On moduli of smoothness and approximation by trigonometric polynomials in weighted Lorentz spaces

Sadulla Z. JAFAROV*

Abstract
We investigate the approximation properties of the functions by trigonometric polynomials in weighted Lorentz spaces with weights satisfying so called Muckenhoupt’s $A_p$ condition. Relations between moduli of smoothness of the derivatives of the functions and those of the functions itself are studied. In weighted Lorentz spaces we also prove a theorem on the relationship between the derivatives of a polynomial of best approximation and the best approximation of the function. Moreover, we study relationship between modulus of smoothness of the function and its de la Vallée-Poussin sums in these spaces.

Keywords: moduli of smoothness, weighted Lorentz spaces, Muckenhoupt weight, trigonometric approximation, best approximation.


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1. Introduction and the main results

Let $\mathbb{T} = [-\pi, \pi]$. A function $\omega : \mathbb{T} \to [0, \infty]$ will be called a weight function if $\omega$ is locally integrable and almost everywhere (a.e.) positive. The function $\omega$ generates the Borel measure

$$\omega(E) = \int_E \omega(x)dx.$$ 

By

$$f^*_\omega(t) = \inf \{\nu \geq 0 : \omega(\{x \in \mathbb{T} : |f(x)| > \nu\}) \leq t\}$$

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*Department of Mathematics, Pamukkale University 20070 Denizli, Turkey; Institute of Mathematics and Mechanics, NAS of Azerbaijan, 9 B. Vakhabzade Street, 1141 Baku, Azerbaijan
Email: sjafarov@pau.edu.tr
we denote the nondecreasing rearrangement of a function \( f : \mathbb{T} \to [0, \infty] \). We denote also

\[
f^{**}(t) := \frac{1}{t} \int_{0}^{t} f_*(u)\,du.
\]

Let \( 0 < p < \infty, 0 < q < \infty \). A measurable and a.e. finite function \( f \) on \( \mathbb{T} \) belongs to the Lorentz space \( L^{p,q}_\alpha(\mathbb{T}) \) if

\[
\| f \|_{L^{p,q}_\alpha} := \left( \int \left( \frac{1}{t^p} f^{**}(t) \right)^q \frac{dt}{t} \right)^{1/q} < \infty.
\]

Note that Lorentz spaces, introduced by G. Lorentz in the 1950s, are involved in terms of the second parameter \( q \). We also consider weighted Lorentz spaces \( L^{p,q}_\alpha(\mathbb{T}) \) with respect to a weight function \( \omega \) belonging to the \( A_p \)-class, introduced by Muckenhoupt [27], and play a very important role in mathematical analysis.

We use \( c, c_1, c_2, \ldots \) to denote constants (which may, in general, differ in different relations) depending only on numbers that are not important for the questions of interest. We shall also employ the symbol \( \omega \ln \omega \), denoting that \( c \omega \ln \omega \leq B \omega \leq C \omega \), where \( c, C > 0 \) are constants.

Let \( \alpha \in \mathbb{Z}^+ \) and \( f \in L^1(\mathbb{T}) \). Suppose that \( x, h \) are real, and let us take into account

\[
\Delta^\alpha_h f(x) := \sum_{j=0}^{\alpha} (-1)^j \binom{\alpha}{j} f(x + (\alpha - j)h),
\]

where \( \binom{\alpha}{j} := \frac{\alpha(\alpha-1)\cdots(\alpha-j+1)}{j!} \). Let \( j > 1 \) be the Binomial coefficients and \( \binom{\alpha}{1} = 1, \binom{\alpha}{0} = \alpha \).

Let \( 1 < p, q < \infty, \omega \in A_p(\mathbb{T}), f \in L^{p,q}_\alpha(\mathbb{T}) \). We put

\[
\sigma_\delta^\alpha f(x) := \frac{1}{\delta} \int_{0}^{\delta} |\Delta^\alpha_h f(x)| \,dt.
\]

If \( f \in L^{p,q}_\alpha(\mathbb{T}), \omega \in A_p(\mathbb{T}) \) according to [6] the Hardy-Littlewood Maximal function is bounded in \( L^{p,q}_\alpha(\mathbb{T}) \). We then have

\[
\| \sigma_\delta^\alpha f \|_{L^{p,q}_\alpha} \leq c_1 \| f \|_{L^{p,q}_\alpha}, \quad \delta < \infty.
\]

For \( 1 < p, q < \infty, \omega \in A_p(\mathbb{T}), f \in L^{p,q}_\alpha(\mathbb{T}) \), we define the \( \alpha \)-th mean modulus of smoothness \( \omega_\alpha(f, h)_{L^{p,q}_\alpha} \) by

\[
\omega_\alpha(f, h)_{L^{p,q}_\alpha} := \sup_{|t| \leq h} \| \sigma_\delta^\alpha f(x) \|_{L^{p,q}_\alpha}.
\]
Let \( f \in L^p(T) \), \( \alpha \in \mathbb{Z}^+ \) the modulus of smoothness \( \omega_\alpha(f, \cdot)_{L^p} \) is a nondecreasing, nonnegative, function and
\[
\omega^p_\alpha(f_1 + f_2, \cdot)_{L^p} \leq \omega^p_\alpha(f_1, \cdot)_{L^p} + \omega^p_\alpha(f_2, \cdot)_{L^p},
\]
\[
\lim_{\delta \to 0^+} \omega_\alpha(f, \delta)_{L^p} = 0.
\]

For \( f \in L^p(T) \), we define the \( \alpha \)-th derivative of \( f \) as function \( g \in L^p(T) \) satisfying
\[
\lim_{h \to 0^+} \left\| \frac{\Delta^\alpha_h(f)}{h^\alpha} - g \right\|_{L^p} = 0,
\]
in which case we write \( g = f^{(\alpha)} \).

Let
\[
a_0 \frac{2^n}{2} + \sum_{k=1}^{\infty} A_k(f, x), \quad A_k(f, x) := a_k(f) \cos kx + b_k(f) \sin kx
\]
be the Fourier series of the function \( L^1(T) \). The \( n \)th partial sums, and de la Vallée-Poussin sum of the series (1.2) are defined, respectively, as
\[
S_n(f) = a_0 \frac{2^n}{2} + \sum_{k=1}^{n} A_k(f, x),
\]
\[
V_n(f) = \frac{2^{n-1}}{n} \sum_{i=1}^{2^n-1} S_i(f).
\]

We denote by \( E_n(f)_{L^p} \) (\( n = 0, 1, 2, \ldots \)) the best approximation of \( f \in L^p(T) \) by trigonometric polynomials of degree not exceeding \( n \), i.e.,
\[
E_n(f)_{L^p} := \inf \{ \| f - T_n \|_{L^p} : T_n \in \Pi_n \},
\]
where \( \Pi_n \) denotes the class of trigonometric polynomials of degree at most \( n \).

Let \( W^\alpha_{pq,\omega}(T) \) (\( r = 1, 2, \ldots \)) be the linear space of functions \( f \in L^p(T) \), \( 1 < p, q < \infty \), \( \omega \in A_T(T) \), such that \( f^{(\alpha)} \in L^p(T) \). It becomes a Banach space with the norm
\[
\| f \|_{W^\alpha_{pq,\omega}(T)} := \| f \|_{L^p} + \| f^{(\alpha)} \|_{L^p}.
\]

The problems of approximation theory in the weighted and nonweighted Lorentz space have been investigated in [1], [21], [33] and [37]. The approximation problems by trigonometric polynomials in different spaces have been investigated by several authors (see, for example, [2-5], [7], [9], [11], [13-19], [22], [23], [28-30], [33] and [34]).

In this work we study the approximation problems of functions by trigonometric polynomials in the weighted Lorentz space \( L^p(T) \) with Muckenhoupt weights. Relations between moduli of smoothness of the derivatives of a function and those of the function itself are investigated. We also prove a theorem on the relationship between derivatives of a polynomial of best approximation and the best approximation of the function in the weighted Lorentz space \( L^p_T \). In addition, in the weighted Lorentz space \( L^p_T \) relationship between moduli of smoothness of the function and its de la Vallée-Poussin sums is studied. Similar problems in different spaces were investigated in [9], [30], [32].

Our main results are the following.

**Theorem 1.1.** Let \( 1 < p, q < \infty \), \( \omega \in A_T(T) \), \( f \in L^p(T) \) and \( T_n \) a trigonometric polynomial of degree \( n \) satisfying the following conditions:
\[
\| f - T_n \|_{L^p} = \Theta \left( \frac{1}{n} \right) \quad \text{and} \quad \| g - T_n' \|_{L^p} = o(1), \quad n \to \infty.
\]
Then we obtain \( f' = g \), that is, the function \( g \) satisfies the condition (1.1).

Using the same method as in the proof of Theorem 1.1 we have the following Corollary.

**Corollary 1.1.** Let \( 1 < p, q < \infty, \omega \in A_p(T) \), \( f, g_1, \ldots, g_k \in L^p_\omega(T) \) and \( T_n \) be a trigonometric polynomial satisfying, for \( i = 1, \ldots, k \), the conditions
\[
\| f - T_n \|_{L^p_\omega} = o \left( \frac{1}{n^s} \right), \quad n \to \infty,
\]
\[
\| g_i - T_n^{(i)} \|_{L^p_\omega} = o \left( \frac{1}{n^{k-1}} \right), \quad n \to \infty.
\]
Then we obtain \( g_i = g_{i-1}^i \) (\( f = g_0 \)) in the sense of (1.1).

**Theorem 1.2.** Let \( 1 < p < \infty \) and \( 1 < q \leq 2 \) or \( p > 2 \) and \( q \geq 2 \). Then, for a given \( \omega \in A_p(T) \), \( f \in L^p_\omega(T) \) and integers \( \alpha, r \) satisfying \( \alpha > r > 0 \) we have
\[
\omega_{\alpha-r} \left[ f^{(r)}(t) \right]_{L^p_\omega} \leq c_2 \left\{ \int_0^1 \frac{\omega \left( f, u \right)^2_{L^p_\omega}}{u^r + 1} \, du \right\}^{1/2},
\]
where \( s = \min \{ q, 2 \} \).

**Theorem 1.3.** Let \( 1 < p, q < \infty \), \( \omega \in A_p(T) \), \( f \in L^p_\omega(T) \), \( \alpha, r \in \mathbb{Z}^+ (\alpha > r > 0) \) and let \( T_n(f) \in \Pi_n \) be the polynomial of best approximation to \( f \) in \( L^p_\omega(T) \). In order that
\[
\left\| T_n^{(\alpha)}(f) \right\|_{L^p_\omega} = O(n^{\alpha-r})
\]
it is necessary and sufficient that
\[
E_n(f)_{L^p_\omega} = O(n^{-r}).
\]

**Theorem 1.4.** Let \( 1 < p, q < \infty \), \( \omega \in A_p(T) \), \( \alpha \in \mathbb{Z}^+ \). If \( f \in L^p_\omega \), then \( f \)
\[
c_3 \omega \left( f, \frac{1}{n} \right)_{L^p_\omega} \leq \left( n^{-\alpha} \left\| V_n^{(\alpha)}(f) \right\|_{L^p_\omega} + \left\| f(x) - V_n(f) \right\|_{L^p_\omega} \right)
\]
(1.3)
\[
\leq c_4 \omega \left( f, \frac{1}{n} \right)_{L^p_\omega}
\]
where the constants \( c_3 \) and \( c_4 \) are dependent on \( \alpha, p \) and \( q \).
\[
c_5 \omega \left( f, \frac{1}{n} \right)_{L^p_\omega} \leq \left( n^{-\alpha} \left\| S_n^{(\alpha)}(f) \right\|_{L^p_\omega} + \left\| f(x) - S_n(f) \right\|_{L^p_\omega} \right)
\]
(1.4)
\[
\leq c_6 \omega \left( f, \frac{1}{n} \right)_{L^p_\omega},
\]
where the constants \( c_5 \) and \( c_7 \) are dependent on \( \alpha, p \) and \( q \).

### 2. Proofs of main results

We need the following results obtained in [35].

**Lemma 2.1.** Let \( \omega \in A_p(T) \), \( 1 < p, q < \infty \). If \( f \in L^p_\omega(T) \) and \( \alpha = 1, 2, \ldots \), then there exists a constant \( c_\alpha > 0 \) depending \( \alpha, p \) and \( q \) such that
\[
E_n(f)_{L^p_\omega} \leq c_\alpha \omega \left( f, \frac{1}{n} \right)_{L^p_\omega}.
\]
holds where \( n = 0, 1, 2, \ldots \),
Lemma 2.2. Let \( \omega \in A_p(T) \) and \( \alpha \in \mathbb{Z}^+ \), \( 1 < p, q < \infty \). If \( T_n \in \Pi_n \), \( n \geq 1 \), then there exists a constant \( c_0 > 0 \) depending only on \( \alpha, p \) and \( q \) such that

\[
\omega_n(T_n, h)_{L^q} \leq c_0 h^{\alpha} \| T_n^{(\alpha)} \|_{L^q}, \quad 0 < h \leq \pi
\]

Lemma 2.3. Let \( \omega \in A_p(T) \), \( 1 < p, q < \infty \). If \( T_n \in \Pi_n \), \( n \geq 1 \) and \( \alpha \in \mathbb{Z}^+ \), then there exists a constant \( c_0 > 0 \) depending only on \( \alpha, p \) and \( q \) such that

\[
\| T_n^{(\alpha)} \|_{L^q} \leq c_0 n^{\alpha} \| T_n \|_{L^q}.
\]

Proof of Theorem 1.1. We take \( \varepsilon > 0 \). We choose a natural number \( n_0 = n_0(\varepsilon) \) such that for \( n \geq n_0 \)

\[
(2.1) \quad \| f - T_n \|_{L^p} \leq \varepsilon \frac{1}{n}, \quad \| g - T_n \|_{L^p} \leq \varepsilon.
\]

Taking account of (2.1) for \( h \) satisfying the condition \( \frac{\sqrt{\varepsilon}}{n} \leq h \leq \frac{1}{n} \) we obtain

\[
(2.2) \quad \left\| \frac{f(\cdot + h) - f(\cdot)}{h} - \frac{T(\cdot + h) - T(\cdot)}{h} \right\|_{L^q} \leq 2 \pi
\]

Considering [8] we have

\[
\Delta_n T_n(x) = \sum_{i=0}^{r} \binom{r}{i} (-1)^i T_n \left( x + \left( \frac{r}{2} - i \right) h \right) = \sum_{j=r}^{\infty} \sum_{i=0}^{r} \binom{r}{i} (-1)^i \left( \frac{r}{2} - i \right)^j h^j T_n^{(j)}(x) = h^r T_n^{(r)}(x) + \sum_{j=r+1}^{\infty} \eta(r, j) h^{j-r} T_n^{(j)}(x),
\]

where \( -\frac{\sqrt{\varepsilon}}{n} < \eta(r, j) < \frac{\sqrt{\varepsilon}}{n} \) and \( \eta(r, j) = 0 \) if \( j - r \) is odd. Then using (2.3) and Lemma 2.3 for \( \frac{\sqrt{\varepsilon}}{n} \leq h < \frac{\sqrt{2\pi \varepsilon}}{n} \) we find that

\[
(2.4) \quad \sum_{m=2}^{\infty} \left( \frac{\pi^m}{m!} \right)^p \| T_n \|_{L^p}^p \leq \frac{\varepsilon}{1 - 2\pi \varepsilon/2} \| T_n \|_{L^p}^p \leq c_{12} \varepsilon^p \| T_n \|_{L^p}^p.
\]

Using (2.2), (2.4) and (2.1) for \( \frac{\sqrt{\varepsilon}}{n} \leq h < \frac{\sqrt{2\pi \varepsilon}}{n} \) we reach

\[
\left\| \frac{f(\cdot + h) - f(\cdot)}{h} - g \right\|_{L^p}^p \leq \left\| \frac{f(\cdot + h) - f(\cdot)}{h} - T_n(\cdot + h) - T_n(\cdot) \right\|_{L^p}^p + \left\| T_n(\cdot + h) - T_n(\cdot) \right\|_{L^p}^p + \left\| T_n - g \right\|_{L^p}^p \leq c_{10} \left( \varepsilon^{p/2} + \varepsilon^p \| f \|_{L^p}^p + \varepsilon^p \right)
\]

From the last inequality we have \( g = f' \) in the sense of (1.2). Then the proof of Theorem 1.1 is completed.

Proof of Theorem 1.2. The function \( \omega_m(F, t)_{L^p} \) non-decreasing and according to reference [34] the following inequality holds:

\[
(2.5) \quad \omega_m(F, 2t)_{L^p} \leq c_{11} \omega_m(F, t)_{L^p}
\]
It is sufficient to prove theorem for \( t = 2^{-n} \). Then using of (2.5) we obtain
\[
\left\{ \int_0^{2^{-n}} \frac{\omega_n(f, u)^s}{u^{n+1}} \frac{du}{u} \right\}^{1/s} \leq \left\{ \sum_{\nu=1}^{\infty} 2^{\nu \alpha} \omega_n(f, 2^{-\nu}) \right\}^{1/s}.
\]
Therefore for all \( n \) it is sufficient to prove the following inequality:
\[
(2.6) \quad \omega_{n-r}(f^{(r)}, 2^{-n}) \leq \left\{ \sum_{\nu=1}^{\infty} 2^{\nu \alpha} \omega_n(f, 2^{-\nu}) \right\}^{1/s}
\]
By [34] for any trigonometric polynomial \( Q_n \) of degree \( cn \) and \( F \in L_2^p(T) \) we obtain
\[
(2.7) \quad \omega_n(F, 1/n) \leq c_{12} \left( \|F - Q_n\|_{L_p^q} + n^{-\alpha} \|Q_n^{(\nu)}\|_{L_p^q} \right).
\]
Therefore we aim to find \( Q_{2^n} \) of degree \( c2^n \) such that both \( \|f^{(r)} - Q_{2^n}\|_{L_p^q} \) and \( 2^{-n(\alpha-r)} \|Q_{2^n}^{(\alpha-r)}\|_{L_p^q} \) are bounded by the right-hand side of inequality (2.6). Let \( T_n \in \Pi_n \) \( (n = 0, 1, 2, \ldots) \) be the polynomial of best approximation to \( f \). It is known that [34] the set of trigonometric polynomials is dense in \( L_2^p(T) \). Then we have \( \|f - T_n\|_{L_p^q} \to 0 \) as \( n \to \infty \).

Let \( f \in L_2^p(T) \) has the Fourier series
\[
f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} \left( a_k \cos kx + b_k \sin kx \right) = \sum_{k=0}^{\infty} A_k(f).
\]

We define trigonometric polynomial \( \nu_N f \) as
\[
\nu_N f = \sum_{k=0}^{\infty} \nu\left( \frac{k}{N} \right) A_k(f),
\]
where \( \nu \in C^\infty[0, \infty), \nu(x) = 1 \) for \( x \leq 1 \) and \( \nu(x) = 0 \) for \( x \geq 1 \). Note that trigonometric polynomial \( \nu_N f \) has the following properties:

I) \( \nu_N f \) is a trigonometric polynomial of degree smaller than \( N \);
II) If \( g \) is a trigonometric polynomial of degree \([N/2] \), then \( \nu_N g = g \);
III) \( \|\nu_N f\|_{L_p^q} \leq c \|f\|_{L_p^q} \).

According to reference [34] we have
\[
\|\nu_N f - f\|_{L_p^q} \leq c_{13} E_{N/2}(f)_{L_p^q},
\]
where \( E_k(f)_{L_p^q} \) is the best approximation of \( f \in L_2^p(T) \) trigonometric polynomials of degree not exceeding \( k \). We now choose the \( Q_n \) of (2.7) for \( F = f^{(r)} \) to be \( (\nu_N f)^{(r)} \). It is clear that \( \|f - \nu_N f\|_{L_p^q} = o(1) \) as \( n \to \infty \).

The following identity holds:
\[
(\nu_{2k} f - \nu_{2n} f) = \sum_{m=n}^{k-1} (\nu_{2m+1} f - \nu_{2m} f) = \sum_{m=n}^{k-1} \gamma_m f.
\]

Then
\[
(\nu_{2k} f)^{(r)} - (\nu_{2n} f)^{(r)} = \sum_{m=1}^{k-1} (\gamma_m f)^{(r)}.
\]
Using the Littlewood-Paley inequality for the weighted Lorentz spaces $L^{pq}_u(\mathbb{T})$ in [21] we have

$$
\begin{align*}
&c_{14} \left\| (\nu_{2k} f)^{(r)} - (\nu_{2^n} f)^{(r)} \right\|_{L^{pq}_u} \\
&\leq \left\| \left( \sum_{m=n}^{k-1} \left( \gamma_m f \right)^{(r)} \right)^{1/2} \right\|_{L^{pq}_u} \\
&\leq c_{15} \left\| (\nu_{2k} f)^{(r)} - (\nu_{2^n} f)^{(r)} \right\|_{L^{pq}_u}.
\end{align*}
$$

(2.8)

According to [21, Lemma 4.2 and 4.3] we get

$$
(2.9) \quad \left\| \left( \sum_{m=n}^{k-1} \left( \gamma_m f \right)^{(r)} \right)^{1/2} \right\|_{L^{pq}_u} \leq \left( \sum_{m=n}^{k-1} \left\| (\gamma_m f)^{(r)} \right\|^s_{L^{pq}_u} \right)^{1/s}.
$$

where $s = \min(q, 2)$.

Note that $\nu_n f$ is the near best approximation to $f$ in $L^{pq}_u$. Then using [35] we reach the following equivalence

$$
(2.10) \quad \omega_n(f, 1/n) \asymp \|f - \nu_n f\|_{L^{pq}_u} + n^{-\alpha} \left\| (\nu_n f)^{(\alpha)} \right\|_{L^{pq}_u}.
$$

From (2.8) - (2.10) and Lemma 2.3 we conclude that

$$
\begin{align*}
&\left\| (\nu_{2k} f)^{(r)} - (\nu_{2^n} f)^{(r)} \right\|_{L^{pq}_u} \\
&\leq c_{16} \left( \sum_{m=n}^{k-1} 2^{mrs} \left\| (\gamma_m f)^{(r)} \right\|^s_{L^{pq}_u} \right)^{1/s} \\
&\leq c_{17} \left( \sum_{m=n}^{k-1} 2^{mrs} \omega_n(f, 2^{-m})^s_{L^{pq}_u} \right)^{1/s},
\end{align*}
$$

where $c_1$ independent of $m, k$ and $f$.

Use of $Q_{2^n} = \nu_{2^n} f$ and (2.10) gives us

$$
2^{-n(\alpha - r)} \left\| (\nu_{2^n} f)^{(r)}(\alpha - r) \right\|_{L^{pq}_u} = 2^{-n(\alpha - r)} \left\| (\nu_{2^n} f)^{(\alpha)} \right\|_{L^{pq}_u} \leq 2^{mrs} \omega_n(f, 2^{-n})^s_{L^{pq}_u} \leq c_{18} \left( \sum_{m=n}^{\infty} 2^{mrs} \omega_n(f, 2^{-m})^s_{L^{pq}_u} \right)^{1/s}.
$$

The proof of Theorem 1.2 is completed.

**Proof of Theorem 1.3.** We suppose that

$$
(2.11) \quad E_n(f)_{L^{pq}_u} = \left\| f - T_n(f) \right\|_{L^{pq}_u} = O(n^{-r}), \quad (r > 0).
$$

Taking into account Lemma 2.3 and the relations (2.11) we obtain

$$
\left\| T_n^{(\alpha)}(f) \right\|_{L^{pq}_u} \leq c_{19} n^\alpha \left\| T_n(f) \right\|_{L^{pq}_u} \leq n^\alpha \left\| f - T_n(f) \right\|_{L^{pq}_u} + \left\| T_n(f) \right\|_{L^{pq}_u} \leq c_{20} n^{\alpha - r}.
$$

Now we suppose that

$$
(2.12) \quad \left\| T_n^{(\alpha)}(f) \right\|_{L^{pq}_u} = O(n^{\alpha - r}).
$$
Using Lemma 2.1, Lemma 2.2 and (2.2) we get

\[ \|T_{2n}(f) - T_n(T_{2n}(f))\|_{L^p} \leq E_n(T_{2n}(f))_{L^p} \leq c_{21}\omega_n(T_{2n}, \frac{1}{n})_{L^p}. \]

(2.13)

\[ \leq c_{22}n^{-\alpha}\|T_n^{(a)}\|_{L^p} \leq c_{23}n^{-\alpha}(n^{\alpha-r}) \leq c_{24}n^{-r}. \]

On the other hand, since \( T_n(T_{2n}(f)) \) is a polynomial of order \( n \) the following inequality holds:

\[ \|T_{2n}(f) - T_n(T_{2n}(f))\|_{L^p} = \|f - T_n(T_{2n}(f)) - (f - T_{2n}(f))\|_{L^p} \]

(2.14)

\[ \geq \|f - T_n(T_{2n}(f))\|_{L^p} - \|f - T_{2n}(f)\|_{L^p} \]

Use of (2.13) and (2.14) gives us

\[ 0 \leq E_n(f)_{L^p} - E_{2n}(f)_{L^p} \leq c_{25}n^{-r}. \]

(2.15)

Since \( E_n(f)_{L^p} \to 0 \) from the inequality (2.15) we conclude that

\[ \sum_{k=0}^{\infty} \{E_{2^k}(f)_{L^p} - E_{2^{k+1}}(f)_{L^p}\} \leq c_{26} \sum_{k=0}^{\infty} 2^{-kr}. \]

Then from the last inequality we obtain

\[ E_{2^k}(f)_{L^p} \leq c_{27}2^{-nk}. \]

(2.16)

It is clear that inequality (2.16) is equivalent to \( E_n(f)_{L^p} \leq c_{28}(n^{-r}) \). This completes the proof.

Proof of Theorem 1.4. In view of Lemma 2.2 the inequality

\[ \omega_n(T_n, \frac{1}{n})_{L^p} \leq c_{29}n^{-\alpha}\|T_n^{(a)}\|_{L^p}, \]

(2.17)

holds, where \( T_n \) is a trigonometric polynomial of order \( n \). Using the properties of smoothness \( \omega_n(f, .)_{L^p} \) and (2.17), we reach

\[ \omega_n(f, \frac{1}{n})_{L^p} \leq \left( \omega_n(f - T_n, \frac{1}{n})_{L^p} + \omega_n(T_n, \frac{1}{n})_{L^p} \right) \]

(2.18)

\[ \leq c_{30}\left(\|f - T_n\|_{L^p} + n^{-\alpha}\|T_n^{(a)}\|_{L^p}\right). \]

Considering [34] there exists a constant \( c > 0 \) depending only on \( \alpha, p \) and \( q \) such that

\[ n^{-\alpha}\|T_n^{(a)}\|_{L^p} \leq c_{31}\omega_n(T_n, \frac{1}{n})_{L^p}. \]

(2.19)

By virtue of Lemma 2.1

\[ E_n(f)_{L^p} \leq c_{32}\omega_n(f, \frac{1}{n})_{L^p}. \]

(2.20)

It is known that [34] for the de la Vallée-Poussin mean the inequality

\[ \|f - V_n(f)\|_{L^p} \leq c_{33}E_n(f)_{L^p}. \]

(2.21)
holds. Use of (2.19)-(2.21) gives us
\[ n^{-\alpha} \left\| V_n^{(\alpha)}(f) \right\|_{L^{pq}_\omega} + \left\| f - V_n(f) \right\|_{L^{pq}_\omega} \]
\[ \leq c_{34} \left( \omega_n(1, V_n, \frac{1}{n})_{L^{pq}_\omega} + E_n(f)_{L^{pq}_\omega} \right) \]
\[ \leq c_{35} \left( \omega_n \left( f, \frac{1}{n} \right)_{L^{pq}_\omega} + \omega_n(f - V_n, 1, \frac{1}{n})_{L^{pq}_\omega} + E_n(f)_{L^{pq}_\omega} \right) \]
\[ \leq c_{36} \omega_n \left( f, \frac{1}{n} \right)_{L^{pq}_\omega}. \]
The last inequality and (2.18) imply that (1.3).
According to [35] there exists a constant \( c_{25} \) such that
\[ \| f - S_n(f) \|_{L^{pq}_\omega} \leq c_{37} E_n(f)_{L^{pq}_\omega}. \]
If the inequality (2.22) and the scheme of proof of the estimation (1.3) is used we obtain
the estimation (1.4).
Theorem 1.4 is proved.
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