SOME SUBCLASSES OF ANALYTIC FUNCTIONS INVOLVING THE GENERALIZED SRIVASTAVA-ATTIYA OPERATOR

Zhi-Hong Liu*, Zhi-Gang Wang†‡, Feng-Hua Wen§ and Yong Sun¶

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Abstract
In the present paper, we introduce and investigate some new subclasses of multivalent analytic functions involving the generalized Srivastava-Attiya operator. Such results as inclusion relationships, subordination and superordination properties, integral-preserving properties and convolution properties are proved.

Keywords: Analytic functions, Multivalent functions, Differential subordination, Superordination, Hadamard product (or convolution), Generalized Srivastava-Attiya operator.


1. Introduction
Let \( A_p(n) \) denote the class of functions of the form

\[
f(z) = z^p + \sum_{k=n}^{\infty} a_{p+k} z^{p+k}
\]

which are analytic in the open unit disk

\( \mathbb{U} := \{z: z \in \mathbb{C} \text{ and } |z| < 1\} \).

*Department of Mathematics, Honghe University, Mengzi 661100, Yunnan, People’s Republic of China. E-mail: liuzhihongmath@163.com
†School of Mathematics and Statistics, Anyang Normal University, Anyang 455002, Henan, People’s Republic of China. E-mail: wangmath@163.com
‡Corresponding Author.
§School of Econometrics and Management, Changsha University of Science and Technology, Changsha 410114, Hunan, People’s Republic of China. E-mail: wfh@126.com
¶Department of Mathematics, Huaihua University, Huaihua 418008, Hunan, People’s Republic of China. E-mail: sy785153@126.com
For simplicity, we write
\[ A_1(1) := \mathcal{A}. \]
Also let \( H[a, n] \) be the class of analytic functions of the form
\[ h(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots \quad (z \in \mathbb{U}). \]
Let \( f, g \in \mathcal{A}_p(n) \), where \( f \) is given by (1.1) and \( g \) is defined by
\[ g(z) = z^p + \sum_{k=n}^{\infty} b_{p+k} z^{p+k}. \]
Then the Hadamard product (or convolution) \( f \ast g \) of the functions \( f \) and \( g \) is defined by
\[ (f \ast g)(z) := z^p + \sum_{k=n}^{\infty} a_{p+k} b_{p+k} z^{p+k} =: (g \ast f)(z). \]
Let \( \mathcal{P} \) denote the class of functions of the form
\[ p(z) = 1 + \sum_{k=n}^{\infty} p_k z^k \quad (n \in \mathbb{N}), \]
which are analytic and convex in \( \mathbb{U} \) and satisfy the condition
\[ \Re(p(z)) > 0 \quad (z \in \mathbb{U}). \]
For two functions \( f \) and \( g \), analytic in \( \mathbb{U} \), the function \( f \) is said to be subordinate to \( g \) in \( \mathbb{U} \), or the function \( g \) is said to be superordinate to \( f \) in \( \mathbb{U} \), and write
\[ f(z) \prec g(z) \quad (z \in \mathbb{U}), \]
if there exists a Schwarz function \( \omega \), which is analytic in \( \mathbb{U} \) with
\[ \omega(0) = 0 \quad \text{and} \quad |\omega(z)| < 1 \quad (z \in \mathbb{U}) \]
such that
\[ f(z) = g(\omega(z)) \quad (z \in \mathbb{U}). \]
Indeed, it is known that
\[ f(z) \prec g(z) \quad (z \in \mathbb{U}) \iff f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}). \]
Furthermore, if the function \( g \) is univalent in \( \mathbb{U} \), then we have the following equivalence:
\[ f(z) \prec g(z) \quad (z \in \mathbb{U}) \iff f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}). \]
The following we recall a general Hurwitz-Lerch Zeta function \( \Phi(z, s, a) \) defined by (cf., e.g., [22, p. 121 et sep.])
\[ \Phi(z, s, a) := \sum_{k=0}^{\infty} \frac{z^k}{(k + a)^s} \]
\[ (a \in \mathbb{C} \setminus \mathbb{Z}_0^{-}; \quad s \in \mathbb{C} \text{ when } |z| < 1; \quad \Re(s) > 1 \text{ when } |z| = 1), \]
where, as usual,
\[ \mathbb{Z}_0^{-} := \mathbb{Z} \setminus \mathbb{N} \quad (\mathbb{Z} := \{0, \pm 1, \pm 2, \ldots\}; \quad \mathbb{N} := \{1, 2, 3, \ldots\}). \]
Several interesting properties and characteristics of the Hurwitz-Lerch Zeta function \( \Phi(z, s, a) \) can be found in recent investigations by (for example) Choi and Srivastava [1], Ferreira and López [3], Garg et al. [4], Lin et al. [6], Luo and Srivastava [10], Wen and Liu [26], Wen and Yang [27] and others.
Recently, Srivastava and Attiya [21] (see also [2, 5, 8, 9, 14, 15, 16, 17, 18, 23, 24, 25, 28, 29]) introduced and investigated the linear operator
\[ J_s, b(f) : A \rightarrow A \]
defined, in terms of the Hadamard product (or convolution), by
\[ J_s, b(f) := G_s, b(z) * f(z) \quad (z \in U; \ b \in \mathbb{C} \setminus \mathbb{Z}_0^+; \ s \in \mathbb{C}; \ f \in A), \]
where, for convenience,
\[ G_s, b(z) := (1 + b)^s \Phi(z, s, b) - b^{-s} \quad (z \in U). \]
It is easy to observe from (1.2) and (1.3) that
\[ J_s, b(f) = z + \sum_{k=2}^{\infty} \left( \frac{1 + b}{k + b} \right)^s a_k z^k. \]
By setting
\[ f_{s, b}^p, n(z) := z^p + \sum_{k=n}^{\infty} \left( \frac{p + b}{p + k + b} \right)^s z^{p+k} \quad (z \in U; \ n \in \mathbb{N}). \]
Then, motivated essentially by the above-mentioned Srivastava-Attiya operator, we introduce the operator
\[ J_{s, b}^p, n(f) : A_p(n) \rightarrow A_p(n), \]
which is defined as
\[ J_{s, b}^p, n(f) := f_{s, b}^p, n(z) * f(z) = z^p + \sum_{k=n}^{\infty} \left( \frac{p + b}{p + k + b} \right)^s a_{p+k} z^{p+k}, \]
where (and throughout this paper unless otherwise mentioned) the parameters \( s, b, p \) and \( n \) are constrained as follows:
\[ s \in \mathbb{C}; \ b \in \mathbb{C} \setminus \mathbb{Z}_0^+ \text{ and } p, n \in \mathbb{N}. \]
It is easily verified from (1.4) that
\[ J_{s, b}^p, n(f) = (p + b) J_{s, b}^p, n(f) - b J_{s+1, b}^p, n(f). \]
In this paper, by making use of the operator \( J_{s, b}^p, n \) and the above-mentioned principle of subordination between analytic functions, we introduce and investigate the following subclasses of the class \( A_p(n) \) of \( p \)-valent analytic functions.

1.1. Definition. A function \( f \in A_p(n) \) is said to be in the class \( S_{s, b}^p, n(\eta; \phi) \) if it satisfies the subordination condition
\[ \frac{1}{p - \eta} \left( z \frac{J_{s, b}^p, n(f)}{J_{s, b}^p, n(f)} ' (z) \right) - \eta < \phi(z) \quad (z \in U; \ 0 \leq \eta < p; \ \phi \in \mathbb{P}). \]

1.2. Definition. A function \( f \in A_p(n) \) is said to be in the class \( K_{s, b}^p, n(\lambda; \phi) \) if it satisfies the subordination condition
\[ (1 - \lambda) \frac{J_{s, b}^p, n(f)}{z^p} + \lambda \frac{J_{s+1, b}^p, n(f)}{z^p} < \phi(z) \quad (z \in U; \ \lambda \in \mathbb{C}; \ \phi \in \mathbb{P}). \]

In the present paper, we aim at proving such results as inclusion relationships, subordination and superordination properties, integral-preserving properties and convolution properties for the classes \( S_{s, b}^p, n(\eta; \phi) \) and \( K_{s, b}^p, n(\lambda; \phi) \).
2. Preliminary results

In order to prove our main results, we need the following lemmas.

2.1. Lemma. (see [11]) Let \( \vartheta, \gamma \in \mathbb{C} \). Suppose that \( \varphi \) is convex and univalent in \( U \) with \( \varphi(0) = 1 \) and \( \Re(\vartheta \varphi(z) + \gamma) > 0 \) (\( z \in U \)).

If \( p \) is analytic in \( U \) with \( p(0) = 1 \), then the following subordination

\[
p(z) + \frac{zp'(z)}{\varphi(z) + \gamma} \prec \varphi(z) \quad (z \in U)
\]

implies that

\[
p(z) \prec \varphi(z) \quad (z \in U).
\]

2.2. Lemma. (see [12]) Let the function \( \Omega \) be analytic and convex (univalent) in \( U \) with \( \Omega(0) = 1 \). Suppose also that the function \( \Theta \) given by

\[
\Theta(z) = 1 + c_nz^n + c_{n+1}z^{n+1} + \cdots
\]

is analytic in \( U \). If

\[
(2.1) \quad \Theta(z) + \frac{z\Theta'(z)}{\zeta} \prec \Omega(z) \quad (\Re(\zeta) > 0; \zeta \neq 0; z \in U),
\]

then

\[
\Theta(z) \prec \chi(z) = \frac{\zeta}{n} z^{-\frac{\zeta}{n}} \int_{0}^{1} t^{\frac{\zeta}{n} - 1} h(t) \, dt \prec \Omega(z) \quad (z \in U),
\]

and \( \chi \) is the best dominant of (2.1).

Denote by \( Q \) the set of all functions \( f \) that are analytic and injective on \( U - E(f) \), where

\[
E(f) = \left\{ \varepsilon \in \partial U : \lim_{z \to \varepsilon} f(z) = \infty \right\},
\]

and such that \( f'(z) \neq 0 \) for \( \varepsilon \in \partial U - E(f) \).

2.3. Lemma. (see [13]) Let \( q \) be convex univalent in \( U \) and \( \kappa \in \mathbb{C} \). Further assume that \( \Re(\kappa) > 0 \). If

\[
p \in \mathcal{H}[q(0), 1] \cap Q,
\]

and \( p + \kappa z p' \) is univalent in \( U \), then

\[
q(z) + \kappa z q'(z) \prec p(z) + \kappa z p'(z)
\]

implies \( q \prec p \) and \( q \) is the best subdominant.

2.4. Lemma. (see [19]) Let \( q \) be a convex univalent function in \( U \) and let \( \sigma, \eta \in \mathbb{C} \) with

\[
\Re \left( 1 + \frac{z q''(z)}{q'(z)} \right) > \max \left\{ 0, -\Re \left( \frac{\sigma}{\eta} \right) \right\}.
\]

If \( p \) is analytic in \( U \) and

\[
\sigma p(z) + \eta z p'(z) \prec \sigma q(z) + \eta z q'(z),
\]

then \( p \prec q \) and \( q \) is the best dominant.

2.5. Lemma. (see [20]) Let the function \( \Upsilon \) be analytic in \( U \) with

\[
\Upsilon(0) = 1 \quad \text{and} \quad \Re(\Upsilon(z)) > \frac{1}{2} \quad (z \in U).
\]

Then, for any function \( \Psi \) analytic in \( U \), \( (\Upsilon * \Psi)(U) \) is contained in the convex hull of \( \Psi(U) \).
3. Properties of the function class $S_{p, n}^{b, n}(\eta; \phi)$

We begin by stating the following inclusion relationship for the function class $S_{p, n}^{b, n}(\eta; \phi)$.

**3.1. Theorem.** Let $0 \leq \eta < p$ and $\phi \in \mathcal{P}$ with
\[
\Re(\phi(z)) > \max\left\{0, -\frac{\Re(b) + \eta}{p - \eta}\right\} \quad (z \in \mathbb{U}).
\]

Then
\[
S_{p, n}^{p, n}(\eta; \phi) \subset S_{p+1, n}^{p, n + 1}(\eta; \phi).
\]

**Proof.** Let $f \in S_{p, n}^{p, n}(\eta; \phi)$ and suppose that
\[
(3.1) \quad \eta \leq \eta < p \quad \text{and} \quad \phi \in \mathcal{P} \quad \text{with} \quad (3.1)
\]
\[
\Re(\phi(z)) > \max\left\{0, -\frac{\Re(b) + \eta}{p - \eta}\right\} \quad (z \in \mathbb{U}).
\]

Then $\psi$ is analytic in $\mathbb{U}$ with $\psi(0) = 1$. Combining (1.5) and (3.3), we easily find that
\[
(3.4) \quad (p + b)\left. \frac{\partial^p}{\partial z^p} f(z) \right|_{z=0} = (p - \eta)\psi(z) + b + \eta.
\]

Differentiating both sides of (3.4) with respect to $z$ logarithmically and using (3.3), we have
\[
(3.5) \quad \frac{1}{p - \eta} \left( z \left( \frac{\partial^p}{\partial z^p} f(z) \right) - \eta \right) = \psi(z) + \frac{z\psi'(z)}{(p - \eta)\psi(z) + b + \eta} \prec \phi(z).
\]

By noting that (3.1) holds, an application of Lemma 2.1 to (3.5) yields
\[
\psi(z) = \frac{1}{p - \eta} \left( z \left( \frac{\partial^p}{\partial z^p} f(z) \right) - \eta \right) \prec \phi(z),
\]
that is $f \in S_{p+1, n}^{p, n + 1}(\eta; \phi)$, which implies that the assertion (3.2) of Theorem 3.1 holds.

Next, we prove some integral-preserving properties for the function class $S_{p, n}^{p, n}(\eta; \phi)$.

**3.2. Theorem.** Let $f \in S_{p, n}^{p, n}(\eta; \phi)$ with
\[
\Re((p - \eta)\phi(z) + \mu + \eta) > 0 \quad (z \in \mathbb{U}; \mu > -p).
\]

Then the integral operator $F$ defined by
\[
(3.6) \quad F(z) := \frac{\mu + p}{2\mu} \int_0^z t^{\mu-1} f(t) dt \quad (z \in \mathbb{U}; \mu > -p)
\]
belongs to the class $S_{p, n}^{p, n}(\eta; \phi)$.

**Proof.** Let $f \in S_{p, n}^{p, n}(\eta; \phi)$. Then, from (3.6), we find that
\[
(3.7) \quad z \left( \frac{\partial^p}{\partial z^p} F(z) \right) + \mu \frac{\partial^p}{\partial z^p} F(z) = (\mu + p)\frac{\partial^p}{\partial z^p} f(z).
\]

By setting
\[
(3.8) \quad q(z) := \frac{1}{p - \eta} \left( z \left( \frac{\partial^p}{\partial z^p} F(z) \right) - \eta \right),
\]
we observe that \( q \) is analytic in \( U \) with \( q(0) = 1 \). It follows from (3.7) and (3.8) that
\[
\mu + \eta + (p - \eta)q(z) = (\mu + p) \frac{\partial_{s, b}^p \mu f(z)}{\partial_{s, b}^p f(z)}.
\]
Differentiating both sides of (3.9) with respect to \( z \) logarithmically and using (3.8), we get
\[
q(z) + \frac{zq'(z)}{\mu + \eta + (p - \eta)q(z)} = \frac{1}{p - \eta} \left( \frac{z}{\partial_{s, b}^p f(z)} - \eta \right) < \phi(z).
\]
Since
\[
\Re((p - \eta)\phi(z) + \mu + \eta) > 0 \quad (z \in U),
\]
an application of Lemma 2.1 to (3.10) yields
\[
\frac{1}{p - \eta} \left( \frac{z}{\partial_{s, b}^p f(z)} - \eta \right) < \phi(z),
\]
and we readily deduce that the assertion of Theorem 3.2 holds true. \( \square \)

3.3. Theorem. Let \( f \in S_{s, b}^p (\eta, \phi) \) with
\[
\Re((p - \eta)\phi(z) + \mu + \eta \delta) > 0 \quad (z \in U ; \delta \neq 0).
\]
Then the function \( K \in A_\mu(n) \) defined by
\[
\partial_{s, b}^p K(z) := \left( \frac{\mu + p \delta}{z} \int_0^z t^{\mu-1} (\partial_{s, b}^p f(t))^{\delta} \, dt \right)^{1/\delta} \quad (z \in U)
\]
belongs to the class \( S_{s, b}^p (\eta, \phi) \).

Proof. Let \( f \in S_{s, b}^p (\eta, \phi) \). We easily find from (3.11) that
\[
z \left[ (\partial_{s, b}^p K(z))^{\delta} \right]' + \mu (\partial_{s, b}^p K(z))^{\delta} = (\mu + p \delta) (\partial_{s, b}^p f(z))^{\delta}.
\]
By putting
\[
\varrho(z) := \frac{1}{p - \eta} \left( \frac{z}{\partial_{s, b}^p f(z)} - \eta \right) \quad (z \in U),
\]
in view of (3.12) and (3.13), we have
\[
\mu + \eta \delta + (p - \eta) \delta \varrho(z) = (\mu + p \delta) \left( \frac{\partial_{s, b}^p f(z)}{\partial_{s, b}^p K(z)} \right)^{\delta}.
\]
Making use of (3.11), (3.13) and (3.14), we get
\[
\varrho(z) + \frac{zq'(z)}{\mu + \eta \delta + (p - \eta) \delta \varrho(z)} = \frac{1}{p - \eta} \left( \frac{z}{\partial_{s, b}^p f(z)} - \eta \right) < \phi(z).
\]
Since
\[
\Re((p - \eta)\phi(z) + \mu + \eta \delta) > 0 \quad (z \in U),
\]
it follows from (3.15) and Lemma 2.1 that
\[
\varrho(z) < \phi(z) \quad (z \in U),
\]
that is \( K \in S_{p, b}^n(\eta; \phi) \). This completes the proof of Theorem 3.3.

Now, we derive certain convolution properties for the class \( S_{p, b}^n(\eta; \phi) \).

3.4. Theorem. Let \( f \in S_{p, b}^n(\eta; \phi) \). Then

\[
(3.16) \quad f(z) = \left[ z^p \cdot \exp \left( (p - \eta) \int_0^z \frac{\phi(\omega(\xi)) - 1}{\xi} \, d\xi \right) \right] \ast \left( z^p + \sum_{k=n}^{\infty} \left( \frac{p + k + b}{p + b} \right)^s z^{p+k} \right),
\]

where \( \omega \) is analytic in \( U \) with

\[
\omega(0) = 0 \quad \text{and} \quad |\omega(z)| < 1 \quad (z \in U).
\]

Proof. Suppose that \( f \in S_{p, b}^n(\eta; \phi) \). We know that the subordination condition (1.6) can be written as follows:

\[
(3.17) \quad \frac{z \left( \frac{\partial^n_{p, b} f}{\partial z^n} \right)(z)}{\frac{\partial^n_{p, b} f}{\partial z^n}(z)} = (p - \eta) \phi(\omega(z)) + \eta,
\]

where \( \omega \) is analytic in \( U \) with

\[
\omega(0) = 0 \quad \text{and} \quad |\omega(z)| < 1 \quad (z \in U).
\]

We now find from (3.17) that

\[
(3.18) \quad \frac{\left( \frac{\partial^n_{p, b} f}{\partial z^n} \right)(z)}{\frac{\partial^n_{p, b} f}{\partial z^n}(z)} - \frac{p}{z} = (p - \eta) \frac{\phi(\omega(z)) - 1}{z},
\]

which, upon integration, yields

\[
(3.19) \quad \log \left( \frac{\partial^n_{p, b} f}{z^p} \right) = (p - \eta) \int_0^z \frac{\phi(\omega(\xi)) - 1}{\xi} \, d\xi.
\]

It follows from (3.19) that

\[
(3.20) \quad \frac{\partial^n_{p, b} f(z)}{z^p} = z^p \cdot \exp \left( (p - \eta) \int_0^z \frac{\phi(\omega(\xi)) - 1}{\xi} \, d\xi \right).
\]

The assertion (3.16) of Theorem 3.4 can now easily be derived from (1.4) and (3.20). \( \square \)

3.5. Theorem. Let \( f \in A_p(n) \) and \( \phi \in P \). Then \( f \in S_{p, b}^n(\eta; \phi) \) if and only if

\[
(3.21) \quad \frac{1}{z} \left\{ f \ast \left\{ p z^p + \sum_{k=n}^{\infty} \left( \frac{p + b}{p + k + b} \right)^s z^{p+k} \right. \right. \left. \left. - \left[ (p - \eta) \phi \left( e^{i\theta} \right) + \eta \right] \left( z^p + \sum_{k=n}^{\infty} \left( \frac{p + b}{p + k + b} \right)^s z^{p+k} \right) \right\} \right\} \neq 0
\]

\[(z \in U; \ 0 \leq \theta < 2\pi).\]

Proof. Suppose that \( f \in S_{p, b}^n(\eta; \phi) \). We know that (1.6) is equivalent to

\[
(3.22) \quad \frac{1}{p - \eta} \left( z \left( \frac{\partial^n_{p, b} f}{\partial z^n} \right)(z) - \eta \right) \neq \phi \left( e^{i\theta} \right) \quad (z \in U; \ 0 \leq \theta < 2\pi).
\]

It is easy to see that the condition (3.22) can be written as follows:

\[
(3.23) \quad \frac{1}{z} \left\{ z \left( \frac{\partial^n_{p, b} f}{\partial z^n} \right)(z) - \left[ (p - \eta) \phi \left( e^{i\theta} \right) + \eta \right] \frac{\partial^n_{p, b} f}{\partial z^n}(z) \right\} \neq 0 \quad (z \in U; \ 0 \leq \theta < 2\pi).
\]
On the other hand, we find from (1.4) that
\begin{equation}
(3.24) \quad z \left( \frac{p}{z} \right) (z) = p z^p + \sum_{k=n}^{\infty} (p+k) \left( \frac{p}{p+k} \right)^z a_{p+k} z^{p+k}.
\end{equation}
Combining (1.4), (3.23) and (3.24), we readily get the convolution property (3.21) asserted by Theorem 3.5.

\section{Properties of the function class \( \mathcal{K}_{\lambda}^{p, n} (\lambda; \phi) \)}

In this section, we first derive the following subordination property.

\subsection{Theorem}
Let \( f \in \mathcal{K}_{\lambda}^{p, n} (\lambda; \phi) \) with \( \Re(\lambda) > 0 \). Then
\begin{equation}
(4.1) \quad \frac{\partial_{a+1, b} f(z)}{z^p} < \frac{p+b}{n+1} \int_0^z \frac{p+b}{t^p} \phi(t) dt < \phi(z).
\end{equation}
Proof. Let \( f \in \mathcal{K}_{\lambda}^{p, n} (\lambda; \phi) \) and suppose that
\begin{equation}
(4.2) \quad h(z) := \frac{\partial_{a+1, b} f(z)}{z^p} \quad (z \in \mathbb{U}).
\end{equation}
Then \( h \) is analytic in \( \mathbb{U} \). By virtue of (1.5), (1.7) and (4.2), we find that
\begin{equation}
(4.3) \quad h(z) + \frac{\lambda}{p+b} z h'(z) = (1-\lambda) \frac{\partial_{a+1, b} f(z)}{z^p} + \frac{\lambda}{p+b} \frac{\partial_{a, b} f(z)}{z^p} \phi(z).
\end{equation}
Thus, an application of Lemma 2.2 to (4.3) yields the assertion (4.1) of Theorem 4.1.

In view of Theorem 4.1, we easily get the following inclusion relationship.

\subsection{Corollary}
Let \( \Re(\lambda) > 0 \). Then
\( \mathcal{K}_{\lambda}^{p, n} (\lambda; \phi) \subset \mathcal{K}_{\lambda}^{p, n} (\lambda; \phi) \).

Now, we give another inclusion relationship for the function class \( \mathcal{K}_{\lambda}^{p, n} (\lambda; \phi) \).

\subsection{Theorem}
Let \( \lambda_2 > \lambda_1 \geq 0 \). Then
\( \mathcal{K}_{\lambda}^{p, n} (\lambda_2; \phi) \subset \mathcal{K}_{\lambda}^{p, n} (\lambda_1; \phi) \).
Proof. Suppose that \( f \in \mathcal{K}_{\lambda}^{p, n} (\lambda_2; \phi) \). It follows that
\begin{equation}
(4.4) \quad (1-\lambda_2) \frac{\partial_{a+1, b} f(z)}{z^p} + \lambda_2 \frac{\partial_{a, b} f(z)}{z^p} \phi(z) \quad (z \in \mathbb{U}).
\end{equation}
Since
\begin{equation}
0 \leq \frac{\lambda_1}{\lambda_2} < 1
\end{equation}
and the function \( \phi \) is convex and univalent in \( \mathbb{U} \), we deduce from (4.1) and (4.4) that
\begin{equation}
(1-\lambda_1) \frac{\partial_{a+1, b} f(z)}{z^p} + \lambda_1 \frac{\partial_{a, b} f(z)}{z^p}
= \frac{\lambda_1}{\lambda_2} \left[ (1-\lambda_2) \frac{\partial_{a+1, b} f(z)}{z^p} + \lambda_2 \frac{\partial_{a, b} f(z)}{z^p} \right] + \left( 1-\frac{\lambda_1}{\lambda_2} \right) \frac{\partial_{a+1, b} f(z)}{z^p} \phi(z) \quad (z \in \mathbb{U}),
\end{equation}
which implies that \( f \in \mathcal{K}_{\lambda}^{p, n} (\lambda_1; \phi) \). The proof of Theorem 4.3 is evidently completed.

\subsection{Theorem}
Let \( f \in \mathcal{K}_{\lambda}^{p, n} (\lambda; \phi) \). If the function \( F \in \mathcal{A}_p(n) \) is defined by (3.6), then
\begin{equation}
(4.5) \quad \frac{\partial_{a+1, b} F(z)}{z^p} \phi(z) \quad (z \in \mathbb{U}).
\end{equation}
Proof. Let \( f \in \mathcal{K}^{p, n}_{1} \). Suppose also that

\begin{equation}
G(z) := \frac{\partial^{p, n}_{+1, b} F(z)}{z^{p}} \quad (z \in \mathbb{U}).
\end{equation}

From (3.6), we deduce that

\begin{equation}
z \left( \partial^{p, n}_{+1, b} F \right)'(z) + \mu \partial^{p, n}_{+1, b} F(z) = (\mu + p) \partial^{p, n}_{+1, b} f(z).
\end{equation}

Combining (4.1), (4.6) and (4.7), we have

\begin{equation}
G(z) + \frac{1}{\mu + p} zG'(z) = \frac{\partial^{p, n}_{+1, b} f(z)}{z^{p}} < \phi(z).
\end{equation}

Thus, by Lemma 2.2 and (4.8), we conclude that the assertion (4.5) of Theorem 4.4 holds true. \( \Box \)

4.5. Theorem. Let \( f \in \mathcal{K}^{p, n}_{1} \) and \( g \in \mathcal{A}_{p}(1) \) with \( \Re \left( \frac{\mu z}{z^{p}} \right) > \frac{1}{2} \). Then

\begin{equation}
(f \ast g)(z) \in \mathcal{K}^{p, n}_{1}. \end{equation}

Proof. Let \( f \in \mathcal{K}^{p, n}_{1} \) and \( g \in \mathcal{A}_{p}(1) \) with \( \Re \left( \frac{\mu z}{z^{p}} \right) > \frac{1}{2} \). Suppose also that

\begin{equation}
H(z) := (1 - \lambda) \frac{\partial^{p, 1}_{+1, b} f(z)}{z^{p}} + \lambda \frac{\partial^{p, 1}_{+1, b} f(z)}{z^{p}} < \phi(z).
\end{equation}

It follows from (4.9) that

\begin{equation}
(1 - \lambda) \frac{\partial^{p, 1}_{+1, b} f \ast g(z)}{z^{p}} + \lambda \frac{\partial^{p, 1}_{+1, b} f \ast g(z)}{z^{p}} = H(z) \ast g(z).
\end{equation}

Since the function \( \phi \) is convex and univalent in \( \mathbb{U} \), by virtue of (4.9), (4.10) and Lemma 2.5, we conclude that

\begin{equation}
(1 - \lambda) \frac{\partial^{p, 1}_{+1, b} f \ast g(z)}{z^{p}} + \lambda \frac{\partial^{p, 1}_{+1, b} f \ast g(z)}{z^{p}} < \phi(z),
\end{equation}

which implies that the assertion of Theorem 4.5 holds true. \( \Box \)

4.6. Theorem. Let \( f \in \mathcal{K}^{p, n}_{1} \) and suppose that \( F \) is defined by (3.6) with \( f \in \mathcal{A}_{p}(1) \) and \( \mu > -p \). Then \( F \in \mathcal{K}^{p, n}_{1} \).

Proof. Let \( f \in \mathcal{K}^{p, n}_{1} \) and suppose that \( F \) is defined by (3.6) with \( \mu > -p \). We easily find that

\begin{equation}
F(z) = \mu + p \int_{0}^{1} t^{\mu-1} f(t) dt = (f \ast h)(z),
\end{equation}

where

\begin{equation}
h(z) = \frac{\mu + p}{z^{p}} \int_{0}^{z} t^{\mu+p-1} \frac{dt}{1-t} \in \mathcal{A}_{p}(1).
\end{equation}

Moreover, for \( \mu > -p \), we have

\begin{equation}
\Re \left( \frac{h(z)}{z^{p}} \right) = \Re \left( \frac{\mu + p}{z^{p} + p} \int_{0}^{z} t^{\mu+p-1} \frac{dt}{1-t} \right)
= (\mu + p) \int_{0}^{1} u^{\mu+p-1} \Re \left( \frac{1}{1-uz} \right) du
> (\mu + p) \int_{0}^{1} u^{\mu+p-1} \frac{du}{1+u} > \frac{1}{2} \quad (z \in \mathbb{U}).
\end{equation}
Combining (4.12) and Theorem 4.5, we conclude that $F \in \mathcal{K}_{s, b}^p \lambda(\phi)$. The proof of Theorem 4.6 is thus completed. □

4.7. Theorem. Let $f \in \mathcal{X}_{s, b}^p \lambda(\phi)$ and 

\[
S_j(z) := z^p + \sum_{k=1}^{j-1} a_{p+k} z^{p+k} \quad (z \in U; \ j \in \mathbb{N} \setminus \{1\}).
\]

Then the function $W_j$ defined by 

\[
W_j(z) := z^{p-1} \int_0^z \frac{S_j(t)}{t^p} \, dt \quad (z \in U; \ j \in \mathbb{N} \setminus \{1\})
\]

belongs to the class $\mathcal{X}_{s, b}^p \lambda(\phi)$.

Proof. Let $f \in \mathcal{X}_{s, b}^p \lambda(\phi)$ and let $S_j$ be defined by (4.13). We readily get 

\[
W_j(z) = z^{p-1} \int_0^z \frac{S_j(t)}{t^p} \, dt = (f \ast g_j)(z) \quad (z \in U; \ j \in \mathbb{N} \setminus \{1\}),
\]

where 

\[
g_j(z) = z^p + \sum_{k=1}^{j-1} \frac{1}{k+1} z^{p+k} \in \mathcal{A}_p(1).
\]

For $j \in \mathbb{N} \setminus \{1\}$, we know from [20] that 

\[
\Re \left( \frac{g_j(z)}{z^p} \right) = \Re \left( 1 + \sum_{k=1}^{j-1} \frac{1}{k+1} z^k \right) > \frac{1}{2}.
\]

Combining (4.14) and Theorem 4.5, we deduce that $W_j \in \mathcal{X}_{s, b}^p \lambda(\phi)$. We thus complete the proof of Theorem 4.7. □

4.8. Theorem. Let $f \in \mathcal{X}_{s, b}^p \lambda(\phi)$. Then 

\[
\frac{1}{z} \left[ \left( z^p + \sum_{k=n}^{\infty} \left( \frac{p+b}{p+k+b} \right)^{k+1} z^{p+k} \right) \ast f(z) - z^p \phi(e^{i\theta}) \right] \neq 0
\]

\[
(z \in U; \ 0 \leq \theta < 2\pi).
\]

Proof. Suppose that $f \in \mathcal{X}_{s, b}^p \lambda(\phi)$. By virtue of Theorem 4.1, we know that 

\[
\frac{\partial_{s+1}^p f(z)}{z^p} \prec \phi(z) \quad (z \in U).
\]

Thus, by similarly applying the method of Theorem 3.5, we easily get the convolution property (4.15) asserted by Theorem 4.8. □

4.9. Theorem. Let $q_1$ be univalent in $U$ and $\Re(\lambda) > 0$. Suppose also that $q_1$ satisfies 

\[
\Re \left( 1 + \frac{z q''_1(z)}{q'_1(z)} \right) > \max \left\{ 0, \ -\Re \left( \frac{p+b}{\lambda} \right) \right\}.
\]

If $f \in \mathcal{A}_p(n)$ satisfies the following subordination 

\[
(1 - \lambda) \frac{\partial_{s+1}^p f(z)}{z^p} + \lambda \frac{\partial_{s+1}^p f(z)}{z^p} \prec q_1(z) + \frac{\lambda}{p+b} z q'_1(z),
\]

then 

\[
\frac{\partial_{s+1}^p f(z)}{z^p} \prec q_1(z),
\]

and $q_1$ is the best dominant.
Proof. Let the function $h$ be defined by (4.2). We know that (4.3) holds. Combining (4.3) and (4.18), we find that

$$h(z) + \frac{\lambda}{p+b} z h'(z) \prec q_1(z) + \frac{\lambda}{p+b} z q_1'(z).$$

By Lemma 2.4 and (4.19), we easily get the assertion of Theorem 4.9. □

Taking $q_1(z) = \frac{1+A}{1+B}$ in Theorem 4.9, we get the following result.

4.10. Corollary. Let $\Re(\lambda) > 0$ and $-1 \leq B < A \leq 1$. Suppose also that $\frac{1+A}{1+B}$ satisfies the condition (4.17). If $f \in A_p(n)$ satisfies the following subordination

$$(1-\lambda)\frac{\partial_{p,n}^b f(z)}{z^p} + \lambda \frac{\partial_{p,n}^b f(z)}{z^p} \prec \frac{1+A}{1+B} + \frac{\lambda}{p+b} (A-B)z,$$

then

$$\frac{\partial_{p,n}^b f(z)}{z^p} \prec \frac{1+A}{1+B},$$

and $\frac{1+A}{1+B}$ is the best dominant. □

We now derive the following superordination result for the class $K_{p,n}(m,\lambda;l;\beta;\phi)$.

4.11. Theorem. Let $q_2$ be convex univalent in $U$, $\lambda \in \mathbb{C}$ with $\Re(\lambda) > 0$. Also let

$$\frac{\partial_{p,n}^b f(z)}{z^p} \in G[q_2(0), 1] \cap Q$$

and

$$(1-\lambda)\frac{\partial_{p,n}^b f(z)}{z^p} + \lambda \frac{\partial_{p,n}^b f(z)}{z^p}$$

be univalent in $U$. If

$$q_2(z) + \frac{\lambda}{p+b} z q_2'(z) \prec (1-\lambda)\frac{\partial_{p,n}^b f(z)}{z^p} + \lambda \frac{\partial_{p,n}^b f(z)}{z^p},$$

then

$$q_2(z) \prec \frac{\partial_{p,n}^b f(z)}{z^p},$$

and $q_2$ is the best subordinator.

Proof. Let the function $h$ be defined by (4.2). Then

$$q_2(z) + \frac{\lambda}{p+b} z q_2'(z) \prec (1-\lambda)\frac{\partial_{p,n}^b f(z)}{z^p} + \lambda \frac{\partial_{p,n}^b f(z)}{z^p} = h(z) + \frac{\lambda}{p+b} z h'(z).$$

Thus, an application of Lemma 2.3 yields the assertion of Theorem 4.11. □

Taking $q_2(z) = \frac{1+A}{1+B}$ in Theorem 4.11, we get the following corollary.

4.12. Corollary. Let $q_2$ be convex univalent in $U$ and $-1 \leq B < A \leq 1$, $\lambda \in \mathbb{C}$ with $\Re(\lambda) > 0$. Also let

$$\frac{\partial_{p,n}^b f(z)}{z^p} \in G[q_2(0), 1] \cap Q$$

and

$$(1-\lambda)\frac{\partial_{p,n}^b f(z)}{z^p} + \lambda \frac{\partial_{p,n}^b f(z)}{z^p}$$
be univalent in $U$. If
\[
\frac{1 + Az}{1 + Bz} + \frac{\lambda}{p + b} \left( A - B \right) z \prec (1 - \lambda) \frac{\partial^{p, n} \frac{f(z)}{z^p}}{\partial z^p} + \lambda \frac{\partial^{p, n} f(z)}{z^p},
\]
then
\[
\frac{1 + Az}{1 + Bz} \prec \frac{\partial^{p, n} \frac{f(z)}{z^p}}{\partial z^p},
\]
and $\frac{1 + Az}{1 + Bz}$ is the best subordinant. \qed

Finally, combining the above results of subordination and superordination, we easily get the following “sandwich-type result”.

4.13. Corollary. Let $q_3$ be convex univalent and let $q_4$ be univalent in $U$, $\lambda \in \mathbb{C}$ with $\Re(\lambda) > 0$. Suppose also that $q_4$ satisfies
\[
\Re \left( 1 + \frac{z q_4''(z)}{q_4'(z)} \right) > \max \left\{ 0, -\Re \left( \frac{p + b}{\lambda} \right) \right\}.
\]
If
\[
0 \neq \frac{\partial^{p, n} f(z)}{z^p} \in \mathfrak{H}(q_3(0), 1) \cap Q,
\]
and
\[
(1 - \lambda) \frac{\partial^{p, n} f(z)}{z^p} + \lambda \frac{\partial^{p, n} \frac{f(z)}{z^p}}{z^p}
\]
is univalent in $U$, also
\[
q_3(z) + \frac{\lambda}{p + b} z q_4'(z) \prec (1 - \lambda) \frac{\partial^{p, n} \frac{f(z)}{z^p}}{\partial z^p} + \lambda \frac{\partial^{p, n} f(z)}{z^p} \prec q_4(z) + \frac{\lambda}{p + b} z q_4'(z),
\]
then
\[
q_3(z) \prec \frac{\partial^{p, n} f(z)}{z^p} \prec q_4(z),
\]
and $q_3$ and $q_4$ are, respectively, the best subordinant and the best dominant. \qed

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