Some properties of $AFG$ and $CTF$ rings

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Abstract

$R$ is said to be a right $AFG$ ring if the right annihilator of every nonempty subset of $R$ is a finitely generated right ideal. $R$ is called a right $CTF$ ring if every cyclic torsionless right $R$-module embeds in a free module. In this paper, we first give new characterizations of $AFG$ rings and study some closure properties of $AFG$ rings. Then we explore the intimate relationships between $AFG$ rings and $CTF$ rings.

Keywords: $AFG$ ring; $CTF$ ring; pseudo-coherent ring; $FP$-injective module; singly projective module.

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1. Introduction

In [19], we introduced the concept of $AFG$ rings, which is a generalization of Noetherian rings. $R$ is said to be a right $AFG$ ring in case the right annihilator of every nonempty subset of $R$ is a finitely generated right ideal, equivalently, every cyclic torsionless right $R$-module is finitely presented, where a right $R$-module $M$ is called torsionless if $M$ embeds in a direct product of copies of $R$. The concept of $AFG$ rings is very useful in ring theory. For more details about $AFG$ rings, we refer the reader to [19, 20, 21].

In this paper, we gave some new characterizations of $AFG$ rings and further study some properties of $AFG$ rings, such as closure properties under finite direct products, quotients and localizations. On the other hand, we explore the intimate connections between $AFG$ rings and $CTF$ rings, where a ring $R$ is called right $CTF$ [27] if every cyclic torsionless right $R$-module embeds in a free module.

The layout of the paper is as follows:

Section 2 is devoted to $AFG$ rings. We first prove that $R$ is a right $AFG$ ring if and only if the dual module $\text{Hom}_{R}(M, R)$ of any cyclic torsionless left $R$-module $M$ is finitely generated if and only if every cyclic torsionless left $R$-module has a projective preenvelope. It is also shown that $R$ is a right $AFG$ ring if $R$ is a left singly injective left
Next we discuss the closure properties of AFG rings. We prove that: (1) $R$ and $S$ are right AFG rings if and only if $R \times S$ is a right AFG ring. (2) If $R$ is a right AFG ring and $I$ is an ideal which is a right annihilator in $R$, then $R/I$ is a right AFG ring. (3) If $R$ is a commutative AFG ring and $S$ a multiplicative subset of $R$ without zero-divisors, then $S^{-1}R$ is also an AFG ring. Finally we give some examples to clarify the relationships among AFG rings, AC rings, II-coherent rings and pseudo-coherent rings.

In Section 3, we deal with some properties of CTF rings. For example, it is shown that $R$ is a right CTF ring if the dual module of every cyclic torsionless right $R$-module is $H$-finitely generated, and the converse holds if $R$ is a left $f$-injective ring. Furthermore, we explore the close connections between AFG rings and CTF rings. We prove that:

1. If $R$ is a left AFG ring, then $R$ is a right CTF ring.
2. If $R$ is a right CTF right pseudo-coherent ring, then $R$ is a right AFG ring.
3. If $R$ is an AFG ring if and only if $R$ is a right CTF ring and $lr(S)$ is a finitely generated left ideal for any finite subset $S$ of $R$.
4. $R$ is a two-sided AFG two-sided singly injective ring if and only if $R$ is a two-sided CTF two-sided $FP$-injective ring.

Throughout this paper, $R$ is an associative ring with identity and all modules are unitary. $M_R$ (resp. $M_R$) denotes a right (resp. left) $R$-module. For an $R$-module $M$, the dual module $Hom_R(M, R)$ is denoted by $M^*$ and the character module $M_+$ is defined by $M^+ = Hom(R, \mathbb{Q}/\mathbb{Z})$. $E(M)$ denotes the injective envelope of $M$. $M^I$ (resp. $M^{(I)}$) stands for the direct product (resp. direct sum) of copies of $M$ indexed by a set $I$. For a subset $X$ of $R$, the right (resp. left) annihilator of $X$ in $R$ is denoted by $r(X)$ (resp. $l(X)$). We refer to [1, 9, 15, 16, 24, 26] for all undefined notions in this article.

2. AFG rings

In [19], the author gave some characterizations of AFG rings. For example, $R$ is a right AFG ring if and only if the dual module $M^*$ of any cyclic left $R$-module $M$ is finitely generated if and only if every cyclic left $R$-module has a projective preenvelope. The following theorem gives an improvement of the above result.

Recall that that a homomorphism $f : M \to P$ is called a projective preenvelope of a left $R$-module $M$ [9] if $P$ is projective, and for any homomorphism $g$ from $M$ to any projective left $R$-module $P'$, there exists $h : P \to P'$ such that $g = hf$.

We also recall a right $R$-module $M$ is $FP$-injective (or absolutely pure) [25, 17] if $Ext^1_R(N, M) = 0$ for any finitely presented right $R$-module $N$. $M$ is called $A$-injective [18] if $Ext^1_R(R/I, M) = 0$ for any right annihilator $I$ in $R$.

2.1. Theorem. The following are equivalent for a ring $R$:

1. $R$ is a right AFG ring.
2. The dual module $M^*$ of any cyclic torsionless left $R$-module $M$ is finitely generated.
3. For any cyclic torsionless left $R$-module $A$ and $x \in A$, the additive subgroup $H_{dx} = \{f(x) : f \in Hom_R(A, R)\}$ of $R$ is a finitely generated right ideal.
4. Every cyclic torsionless left $R$-module has a projective preenvelope.
5. Every $FP$-injective right $R$-module is $A$-injective.

Proof. (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (4) are obvious by [19, Theorem 2.3].

(2) $\Rightarrow$ (1) Let $I$ be any right annihilator in $R$. Then the exact sequence

$$0 \to I \to R_R \xrightarrow{f} R/I \to 0$$
of right $R$-modules yields the exact sequence of left $R$-modules

$$0 \to (R/I)^* \xrightarrow{f^*} (R_R)^* \xrightarrow{f} I^*.$$  

Let $B = \text{im}(i^*)$. Then we get the exact sequence

$$0 \to (R/I)^* \xrightarrow{f^*} (R_R)^* \to B \to 0,$$

which gives rise to the exactness of the sequence

$$0 \to B^* \to (R_R)^* \to (R/I)^*.$$  

By [24, Exercise 2.7, p.27], there exists $\phi : I \to B^*$ such that the following diagram with exact rows commutes.

$$\begin{array}{ccc}
0 & \longrightarrow & I \\
\downarrow & & \downarrow \phi \\
R_E & \longrightarrow & R/I \\
\downarrow \sigma_R & & \downarrow \sigma_{R/I} \\
0 & \longrightarrow & (R/I)^*.
\end{array}$$

Since $\sigma_{R/I}$ is a monomorphism, $I \cong B^*$ by the Five Lemma. Note that $I^*$ is torsionless by [1, Proposition 20.14], so $B$ is a cyclic torsionless left $R$-module. Thus $I \cong B^*$ is finitely generated by (2), which implies that $R$ is a right AFG ring.

(2) $\Rightarrow$ (3) Let $A$ be any cyclic torsionless left $R$-module and $x \in A$. Then there exist $f_1, f_2, \cdots, f_n \in A^*$ such that

$$A^* = f_1 R + f_2 R + \cdots + f_n R.$$  

So $H_{A,x} = \sum_{k=1}^{n} f_k(x) R$ is a finitely generated right ideal.

(3) $\Rightarrow$ (2) Let $A = Rx$ be a cyclic torsionless left $R$-module. Define a right $R$-homomorphism $\beta : A^* \to H_{A,x}$ via $f \mapsto f(x)$. It is clear that $\beta$ is an isomorphism. Thus $A^*$ is a finitely generated right $R$-module by (3).

(4) $\Rightarrow$ (2) Let $M$ be a cyclic torsionless left $R$-module. Then $M$ has a projective preenvelope $f : M \to P$. We may choose $P$ to be finitely generated since $M$ is cyclic. So we get the exact sequence $P^* \to M^* \to 0$. Thus $M^*$ is finitely generated.

(1) $\Rightarrow$ (5) is clear.

(5) $\Rightarrow$ (1) Let $M$ be a cyclic torsionless right $R$-module. Then $\text{Ext}_R^1(M, N) = 0$ for any $FP$-injective right $R$-module $N$ by (5). Therefore $M$ is finitely presented by [8], and so $R$ is a right AFG ring.

Now we investigate AFG rings in terms of singly projective, singly injective and singly flat modules.

Recall that a left $R$-module $M$ is singly projective [2] in case for any cyclic submodule $N$ of $M$, the inclusion map $N \to M$ factors through a free module.

According to [22], a left $R$-module $M$ (resp. right $R$-module $N$) is called singly injective (resp. singly flat) if $\text{Ext}_R^1(F/C, M) = 0$ (resp. $\text{Tor}_R^1(N, F/C) = 0$) for any cyclic submodule $C$ of any finitely generated free left $R$-module $F$. $R$ is called a left singly injective ring if $R$ is a singly injective left $R$-module.

Recall that $R$ is a left $CF$ ring [13] if every cyclic left $R$-module embeds in a free module.

**2.2. Proposition.** The following are true:

(1) $R$ is a left singly injective ring if and only if every singly projective left $R$-module is singly injective.

(2) $R$ is a left $CF$ ring if and only if every singly injective left $R$-module is singly projective.
(3) If $R$ is a left singly injective left $CF$ ring, then $R$ is a right $AFG$ ring.

Proof. (1) “$\Rightarrow$” Let $M$ be a singly projective left $R$-module. For any cyclic submodule $C$ of any finitely generated free left $R$-module $F$ and any homomorphism $f : C \to M$, there exits a finitely generated free left $R$-module $G$, $g : C \to G$ and $h : G \to M$ such that $f = hg$. Note that $G$ is singly injective, and so there exists $\varphi : F \to G$ such that $\varphi \lambda = g$, where $\lambda : C \to F$ is the inclusion. Hence $(h\varphi)\lambda = hg = f$. Thus $M$ is singly injective.

“$\Leftarrow$” is clear.

(2) “$\Rightarrow$” Let $M$ be a singly injective left $R$-module. For any cyclic submodule $N$ of $M$, there exits a monomorphism $\gamma : N \to R^n, n \in \mathbb{N}$. Thus there is $\theta : R^n \to M$ such that $\iota = \theta \gamma$, where $\iota : N \to M$ is the inclusion. So $M$ is singly projective.

“$\Leftarrow$” is obvious by [19, Lemma 3.6].

(3) Let $\{M_i\}_{i \in I}$ be a family of singly projective left $R$-modules. Then each $M_i$ is singly injective by (1) and so $M_i$ is singly injective. Thus $M^+_i$ is singly projective by (2). Hence $R$ is a right $AFG$ ring by [19, Theorem 2.3].

It is known that any singly projective $R$-module is singly flat for any ring $R$ by [22, Lemma 2.4] and any singly flat $R$-module is singly projective for any commutative domain $R$ by [22, Corollary 2.6]. Here we have the following result.

2.3. Proposition. The following are equivalent for a ring $R$:

(1) $R$ is right $AFG$ and every singly flat left $R$-module is singly projective.

(2) $N^+$ is singly projective for every singly injective right $R$-module $N$.

(3) $M^{++}$ is singly projective for every singly flat left $R$-module $M$.

Proof. (1) $\Rightarrow$ (2) Since $R$ is right $AFG$, $N^+$ is singly flat by [22, Theorem 2.10] for any singly injective right $R$-module $N$. So $N^+$ is simply projective by (1).

(2) $\Rightarrow$ (3) Let $M$ be a singly flat left $R$-module. Then $M^+$ is singly injective by [22, Lemma 2.4]. So $M^{++}$ is singly projective by (2).

(3) $\Rightarrow$ (1) Let $\{M_i\}_{i \in I}$ be a family of singly projective left $R$-modules, then the pure exact sequence

$$0 \to (M^+_i)^{(I)} \to (M^+_i)^I$$

induces the split exact sequence

$$((M^+_i)^{(I)})^+ \to ((M^+_i)^{(I)})^+ \to 0.$$ 

Thus $((M^+_i)^{(I)})^+$ is isomorphic to a direct summand of $((M^+_i)^I)^+$. Note that

$$((M^+_i)^{(I)})^+ \cong (M^+_i)^I, ((M^+_i)^I)^+ \cong (M^+_i)^{(I)})^{++}.$$ 

Thus $(M^+_i)^I$ is simply projective since $(M^+_i)^{(I)})^{++}$ is singly projective by (3). Also $M^+_i$ is a pure submodule of $(M^+_i)^I$ by [6, Lemma 1(2)]. Hence $M^+_i$ is simply projective by [2, Proposition 14], and so $R$ is right $AFG$ by [19, Theorem 2.3].

On the other hand, let $M$ be any singly flat left $R$-module, then $M^{++}$ is simply projective by (3). Note that $M$ is a pure submodule of $M^{++}$, and so $M$ is simply projective by [2, Proposition 14].

Recall that $R$ is a left dual ring if every left ideal is a left annihilator in $R$, equivalently, every cyclic left $R$-module is torsionless.

2.4. Theorem. The following are equivalent for a ring $R$:

(1) $R$ is a right $AFG$ left dual ring.

(2) $R$ is a right $AFG$ ring and the injective envelope of every simple left $R$-module is singly projective.
Proof. (1) ⇒ (5) holds by [19, Theorem 3.7].

(5) ⇒ (4) $R$ is a right AFG ring by [19, Theorem 2.3]. Let $N$ be a cyclic submodule of $(R_R)^+$. Since $N$ embeds in $R^n, n \in \mathbb{N}$ and $(R_R)^+$ is injective, the inclusion $N \to (R_R)^+$ factors through $R^n$. So $(R_R)^+$ is singly projective.

(4) ⇒ (2) Let $M$ be a simple left $R$-module. Then there is a monomorphism $E(M) \to ((R_R)^+)^I$. So $E(M)$ is isomorphic to a direct summand of $((R_R)^+)^I$. Since $((R_R)^+)^I$ is singly projective by [19, Theorem 2.3], $E(M)$ is singly projective.

(2) ⇒ (1) Let $N$ be a cyclic left $R$-module. It is enough to show that for any $0 \neq m \in N$, there exists $f : N \to R$ such that $f(m) \neq 0$. In fact, there is a maximal submodule $K$ of $Rm$, and so $Rm/K$ is simple. Let $i : Rm \to N$ and $i : Rm/K \to E(Rm/K)$ be the inclusions, and $\pi : Rm \to Rm/K$ be the natural map. Then there exists $j : N \to E(Rm/K)$ such that $ji = \pi i$. So $j(m) = ji(m) = i\pi(m) \neq 0$. On the other hand, since $E(Rm/K)$ is singly projective by (2), there exist $n \in \mathbb{N}, g : N \to R^n$ and $h : R^n \to E(Rm/K)$ such that $j = hg$. Therefore $g(m) = (x_1, x_2, \cdots, x_n) \neq 0$. Let $x_i \neq 0$ and $p_i : R^n \to R$ be the $i$th projection. Then $p_ig(m) \neq 0$. So $N$ is torsionless. Thus $R$ is a left dual ring.

(2) ⇔ (3) By [15, Theorem 9.4.3], a left $R$-module $N$ is finitely cogenerated if and only if $E(N) = E(S_1) \oplus E(S_2) \oplus \cdots \oplus E(S_n)$, where $S_1, S_2, \cdots, S_n$ are simple left $R$-modules. So (2) ⇔ (3) follows. □

Next we discuss the closure properties of AFG rings.

2.5. Theorem. $R$ and $S$ are right AFG rings if and only if $R \times S$ is a right AFG ring.

Proof. “⇒” Let $M$ be a cyclic torsionless right $(R \times S)$-module. Then $M$ has a unique decomposition that $M = A \oplus B$, where $A = M(R, 0)$ is a right $R$-module and $B = M(0, S)$ is a right $S$-module via $x r = x (r, 0)$ for $x \in A, r \in R$, and $ys = y(0, s)$ for $y \in B, s \in S$. It is easy to verify that $A$ is a cyclic torsionless right $R$-module and $B$ is a cyclic torsionless right $S$-module. Thus $A$ is a finitely presented right $R$-module and $B$ is a finitely presented right $S$-module by hypothesis. So there exist two exact sequences $P_1 \to P_0 \to A \to 0$ of right $R$-modules and $Q_1 \to Q_0 \to B \to 0$ of right $S$-modules, where each $P_i$ is a finitely generated projective right $R$-module, and each $Q_i$ is a finitely generated projective right $S$-module.

Regarding the above exact sequences as exact sequences of right $(R \times S)$-modules, we have an exact sequence of right $(R \times S)$-modules

$$P_1 \oplus Q_1 \to P_0 \oplus Q_0 \to A \oplus B \to 0.$$ 

Note that each $P_1 \oplus Q_1$ is a finitely generated projective right $(R \times S)$-module. So $M = A \oplus B$ is a finitely presented right $(R \times S)$-module. Thus $R \times S$ is a right AFG ring.

“⇐” Let $M$ be a cyclic torsionless right $R$-module. Note that $M$ may be regarded as a cyclic torsionless right $(R \times S)$-module, so $M$ is a finitely presented right $(R \times S)$-module by hypothesis. Thus there exists an exact sequence $P_1 \to P_0 \to M \to 0$ of right $(R \times S)$-modules, where each $P_i$ is a finitely generated projective right $(R \times S)$-module. Let $P_i = A_i \oplus B_i$, where $A_i$ is a right $R$-module and $B_i$ is a right $S$-module, $i = 0, 1$. Then we have the exact sequence $A_1 \to A_0 \to M \to 0$ of right $R$-modules. Note that each $A_i$ is a finitely generated projective right $(R \times S)$-module, and so is a finitely generated
Then we get the exact sequence of right $R$-modules.

**Proof.** Let $M_{R/I}$ be a cyclic torsionless right $R/I$-module. Then $M_R$ is clearly a cyclic right $R$-module. Let $M_R$ be a torsionless right $R$-module since $I$ is a right annihilator in $R$. Thus $M_R$ is also a torsionless right $R$-module. So $M_R$ is a finitely presented right $R$-module, i.e., there is an exact sequence of right $R$-modules

$$R^n \rightarrow R^n \rightarrow M_R \rightarrow 0.$$  

Then we get the exact sequence of right $R/I$-modules

$$R^n \otimes_R R/I \rightarrow R^n \otimes_R R/I \rightarrow M \otimes_R R/I \rightarrow 0,$$

which yields the exact sequence of right $R/I$-modules

$$(R/I)^n \rightarrow (R/I)^n \rightarrow M_{R/I} \rightarrow 0.$$  

Hence $M_{R/I}$ is a finitely presented right $R/I$-module. It follows that $R/I$ is a right AFG ring.

**2.7. Theorem.** Let $R$ be a commutative AFG ring. If $S$ is a multiplicative subset of $R$ without zero-divisors, then $S^{-1}R$ is also an AFG ring.

**Proof.** Let $M$ be a cyclic $S^{-1}R$-module. Then there exists a cyclic $R$-submodule $N$ of $M$ such that $S^{-1}N = M$. Since $S$ contains no zero-divisors, we get the exact sequence of $R$-modules

$$0 \rightarrow R \rightarrow S^{-1}R \rightarrow S^{-1}R/R \rightarrow 0,$$

which induces the exact sequence

$$0 \rightarrow \text{Hom}_R(N, R) \rightarrow \text{Hom}_R(N, S^{-1}R) \rightarrow \text{Hom}_R(N, S^{-1}R/R).$$

On the other hand, there exists an exact sequence $R \rightarrow N \rightarrow 0$, which induces the exact sequence

$$0 \rightarrow \text{Hom}_R(N, S^{-1}R/R) \rightarrow \text{Hom}_R(R, S^{-1}R/R) \cong S^{-1}R/R.$$  

Since $S^{-1}(S^{-1}R/R) = 0$, we have $S^{-1}(\text{Hom}_R(N, S^{-1}R/R)) = 0$. Thus

$$\text{Hom}_{S^{-1}R}(M, S^{-1}R) \cong \text{Hom}_{S^{-1}R}(S^{-1}R \otimes_R N, S^{-1}R)$$

$$\cong \text{Hom}_R(N, S^{-1}R) \cong S^{-1}\text{Hom}_R(N, S^{-1}R) \cong S^{-1}\text{Hom}_R(N, R).$$

Since $\text{Hom}_R(N, R)$ is a finitely generated $R$-module by [19, Theorem 2.3], we have $\text{Hom}_{S^{-1}R}(M, S^{-1}R)$ is a finitely generated $S^{-1}R$-module. So $R/I$ is an AFG ring by [19, Theorem 2.3] again. 

At the end of this section, we consider several rings related to AFG rings.

Recall that $R$ is said to be a right AC ring [18] if the right annihilator of each nonempty subset of $R$ is a cyclic right ideal. $R$ is called a right II-coherent ring [4] in case every finitely generated torsionless right $R$-module is finitely presented. $R$ is called a right coherent ring [5] if every finitely generated right ideal is finitely presented. $R$ is called a right pseudo-coherent ring [3] if the right annihilator of each finite subset of $R$ is a finitely generated right ideal.

Obviously, we have the following implications:

...
2.8. Example. Let $F$ be a field with an isomorphism $x \mapsto \bar{x}$ from $F$ to a subfield $\bar{F} \neq F$. Let $R$ denote the right $F$-space on a basis $\{1, c\}$ where $c^2 = 0$ and $cx = \bar{x}c$ for all $x \in F$. Then by [3, Example] or [28, Example 2.7], $R$ is right Artinian, and so is right $AFG$. But $R$ is not right $AC$. Otherwise, suppose that $R$ is a right $AC$ ring. Let $t \neq 0$ be an element of the Jacobson radical $J = R \bar{c} = F \bar{c}$, then $J \subseteq r(t) \neq R$. Since $R$ is local, $J = r(t)$. Thus $J = aR$ and so $a = be$ for some $b \in R$. Note that $b$ is a unit since $b \notin J$. So $cR = b^{-1}aR = b^{-1}J = J = F \bar{c}$. But $cR = F \bar{c}$, and so $F \bar{c} = F \bar{c}$, which contradicts the fact that $\bar{F} \neq F$.

In fact, we have the following result.

2.9. Proposition. $R$ is a right $AC$ ring if and only if $R$ is a right $AFG$ ring and $rl(S)$ is a cyclic right ideal for any finite subset $S$ of $R$.

Proof. “$\Leftarrow$” Let $r(T)$ be a right annihilator in $R$ for $T \subseteq R$. Then $r(T) = a_1R + a_2R + \cdots + a_nR$. By [1, Proposition 2.15], we have

$$r(T) = rl(r(T)) = rl\{a_1, a_2, \cdots, a_n\}$$

is a cyclic right ideal of $R$. So $R$ is a right $AC$ ring.

“$\Rightarrow$” is trivial.

2.10. Example. Let $F$ be a field and $R$ the subring of $F^N$ consisting of “sequences” $(a_1, a_2, \cdots) \in F^N$ that are eventually constant. Then $R$ is a commutative von Neumann regular ring (see [16, Example 7.54]) and so is pseudo-coherent.

Let $e_i \in R$ denote the $i^{th}$ unit vector $(0, \cdots, 1, 0, \cdots)$ and $S = \{e_1, e_3, e_5, \cdots\}$. Then $r(S)$ consists of sequences $(a_1, a_2, \cdots)$ that are eventually zero and such that $a_{2n} = 0$ for $n$ odd. Clearly, $r(S)$ is not a finitely generated ideal of $R$. Thus $R$ is not an $AFG$ ring.

Björk proved that $R$ is a right $AFG$ ring if $R$ is a right pseudo-coherent left perfect ring (see [3, Proposition 4.3]).

2.11. Example. Let $x, y_1, y_2, \cdots$ be indeterminates over a field $K$, $S = K[x, y_i]$ and $R = K[x^2, x^3, y_i, x y_i]$. Then $R$ is a subring of the commutative domain $S$. Hence $R$ is also a commutative domain, and so is an $AFG$ ring. But $R$ is not a $\Pi$-coherent ring (see [12, p.110]).

It is known that $R$ is a right $\Pi$-coherent ring if and only if every $n \times n$ matrix ring $M_n(R)$ ($n \geq 1$) is a right $AFG$ ring (see [20, Corollary 2.5]). Although being right $\Pi$-coherent ring is Morita invariant, it is false for right $AFG$ rings.

3. $CTF$ rings

In [27], Xue introduced the concept of right $CTF$ rings. He called a ring $R$ right $CTF$ if every cyclic torsionless right $R$-module embeds in a free module. This concept is a generalization of right $FGTF$ rings introduced by Faith [11]. Recall that a ring $R$ is right $FGTF$ if every finitely generated torsionless right $R$-module embeds in a free module.

3.1. Lemma. The following are true:
(1) \( R \) is a right \( CTF \) ring if and only if every right annihilator in \( R \) is a right annihilator of a finite subset of \( R \).

(2) A ring \( R \) is right \( FGTF \) if and only if every \( n \times n \) matrix ring \( M_n(R) \) is right \( CTF \) for every \( n \geq 1 \).

Proof. (1) \(" \Rightarrow \)" Let \( I \) be a right annihilator in \( R \). Then there is a monomorphism \( f : R/I \rightarrow R^n, n \in \mathbb{N} \). Put \( f(I) = (a_1, a_2, \ldots, a_n) \). It is easy to check that \( I = r\{a_1, a_2, \ldots, a_n\} \).

"\( \Leftarrow \)" Let \( I \) be a right annihilator in \( R \). Then \( I = r\{b_1, b_2, \ldots, b_n\} \) by hypothesis. Define \( g : R/I \rightarrow R^n \) by

\[
g(r) = (b_1r, b_2r, \ldots, b_nr).
\]

It is easy to verify that \( g \) is a monomorphism. So \( R \) is a right \( CTF \) ring.

(2) follows from (1) and [11, Theorem 1.1]. \( \square \)

3.2. Remark. (1) Although being right \( FGTF \) is Morita invariant, being right \( CTF \) is not Morita invariant by Lemma 3.1(2).

(2) If \( R \) has the a.c.c. on left annihilators, then \( R \) is a right \( CTF \) ring by Lemma 3.1(1) and [10, Corollary 2].

(3) Clearly, any right \( CF \) ring is right \( CTF \). But the converse is not true in general.

3.3. Example. Let \( k \) be a division ring and \( V_k \) be a right \( k \)-vector space of infinite dimension. Let \( R = \text{End}(V_k) \). Then \( R \) is a right self-injective von Neumann regular ring but not semisimple Artinian (see [16, Example 3.74B]). Note that \( R \) is a Baer ring, so \( R \) is a right \( CTF \) ring. Clearly \( R \) is not a right \( CF \) ring.

In fact, we have the following easy observation.

3.4. Proposition. \( R \) is a right \( CF \) ring if and only if \( R \) is a right \( CTF \) right dual ring.

Recall that a left \( R \)-module \( M \) is \( H \)-finitely generated [7] if there is a finitely generated submodule \( N \) of \( M \) such that \( (M/N)^* = 0 \).

\( R \) is called a left \( f \)-injective ring if \( \text{Ext}^1_R(R/I, R) = 0 \) for any finitely generated left ideal \( I \).

3.5. Theorem. If the dual module of every cyclic torsionless right \( R \)-module is \( H \)-finitely generated, then \( R \) is a right \( CTF \) ring. The converse holds if \( R \) is a left \( f \)-injective ring.

Proof. Let \( M \) be a cyclic torsionless right \( R \)-module. Then there exists a finitely generated submodule \( N \) of \( M^* \) such that \( (M^*/N)^* = 0 \) by hypothesis.

Let \( N = Rf_1 + Rf_2 + \cdots + Rf_n \). Define \( \alpha : M \rightarrow R^n \) by

\[
\alpha(x) = (f_1(x), f_2(x), \ldots, f_n(x)), x \in M.
\]

We next prove that \( \alpha \) is a monomorphism.

Let \( \alpha(x) = 0 \), define \( \beta : M^*/N \rightarrow R \) by

\[
\beta(\overline{g}) = g(x), g \in M^*.
\]

It is easy to check that \( \beta \) is well defined, and so \( \beta = 0 \). Thus \( x \in \bigcap_{g \in M^*} \ker(g) \). Since \( M \) is torsionless, we have \( x = 0 \). So \( \alpha \) is a monomorphism and hence \( R \) is a right \( CTF \) ring.

Conversely, suppose that \( R \) is a right \( CTF \) ring and \( R \) is left \( f \)-injective. For any cyclic torsionless right \( R \)-module \( M \), there exists an exact sequence \( 0 \rightarrow M \xrightarrow{\rho} R^n \rightarrow L \rightarrow 0 \).

Let \( \pi_i : R^n \rightarrow R \) be the \( i \)th projection, \( \varphi_i = \pi_i \gamma \in M^* \) and \( N = R\varphi_1 + R\varphi_2 + \cdots + R\varphi_n \).

We claim that \( (M^*/N)^* = 0 \). Otherwise, if there exists \( 0 \neq \xi \in (M^*/N)^* \), then there exists \( \theta \in M^* \) such that \( \xi(\overline{\theta}) \neq 0 \). Write \( \lambda : N \rightarrow R\theta + N \) and \( \iota : R\theta + N \rightarrow M^* \) to be the inclusions. Since \( M \) is cyclic, there is an exact sequence \( R \xrightarrow{\phi} M \rightarrow 0 \), which induces
the exact sequence \( 0 \to M^* \stackrel{\lambda^*}{\to} R^* \). Since \( R \) is a left \( f \)-injective ring, the exact sequence \( 0 \to N \stackrel{\sigma}{\to} R^* \) induces the exact sequence \( R^* \to \lambda^* \rho^* \sigma R \to 0 \). Thus \( \lambda^* \rho^* \sigma R \) is epic, and so \( \lambda^* \rho^* \sigma R \) is injective. The next show that \( \lambda^* \rho^* \sigma R \) is also monic. In fact, if \( \lambda^* \rho^* \sigma R(x) = 0 \), then \( \sigma R(x) = 0 \), and so \( \sigma R(x) = 0 \), and so \( \gamma(x) = 0 \). Since \( \gamma \) is monic, \( x = 0 \). Hence \( \lambda^* \rho^* \sigma R \) is an isomorphism.

Similarly, the exact sequence \( 0 \to R \theta + N \stackrel{\lambda^*}{\to} R^* \) induces the exact sequence \( R^* \to \lambda^* \rho^* \sigma R \to 0 \). Then \( \lambda^* \rho^* \sigma R \) is an epimorphism. So \( \lambda^* \rho^* \sigma R \) is a monomorphism. Thus \( \lambda^* \rho^* \sigma R \) is an isomorphism. Hence \( \lambda^* : (R \theta + N)^* \to N^* \) is an isomorphism. Note that the exact sequence

\[
0 \to N \stackrel{\lambda^*}{\to} R \theta + N \to (R \theta + N)/N \to 0
\]

induces the exact sequence

\[
0 \to ((R \theta + N)/N)^* \to (R \theta + N)^* \stackrel{\lambda^*}{\to} N^*.
\]

So \( ((R \theta + N)/N)^* \neq 0 \). But \( \xi_{(R \theta + N)/N} \neq 0 \), a contradiction. Thus \( (M^*/N)^* = 0 \). Therefore \( M^* \) is \( H \)-finitely generated.

\[\Box\]

3.6. Corollary. \( R \) is a quasi-Frobenius ring if and only if \( R \) is a two-sided dual ring and the dual module of every cyclic right \( R \)-module is \( H \)-finitely generated.

Proof. It follows from Theorem 3.5 and [13, Theorem 2.1].

Next we consider the relationships between \( AFG \) rings and \( CTF \) rings.

3.7. Lemma. The following are true:

1. If \( R \) is a left \( AFG \) ring, then \( R \) is a right \( CTF \) ring.
2. If \( R \) is a right \( CTF \) right pseudo-coherent ring, then \( R \) is a right \( AFG \) ring.

Proof. (1) By Theorem 2.1, the dual module of every cyclic torsionless right \( R \)-module is \( H \)-finitely generated and so is \( H \)-finitely generated. Thus \( R \) is a right \( CTF \) ring by Theorem 3.5.

(2) is clear by Lemma 3.1(1).

In general, a right or left \( CTF \) ring need not be a left \( AFG \) ring.

3.8. Example. Let \( K \) be a field with a subfield \( L \) such that \( \dim_L K = \infty \), and there exists a field isomorphism \( \varphi : K \to L \) (for instance, \( K = \mathbb{Q}(x_1, x_2, x_3, \cdots), L = \mathbb{Q}(x_2, x_3, \cdots) \)). Let \( R = K \times K \) with multiplication

\[
(x, y)(x', y') = (xx', \varphi(x)y' + yx'), x, y, x', y' \in K.
\]

Then it is easy to see that \( R \) has exactly three right ideals: 0, \( R \) and \( (0, K) \). Therefore \( R \) has the a.c.c and the d.c.c on right annihilators and so has the a.c.c on left annihilators. Thus \( R \) is a two-sided \( CTF \) ring by Remark 3.2(2).

On the other hand, let \( a = (0, 1) \) in \( R \). Then \( l(a) \) is not finitely generated (see [16, Example 4.46 (e)]). Thus \( R \) is not a left \( AFG \) ring.

However we have the following result.

3.9. Proposition. Let \( R \) be a two-sided pseudo-coherent ring. Then the following are equivalent:

1. \( R \) is a left \( AFG \) ring.
2. \( R \) is a right \( AFG \) ring.
3. \( R \) is a left \( CTF \) ring.
4. \( R \) is a right \( CTF \) ring.
Proof. (1) ⇒ (4) and (2) ⇒ (3) follow from Lemma 3.7(1).
(4) ⇒ (2) and (3) ⇒ (1) hold by Lemma 3.7(2). □

3.10. Corollary. The following are true for a ring $R$:

(1) $R$ is a two-sided AFG ring if and only if $R$ is a two-sided CTF two-sided pseudo-
coherent ring.

(2) $R$ is a two-sided II-coherent ring if and only if $R$ is a two-sided FGTF two-sided
coherent ring.

Proof. (1) is an immediate consequence of Proposition 3.9.
(2) follows from (1), Lemma 3.1(2) and [20, Corollary 2.5]. □

Recall that $R$ is a right $FP$-injective ring if $RR$ is an $FP$-injective right $R$-module. Clearly, any right $FP$-injective ring is right singly injective.

3.11. Proposition. The following are true:

(1) $R$ is a left AFG ring if and only if $R$ is a right CTF ring and $lr(S)$ is a finitely
generated left ideal for any finite subset $S$ of $R$.

(2) A right singly injective ring $R$ is left AFG if and only if $R$ is right CTF.

(3) [27, Corollary 3.4] A right $FP$-injective ring $R$ is left II-coherent if and only if
$R$ is right FGTF.

Proof. (1) By Lemma 3.7(1), it is enough to show the sufficiency.
Let $l(T)$ be a left annihilator in $R$ for $T \subseteq R$. By Lemma 3.1(1), $rl(T) = r(S)$ for
a finite subset $S$ of $R$. So by [1, Proposition 2.15], $l(T) = lr(T) = lr(S)$ is a finitely
generated left ideal. Hence $R$ is a left AFG ring.

(2) For any finite subset $S = \{r_1, r_2, \ldots, r_n\}$ of $R$, $RR_1 + RR_2 + \cdots + RR_n = l(T)$
for some $T \subseteq R$ by [22, Proposition 2.8] since $R$ is a right singly injective ring. So
$$lr(S) = lr(RR_1 + RR_2 + \cdots + RR_n) = lr(T) = l(T)$$
is a finitely generated left ideal. Thus the result holds by (1).

(3) By [23, Theorem 5.41 and Corollary 5.42], $R$ is a right $FP$-injective ring if and
only if every $n \times n$ matrix ring $M_n(R)$ is right singly injective for every $n \geq 1$. So (3)
follows from (2), Lemma 3.1(2) and [20, Corollary 2.5]. □

3.12. Corollary. The following are equivalent for a ring $R$:

(1) $R$ is a two-sided AFG two-sided singly injective ring.

(2) $R$ is a two-sided AFG two-sided $FP$-injective ring.

(3) $R$ is a two-sided CTF two-sided $FP$-injective ring.

Proof. (1) ⇒ (2) We first prove that $R$ is a right coherent ring. Let $I$ and $J$ be two
finitely generated right ideals of $R$. Then $I = r(X)$ and $J = r(Y)$ for some finitely
generated left ideals $X$ and $Y$ of $R$ by [22, Proposition 2.8] and Proposition 3.11. Thus
$I \cap J = r(X + Y)$ is finitely generated. Also $r(a)$ is finitely generated for any $a \in R$. So
$R$ is a right coherent ring by [5, Theorem 2.2].

On the other hand, $l(I \cap J) = lr(X) \cap lr(Y) = lr(X + Y) = X + Y = l(I) + l(J)$.
Thus $R$ is a right $f$-injective ring by [14, Theorem 1]. So $R$ is a right $FP$-injective ring
by [25, Lemma 3.1]. Similarly, $R$ is a left $FP$-injective ring.

(2) ⇒ (3) follow from Proposition 3.11. □

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