb{Y},k}(x, t; a, b, m) = \frac{e^{(b-a)t}}{(b-a)^m}.$$

From the above equation, we have the sum of the Bernstein basis functions:

$$\sum_{k=0}^n \mathbb{Y}_k^n(b; a, b, m) = (b-a)^{n-m}.$$

Observe that by substituting $n = m$ into the above equation, we obtain sum of the Bernstein basis function as follows:

$$\sum_{k=0}^n B_k^n(b; a, b) = 1.$$

2.6. Alternating sum of the Bernstein type basis functions. Using the same method proposed in [18], we get the following functional equation:

$$(2.14) \quad \sum_{k=0}^{\infty} (-1)^k f_{\mathbb{Y},k}(x, t; a, b, m) = \frac{e^{(b-a-2x)t}}{(b-a)^m}.$$

By using this equation, we easily arrive at the following alternating sum for the Bernstein type basis functions:

2.4. Theorem.

$$(2.15) \quad \sum_{k=0}^n (-1)^k \mathbb{Y}_k^n(b; a, b, m) = \frac{(b-a-2x)^n}{(b-a)^m}.$$

Remark 7. Substituting $m = n$ in (2.1), we get

$$\sum_{n=0}^{\infty} \left(\sum_{k=0}^n (-1)^k B_k^n(x; a, b, n) \right) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left(\sum_{k=0}^n \frac{\left(\frac{a-x}{b-a}\right)^k \left(\frac{b-x}{b-a}\right)^{n-k}}{k!(n-k)!} \right) t^n.$$

By using the Cauchy product in the above equation, we have

$$\sum_{n=0}^{\infty} \left(\sum_{k=0}^n (-1)^k B_k^n(x; a, b) \right) \frac{t^n}{n!} = e^{\left(\frac{a+b-2x}{b-a}\right)t}.$$

From this relation, we also arrive at the following alternating sum for the Bernstein basis functions:

$$\sum_{k=0}^n (-1)^k B_k^n(x; a, b) = \left(\frac{a+b-2x}{b-a} \right)^n.$$

2.7. Differentiating the generating function. Here, we give higher order derivatives of the Bernstein type basis functions by differentiating the generating function in (2.4) with respect to x . Using Leibnitz's formula for the l th derivative, with respect to x , of the product $f_{\mathbb{Y},k}(x, t; a, b, m)$ of two functions

$$g(t, x; a, b) = \frac{t^k (x - a)^k}{(b - a)^m k!} \quad (a \neq b)$$

and

$$h(t, x; b) = e^{(b-x)t},$$

we obtain the following *higher order partial derivative equation*:

$$(2.16) \quad \frac{\partial^l f_{\mathbb{Y},k}(x, t; a, b, m)}{\partial x^l} = \sum_{j=0}^l \binom{l}{j} \left(\frac{\partial^j g(t, x; a, b)}{\partial x^j} \right) \left(\frac{\partial^{l-j} h(t, x; b)}{\partial x^{l-j}} \right).$$

By using induction on l , Equation (2.16) is easily obtained.

2.5. Theorem. *Let l be a non-negative integer. Then*

$$\frac{\partial^l f_{\mathbb{Y},k}(x, t; a, b, m)}{\partial x^l} = \sum_{j=0}^l \binom{l}{j} (-1)^{l-j} \frac{t^l}{(b-a)^j} f_{\mathbb{Y},k-j}(x, t; a, b, m-j).$$

Proof. By using (2.16), we easily arrive at the desired result. \square

By using Theorem 2.5, we obtain higher order derivatives of the Bernstein type basis functions by the following theorem:

2.6. Theorem. *Let a and b be nonnegative real parameters with $a \neq b$. Let m be a positive integer and let $x \in [a, b]$. Let k, l and n be nonnegative integers with $n \geq k$. Then*

$$\frac{d^l \mathbb{Y}_k^n(x; a, b, m)}{dx^l} = \frac{n!}{(n-l)!} \sum_{j=0}^l (-1)^{l-j} \binom{l}{j} \frac{\mathbb{Y}_{k-j}^{n-l}(x; a, b, m-j)}{(b-a)^j}.$$

Remark 8. Substituting $a = 0, b = 1$ and $m = n$ into Theorem 2.6, we have

$$\frac{d^l B_k^n(x)}{dx^l} = \frac{n!}{(n-l)!} \sum_{j=0}^l (-1)^{l-j} \binom{l}{j} B_{k-j}^{n-l}(x),$$

Substituting $l = 1$ into the above equation, we have

$$\frac{d}{dx} B_k^n(x) = n (B_{k-1}^{n-1}(x) - B_k^{n-1}(x))$$

(cf. [9], [8, Chapter 5, pp. 299-306], [18]) and (cf. [1]-[19]).

2.8. Recurrence Relation. Here, by using higher order derivatives of the novel generating function with respect to t , we derive a partial differential equation. Using this equation, we shall give a new proof of the recurrence relation for the Bernstein type basis functions.

Differentiating Equation (2.4) with respect to t , we prove a recurrence relation for the Bernstein type basis functions. This recurrence relation can also be obtained from Equation (2.1). By using Leibnitz's formula for the v th derivative, with respect to t , of the product $f_{\mathbb{Y},k}(x, t; a, b, m)$ of two function

$$g(t, x; a, b) = \frac{t^k (x - a)^k}{(b - a)^m k!} \quad (a \neq b)$$

and

$$h(t, x; b) = e^{(b-x)t},$$

we obtain another higher order partial differential equation as follows:

$$(2.17) \quad \frac{\partial^v f_{\mathbb{Y},k}(x, t; a, b, m)}{\partial t^v} = \sum_{j=0}^v \binom{v}{j} \left(\frac{\partial^j g(t, x; a, b)}{\partial t^j} \right) \left(\frac{\partial^{v-j} h(t, x; b)}{\partial t^{v-j}} \right).$$

By using induction on v , Equation (2.17) is easily obtained.

2.7. Theorem. *Let v be an integer number. Then*

$$\frac{\partial^v f_{\mathbb{Y},k}(x, t; a, b, m)}{\partial t^v} = \sum_{j=0}^v (b-a)^{v-j} B_j^v(x; a, b) f_{\mathbb{Y},k-j}(x, t; a, b, m-j),$$

where $f_{\mathbb{Y},k}(x, t; a, b, m)$ and $B_j^v(x; a, b)$ are defined in (2.4) and (2.1), respectively.

Proof. Proof of Theorem 2.7 follows immediately from (2.17). \square

Using definition (2.3), (2.1), and Theorem 2.7, we obtain a recurrence relation for the Bernstein type basis functions by the following theorem:

2.8. Theorem. *Let a and b be nonnegative real parameters with $a \neq b$. Let m be a positive integer and let $x \in [a, b]$. Let k, v and n be nonnegative integers with $n \geq k$. Then*

$$\mathbb{Y}_k^n(x; a, b, m) = \sum_{j=0}^v (b-a)^{v-j} B_j^v(x; a, b) \mathbb{Y}_{k-j}^{n-v}(x; a, b, m-j).$$

Remark 9. Substituting $a = 0$ and $b = 1$ into Theorem 2.8, we obtain the following result (cf. [18]):

$$B_k^n(x) = \sum_{j=0}^v B_j^v(x) B_{k-j}^{n-v}(x).$$

Substituting $v = 1$ into above equation, we have (cf. [1]-[19])

$$B_k^n(x) = (1-x)B_k^{n-1}(x) + xB_{k-1}^{n-1}(x).$$

2.9. Multiplication and division by powers of $\left(\frac{x-a}{b-a}\right)^d$ and $\left(\frac{b-x}{b-a}\right)^d$. In [4], Buse and Goldman present much background material on computations with Bernstein polynomials. They provide formulas for multiplication and division of Bernstein polynomials by powers of x and $1-x$ and for degree elevation of Bernstein polynomials. Our method is similar to that of Buse and Goldman's [4]. Here, we find two functional equations. Using these equations, we also give new proofs of both the multiplication and division properties for the Bernstein polynomials.

By using the generating function in (2.4), we provide formulas for multiplying Bernstein polynomials by powers of $\left(\frac{x-a}{b-a}\right)^d$ and $\left(\frac{b-x}{b-a}\right)^d$ and for degree elevation of the Bernstein polynomials.

Using (2.4), we obtain the following functional equation:

$$\left(\frac{x-a}{b-a}\right)^d f_{\mathbb{Y},k}(x, t; a, b, n) = \frac{(k+d)!}{k!t^d} f_{\mathbb{Y},k}(x, t; a, b, n).$$

After elementary manipulations in this equation, we get

$$(2.18) \quad \left(\frac{x-a}{b-a}\right)^d B_k^n(x; a, b) = \frac{n!(k+d)!}{k!(n+d)!} B_{k+d}^{n+d}(x; a, b).$$

Substituting $d = 1$, we have

$$(2.19) \quad \left(\frac{x-a}{b-a}\right) B_k^n(x; a, b) = \frac{k+1}{n+1} B_{k+1}^{n+1}(x; a, b).$$

Remark 10. Substituting $a = 0$ and $b = 1$ into (2.19), we have

$$x B_k^n(x) = \frac{k+1}{n+1} B_{k+1}^{n+1}(x).$$

The above relation can also be proved by (2.2) (cf. [4]).

Similarly, using (2.1), we obtain

$$\left(\frac{b-x}{b-a}\right)^d B_k^n(x; a, b) = \frac{n!(n+d-k)!}{(n+d)!(n-k)!} B_k^{n+d}(x; a, b).$$

Substituting $d = 1$ into the above equation, we have

$$(2.20) \quad \left(\frac{b-x}{b-a}\right) B_k^n(x; a, b) = \frac{n+1-k}{n+1} B_k^{n+1}(x; a, b).$$

Consequently, by the same method as in [4], if we have (2.8), then

$$(2.21) \quad \left(\frac{x-a}{b-a}\right)^d \mathcal{P}(x, a, b) = \sum_{k=0}^n c_k^n \frac{n!(k+d)!}{k!(n+d)!} B_{k+d}^{n+d}(x; a, b),$$

and

$$(2.22) \quad \left(\frac{b-x}{b-a}\right)^d \mathcal{P}(x, a, b) = \sum_{k=0}^n c_k^n \frac{n!(n+d-k)!}{(n+d)!(n-k)!} B_k^{n+d}(x; a, b).$$

We now consider *division* properties. We assume that (2.8) holds and that we are given an integer $j > 0$. Since $\left(\frac{x-a}{b-a}\right)^j$ divides $B_k^n(x; a, b)$ for all $k \geq j$, it follows that $\left(\frac{x-a}{b-a}\right)^j$ divides $\mathcal{P}(x, a, b)$. Similarly, using (2.4), we obtain the following functional equation:

$$\frac{f_{\mathbb{V},k}(x, t; a, b, n)}{\left(\frac{x-a}{b-a}\right)^j} = \frac{(k-f)!t^j}{k!} f_{\mathbb{V},k-j}(x, t; a, b, n-j).$$

For $k \geq j$, from the above equation, we have

$$\frac{B_k^n(x; a, b)}{\left(\frac{x-a}{b-a}\right)^j} = \frac{n!(k-j)!}{k!(n-j)!} B_{k-j}^{n-j}(x; a, b).$$

By a calculation similar to that in [4], for $j \leq n-k$, we have

$$\frac{B_k^n(x; a, b)}{\left(\frac{b-x}{b-a}\right)^j} = \frac{n!(n-j-k)!}{(n-k)!(n-j)!} B_k^{n-j}(x; a, b).$$

Therefore

$$(2.23) \quad \frac{\mathcal{P}(x, a, b)}{\left(\frac{x-a}{b-a}\right)^j} = \sum_{k=j}^n c_k^n \frac{n!(k-j)!}{k!(n-j)!} B_{k-j}^{n-j}(x; a, b),$$

and

$$(2.24) \quad \frac{\mathcal{P}(x, a, b)}{\left(\frac{b-x}{b-a}\right)^j} = \sum_{k=0}^{n-j} c_k^n \frac{n!(n-j-k)!}{(n-k)!(n-j)!} B_k^{n-j}(x; a, b).$$

2.10. Degree elevation. According to Buse and Goldman [4], given a polynomial represented in the univariate Bernstein basis of degree n , degree elevation computes representations of the same polynomial in the univariate Bernstein bases of degree greater than n . Degree elevation allows us to add two or more Bernstein polynomials which are not represented in the same degree Bernstein basis functions.

Adding (2.19) and (2.20), we obtain the degree elevation formula for the Bernstein basis functions:

$$B_k^n(x; a, b) = \frac{k+1}{n+1} B_{k+1}^{n+1}(x; a, b) + \frac{n+1-k}{n+1} B_k^{n+1}(x; a, b).$$

Substituting $d = 1$ into (2.22), and adding it with the latter equations gives the following degree elevation formula for the Bernstein polynomials:

$$(2.25) \quad \mathcal{P}(x, a, b) = \sum_{k=0}^n \left(\frac{k}{n+1} c_{k-1}^n + \frac{n+1-k}{(n+1)} c_k^n \right) B_k^{n+1}(x; a, b),$$

where

$$c_k^{n+1} = \frac{k}{n+1} c_{k-1}^n + \frac{n+1-k}{(n+1)} c_k^n.$$

Remark 11. If we set $a = 0$ and $b = 1$, then Equation (2.25) reduces to Equation (2.5) in [4, p. 853].

3. Application of the Fourier and the Laplace transforms to the generating functions

In this section, by applying the Fourier transform and the Laplace transform to the generating function for the Bernstein basis functions, we obtain some interesting series representations for the Bernstein basis functions.

In [18, p. 5, Eq. (11)], the following functional equation was derived:

$$(3.1) \quad f_{\mathbb{B},j}(xy, t) = f_{\mathbb{B},j}(x, ty) e^{t(1-y)}.$$

From this generating function, we obtain subdivision property for the Bernstein basis functions (see [18]):

$$B_j^n(xy) = \sum_{k=j}^n B_j^k(x) B_k^n(y)$$

cf. (see also [9]-[8, Chapter 5, pp. 299-306]).

By using (3.1), we obtain functional equation

$$f_{\mathbb{B},k}(xy, t) e^{-t} = f_{\mathbb{B},k}(x, ty) e^{-ty}.$$

For $a = 0$ and $b = 1$, combining (2.4) with the above equation, we get

$$(3.2) \quad \sum_{n=0}^{\infty} B_k^n(xy) \frac{t^n}{n!} e^{-t} = \sum_{n=0}^{\infty} B_k^n(x) y^n \frac{t^n}{n!} e^{-ty}.$$

Integrate this equation (by parts) with respect to t from 0 to ∞ , we get

$$\sum_{n=0}^{\infty} \frac{B_k^n(xy)}{n!} \int_0^{\infty} t^n e^{-t} dt = \sum_{n=0}^{\infty} \frac{B_k^n(x) y^n}{n!} \int_0^{\infty} t^n e^{-ty} dt.$$

By using the Laplace transform in the above equation, we arrive at the following Theorem:

3.1. Theorem. Let $x, y \in [0, 1]$. The following relationship holds true:

$$\sum_{n=0}^{\infty} B_k^n(xy) = \sum_{n=0}^{\infty} \frac{1}{y} B_k^n(x).$$

From (2.4), we define the following functional equation:

$$\frac{t^k (x-a)^k}{(b-a)^m k!} e^{-xt} = \sum_{n=0}^{\infty} \mathbb{Y}_k^n(x; a, b, m) \frac{t^n}{n!} e^{-bt}.$$

By applying the Fourier transform to the above equation,

$$\frac{(x-a)^k}{(b-a)^m k!} \int_0^{\infty} t^k e^{-xt} e^{-ist} dt = \sum_{n=0}^{\infty} \mathbb{Y}_k^n(x; a, b, m) \frac{1}{n!} \int_0^{\infty} t^n e^{-bt} e^{-ist} dt.$$

From this equation, we arrive at the following Theorem:

3.2. Theorem. Let $x \in [a, b]$ and $s \in \mathbb{R}$. We have

$$\sum_{n=0}^{\infty} \frac{\mathbb{Y}_k^n(x; a, b, m)}{(b+is)^{n+1}} = \frac{(x-a)^k}{(b-a)^m (x+is)^{k+1}},$$

where $\left| \frac{b-x}{b+is} \right| < 1$.

4. New Identities

By using novel generating functions, we derive some new identities related to the Bernstein type basis function.

4.1. Theorem.

$$\sum_{j=0}^n \sum_{k=0}^j (-1)^k \binom{n}{j} \mathbb{Y}_k^j(x; a, b, m) (2x)^{n-j} = (b-a)^{n-m}.$$

Proof. By using (2.14), we obtain

$$(4.1) \quad \sum_{k=0}^{\infty} (-1)^k f_{\mathbb{Y},k}(x, t; a, b, m) e^{2xt} = \frac{1}{(b-a)^m} e^{(b-a)t}.$$

From this equation, we get

$$\sum_{n=0}^{\infty} \sum_{k=0}^n (-1)^k \mathbb{Y}_k^n(x; a, b, m) \frac{t^n}{n!} \sum_{n=0}^{\infty} (2x)^n \frac{t^n}{n!} = \sum_{n=0}^{\infty} (b-a)^{n-m} \frac{t^n}{n!}.$$

Therefore

$$\sum_{n=0}^{\infty} \left(\sum_{j=0}^n \sum_{k=0}^j (-1)^k \binom{n}{j} \mathbb{Y}_k^j(x; a, b, m) (2x)^{n-j} \right) \frac{t^n}{n!} = \sum_{n=0}^{\infty} (b-a)^{n-m} \frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on the both sides of the above equation, we arrive at the the desired result. \square

4.2. Theorem.

$$\sum_{k=j}^n (-1)^{n-k} \binom{n}{k} B_j^k(xy) = y^n \sum_{k=j}^n (-1)^{n-k} \binom{n}{k} B_j^k(x).$$

Proof. Using (3.2), we obtain

$$\sum_{n=0}^{\infty} B_k^n(xy) \frac{t^n}{n!} \sum_{n=0}^{\infty} (-1)^n \frac{t^n}{n!} = \sum_{n=0}^{\infty} B_k^n(x) y^n \frac{t^n}{n!} \sum_{n=0}^{\infty} (-y)^n \frac{t^n}{n!}.$$

From the above equation, we get

$$\sum_{n=j}^{\infty} \left(\sum_{k=j}^n (-1)^{n-k} \binom{n}{k} B_j^k(xy) \right) \frac{t^n}{n!} = \sum_{n=j}^{\infty} \left(y^n \sum_{k=j}^n (-1)^{n-k} \binom{n}{k} B_j^k(x) \right) \frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on the both sides of the above equation, we arrive at the the desired result. \square

5. Further remarks and observations on the generating functions $f_{\mathbb{Y},k}(x, t; a, b, m)$, Poisson distribution and Szasz-Mirakjan type basis functions

The identity of Jetter and Stöckler represents a pointwise orthogonality relation for the multivariate Bernstein polynomials on a simplex. This identity give us a new representation for the dual basis which can be used to construct general quasi-interpolant operators (cf., see, for details, [10] and [1]). As an application of the generating functions for the basis functions to the identity of Jetter and Stöckler, Abel and Li [1] proved Proposition 5.1, which is given in this section. Applying our generating functions to Proposition 5.1, we give pointwise orthogonality relations for the Bernstein polynomials and the Szasz-Mirakjan basis functions.

In this section, we give relations between the Bernstein basis functions, the binomial distribution and the Poisson distribution. First we consider the generalized binomial or Newton distribution (probability function). Suppose that $0 \leq \frac{x-a}{b-a} \leq 1$ and $0 \leq \frac{b-x}{b-a} \leq 1$. Set

$$(5.1) \quad B_k^n(x; a, b) = \binom{n}{k} \left(\frac{x-a}{b-a} \right)^k \left(\frac{b-x}{b-a} \right)^{n-k}.$$

Remark 12. If we set $a = 0$ and $b = 1$, then (5.1) reduces to

$$B_k^n(x) = \binom{n}{k} x^k (1-x)^{n-k}$$

which is the binomial or Newton distribution (probabilities) function. If $0 \leq x \leq 1$ is the probability of an event E , then $B_k^n(x)$ is the probability that E will occur exactly k times in n independent trials (cf. [13]).

Expected value or mean and variance of $B_k^n(x; a, b)$ are given by

$$\mu = \sum_{k=0}^n k B_k^n(x; a, b) = n \left(\frac{x-a}{b-a} \right),$$

and

$$\sigma^2 = \sum_{k=0}^n k^2 B_k^n(x; a, b) - \mu^2 = \frac{n(x-a)(b-x)}{(b-a)^2}.$$

If we let $n \rightarrow \infty$ in (5.1), then we arrive at the well-known Poisson distribution:

$$(5.2) \quad B_k^n \left(\frac{b-a}{n} \mu + a; a, b \right) \rightarrow \frac{\mu^k e^{-\mu}}{k!}.$$

The following proposition is proved by Abel and Li [1, p. 300, Proposition 3]:

5.1. Proposition. *Let the system $\{f_n(x)\}$ of functions be defined by the generating function*

$$A_t(x) = \sum_{n=0}^{\infty} f_n(x)t^n.$$

If there exists a sequence $w_k = w_k(x)$ such that

$$\sum_{k=0}^{\infty} w_k \mathcal{D}^k A_t(x) \mathcal{D}^k A_z(x) = A_{tz}(x)$$

with $\mathcal{D} = \frac{d}{dx}$, then we have

$$\sum_{k=0}^{\infty} w_k \mathcal{D}^k f_i(x) \mathcal{D}^k f_j(x) = \delta_{i,j} f_i(x), (i, j = 0, 1, \dots).$$

As an application of Proposition 5.1, Abel and Li [1] use the generating function in Equation (2.7) for the Bernstein basis functions. They also use generating functions for the Szasz-Mirakjan basis functions and Baskakov basis functions.

In this section, we apply our novel generating functions to Proposition 5.1, which give pointwise orthogonality relations for the Bernstein polynomials and the Szasz-Mirakjan type basis functions, respectively.

As applications of Proposition 5.1, we give the following examples:

Example 1. For given n and k , the Bernstein basis functions

$$f_k(x, n; a, b) = B_k^n(x; a, b) = \binom{n}{k} \left(\frac{x-a}{b-a}\right)^k \left(\frac{b-x}{b-a}\right)^{n-k}$$

are generated by the function in (2.4), that is

$$A_t(x) = \frac{t^k (x-a)^k e^{(b-x)t}}{(b-a)^n k!} = \sum_{k=0}^{\infty} \frac{f_k(x, n; a, b)}{k!} t^k.$$

It is easy to check that Proposition 5.1 holds with $w_k = w_k(x) = B_k^n(x; a, b)$.

Example 2. Using (5.2), for $j \geq 0$, we generalize the *Szasz-Mirakjan type basis functions* as follows

$$f_j(x, n; a, b) = \frac{\left(n \frac{x-a}{b-a}\right)^j e^{-n \frac{x-a}{b-a}}}{j!},$$

where a and b are nonnegative real parameters with $a \neq b$, n is a positive integer and $x \in [a, b]$. The functions $f_j(x, n; a, b)$ are generated by

$$A_t(x) = \exp\left((t-1)n \left(\frac{x-a}{b-a}\right)\right) = \sum_{i=0}^{\infty} f_i(x, n; a, b) t^i,$$

where $\exp(x) = e^x$. In this case, Proposition 5.1 holds with $w_k = w_k(x) = \frac{\left(\frac{x-a}{b-a}\right)^k}{n^k k!}$. Therefore, we have

$$\sum_{k=0}^{\infty} \frac{\left(\frac{x-a}{b-a}\right)^k}{n^k k!} \mathcal{D}^k f_i(x, n; a, b) \mathcal{D}^k f_j(x, n; a, b) = \delta_{i,j} f_i(x, n; a, b).$$

Remark 13. If $a = 0$ and $b = 1$ in Example 2, then we arrive at the *Szasz-Mirakjan basis functions* which are given in [1, p. 300, Example 2].

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