On groups with relatively small normalizers of nonprimary subgroups

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Abstract

We consider the structure of a finite nonsolvable group \( G \) in which for any nonprimary subgroup \( A \) the index \(|N_G(A) : A \cdot C_G(A)|\) is equal unit or a prime number.

Keywords: finite group, subgroup, normalizer, centralizer.

If \( A \) is an arbitrary subgroup of a group \( G \), then \( N(A) \geq A \cdot C(A) \), and the index \(|N(A) : A \cdot C(A)|\) equals to the order of a subgroup of \( \text{Out}(A) \), which is induced by elements of \( G \). In this paper we consider the structure of finite groups \( G \) in which for any nonprimary subgroup \( A \) the index \(|N(A) : A \cdot C(A)|\) is a divisor of a certain prime number, i.e., it is equal to 1 or a prime number. We’ll call these groups \( NP \)-groups.

Note that any subgroup and factor-group of a \( NP \)-group is also a \( NP \)-group.

The aim of this article is to describe the structure of nonsolvable \( NP \)-groups.

1.1. Lemma. If a nonsolvable \( NP \)-group \( G \) is a central product of two subgroups \( G_1 \) and \( G_2 \), then one of the factors is abelian.

Proof. Suppose that \( G_1 \) is nonabelian. Then ([1], Corollary of Lemma 2) there exists a subgroup \( A \) of \( G_1 \) such that \(|N_{G_1}(A) : A \cdot C_{G_1}(A)| = p \) for a prime \( p \). If \( A \) is nonprimary and \( B \) is an arbitrary subgroup of \( G_2 \), then from the fact that \(|N(AB) : AB \cdot C(AB)|\) divides a prime number, it follows that \( N_{G_2}(B) = B \cdot C_{G_2}(B) \). Then \( G_2 \) is abelian (see [1]). If \( A \) is primary and \(|A| = q^n \) for a prime \( q \), then the equality \( N_{G_2}(B) = B \cdot C_{G_2}(B) \) holds for any \( q' \)-subgroup \( B \) of \( G_2 \). By Lemma 4 from [1], \( G_2 = Q \times H \), where \( H \) is an abelian Hall \( q' \)-subgroup of \( G_2 \), i.e. \( G_2 \) is solvable. If \( G_2 \) is nonabelian, then for any \( q' \)-subgroup \( A \) of \( G_1 \), the equality \(|N_{G_1}(A) : A \cdot C_{G_1}(A)| \) holds too. But then the group \( G_1 \) is also solvable, which is impossible.

1.2. Lemma. If \( Q \) is a Sylow \( q \)-subgroup of a \( NP \)-group \( G \), \( C(Q) \leq Q \) and \( N(Q) = (Q \times \langle a \rangle) \times \langle b \rangle \), where \( a \neq 1 \neq b \), then \( a \) and \( b \) are elements of prime orders, and if \( N(Q) = Q \times \langle x \rangle \), then \(|x| \) is the product of no more than two prime factors.

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Proof. In the first case, if we let \( A = Q \triangleleft \langle a \rangle \), we get that \( |b| = |N(A) : A \cdot C(A)| \) is a prime. And supposing \( A = Q \triangleleft \langle c \rangle \), where \( c \) is an element of prime order \( r \) from \( \langle a \rangle \), then from the equality \( |N(A) : A \cdot C(A)| = \frac{|a|}{r} |b| \) we get that \( |a| = r \). In second case, it’s sufficient to choose a subgroup \( A = Q \triangleleft \langle y \rangle \), where \( y \) is an element of prime order from \( \langle x \rangle \).

Later on we will repeatedly use Frattini’s argument ([7], theorem 1.3.7): if \( H < G \) and \( P \) is a Sylow \( p \)-subgroup of \( H \), then \( G = H \cdot N(P) \). In a solvable group all Hall \( \pi \)-subgroups are conjugate. Therefore a similar proposition is true in a case where \( P \) is a Hall \( \pi \)-subgroup of a solvable group \( H \). We will call this Frattini’s argument as well.

1.3. Theorem. A finite nonabelian simple group \( G \) is a \( NP \)-group if and only if \( G \) satisfies one of the following conditions:

1. \( G \cong PSL(2,q^n) \), \( \frac{q^n-1}{(2,q^n-1)} \) is either a prime or a product of two primes;
2. \( G \cong PSU(3,2^n) \), and either \( n = 2 \) or each of the numbers \( 2^n - 1 \) and \( 2^n + 1 \) are primes;
3. \( G \cong Sz(2^n) \), \( n \in \{3, 5\} \).

Proof. Necessity. Let \( G \) be a finite nonabelian simple \( NP \)-group. It is known that any nonabelian simple group is either an alternating group, a Lie type group, or a sporadic simple group.

First, assume that \( G \cong A_n \). If \( n = 5 \), then \( G \cong PSL(2,4) \), and if \( n = 6 \), then \( G \cong PSL(2,9) \). If, however, \( n > 6 \) then \( G \) contains a subgroup which is isomorphic to \( A_7 \). Let \( G = A_7 \), \( a = (1 2)(3 4) \), \( b = (1 3)(2 4) \), \( c = (5 6 7) \), \( x = (1 2)(5 6) \), \( y = (1 2 3) \) and \( A = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \). Then \( C(A) = A \) and \( N(A) = A \cdot (\langle y \rangle \cdot \langle x \rangle) \), i.e. \( |N(A) : A \cdot C(A)| = 6 \), which is impossible.

Now let \( G \) be a simple Lie type group over the Galois field \( GF(q^n) \), where \( q \) is a prime. Suppose that the Lie rank of \( G \) is more than 2. If \( \mathcal{J} \) is a parabolic subgroup of \( G \), corresponding to two nonadjoint nodes of the Dynkin diagram of \( G \), then ([4], Proposition 2.17) \( \mathcal{J} = J/O_q(J) = (\mathcal{Y}_1 \times \mathcal{Y}_2) \cdot \mathcal{P} \), where \( \mathcal{Y}_1 \) and \( \mathcal{Y}_2 \) are Lie type groups of Lie rank 1 over \( GF(q^n) \) and \( H \) is a Cartan subgroup of \( G \). By Lemma 1.1 each of \( \mathcal{Y}_i \) is a solvable group. Since ([4], Theorem 2.13) solvable Lie type groups are either \( A_2(2) \), \( A_1(3) \), \( 2A_2(2) \) or \( 2B_2(2) \), so \( q^n \in \{2, 3\} \). Let \( p_i \in \pi(\mathcal{Y}_i) \setminus \{q\} \). \( \mathcal{A}_i \) and \( \mathcal{A}_2 \) be Sylow \( p_i \)- and \( p_2 \)-subgroup from \( \mathcal{Y}_1 \) and \( \mathcal{Y}_2 \), respectively, then for the nonprimary subgroup \( A = A_1 \cdot A_2 \) the index \( |N(A) : A \cdot C(A)| \) is divisible by \( q^2 \), which is impossible.

Therefore \( l \leq 2 \). Let \( l = 2 \), i.e., \( G \) is isomorphic to one of the groups \( A_2(q^n) \), \( B_2(q^n) \), \( 2A_3(q^n) \), \( 2A_4(q^n) \), \( 3D_4(q^n) \), \( 2F_4(2^{2n+1}) \), \( n > 0 \), \( (2^{2n}(2))' \).

First suppose that the Cartan subgroup \( H \) of the group \( G \) is trivial. The group \( (2^{2n}(2))' \) contains a subgroup \( K \) isomorphic to \( PSL(2,25) \), which is not \( NP \)-group, because it has a Cartan subgroup of order 12, which contradicts Lemma 1.2. Because of this, \( G \) is a group of classical type over the field \( GF(2) \), i.e., either \( G \cong A_2(2) = PSL(3,2) \), or \( G \cong B_2(2) = PSp(4,2) \). It’s left to be noticed that \( PSL(3,2) \cong PSL(2,7) \), and that the group \( PSp(4,2) \cong S_6 \) is not simple.

Therefore \( H \neq 1 \). Let \( J \) be a proper parabolic subgroup of \( G \). Then \( \mathcal{J} = J/O_q(J) = \mathcal{Y} \cdot \mathcal{P} \), where \( \mathcal{Y} \) is a Lie type group of Lie rank 1. If \( G \cong 2F_4(2^{2n+1}) \), \( n > 0 \), then subgroup \( J \) can be chosen so that \( \mathcal{Y} \cong 2B_2(2^{2n+1}) \), and if \( \mathcal{A} \) is a
If $q^n = 2$ and $\overline{A}$ is a subgroup of order 3 of $\overline{Y}$, then by Frattini’s argument we assume that $H \leq N(\overline{A})$ which also leads to a contradiction. However, if $q^n \neq 2$, then as $\overline{A}$ we can take a Cartan subgroup of $\overline{Y}$.

Therefore $t = 1$. If $Q$ is a Sylow $q$-subgroup of $G$, then $C(Q) \leq Q$ and $N(Q) = Q \times H$, where $H$ is a Cartan subgroup of $G$. From the definition of an $NP$-group and the fact that $H$ is abelian, one of the following is true: $|H| = 1$, $|H|$ is a prime number or $|H| = pr$ where $p$ and $r$ are primes. Since group $A_1(2)$ is solvable, then the first case is impossible.

First, suppose that $G$ is a twisted group. Let $G \cong 2A_2(q^n) = PSU(3, q^{2n})$. Then $|H| = \frac{q^{2n} - 1}{q^n - 1} = (q^n - 1) \cdot \frac{q^n + 1}{q^n - 1}$. If $q > 2$ then $|H|$ is divisible by 8, which is impossible. Therefore $q = 2$ and all of the numbers $(2^n - 1)$ and $\frac{2^n + 1}{3}$ are primes. The primarity of $(2^n - 1)$ implies that either $n = 2$ or $n$ is an odd number and then $(2^n + 1, 3) = 3$, i.e., $G$ is a group of type 2) from this Theorem.

The group $2B_2(2^{n+1})$ contains, as subgroups, the Frobenius groups of orders $(2^{2n+1} - 2n + 1) \cdot 4$. Therefore each of the numbers $2^{2n+1} + 2n + 1$ and $2^{2n+1} - 2n + 1$ must be powers of the primes. Because their product is equal $(2^{2n+1} + 1$ it is divisible by 5. But then either $2^{2n+1} + 2n + 1 = 5^n$, or $2^{2n+1} - 2n + 1 = 5^n$ for some number $m$.

Consider the first case. If $2^{2n+1} + 2n + 1 = 5^n$, then either $n = 4t + 1$ or $n = 4t - 1$ for some $t > 0$. Since $2^t + 2^t + 1 = 145 \neq 5^m$, then $n \geq 4$ in any case. Let $m = 2^k r$, where $r$ is an odd number. Then from

$$2^{n+1}(2^n + 1) = 5^m - 1 = 2^{k+2} \cdot \frac{5^r - 1}{4} \prod_{i=0}^{k-1} \frac{5^{2^r} + 1}{2}$$

it follows that $k = n - 1 \geq 3$. But the inequality

$$\prod_{i=0}^{k-1} \frac{5^{2^r} + 1}{2} > 2^{k+1} + 1 = 2^n + 1$$

is true for $k \geq 3$, which is impossible.

If, however, $2^{2n+1} - 2n + 1 = 5^n$, then either $n = 4t + 1$ or $n = 4t + 2$ for some $t \geq 0$. The equality

$$2^{n+1}(2^n - 1) = 5^m - 1 = 2^{k+2} \cdot \frac{5^r - 1}{4} \prod_{i=0}^{k-1} \frac{5^{2^r} + 1}{2}$$

implies $k = n - 1$. If $k > 1$ then from $k \in \{4t, 4t + 1\}$ it follows that $k \geq 4$ and we have the contradiction again. Therefore, $k \in \{0, 1\}$ and, consequently, $n \in \{1, 2\}$, i.e., $G$ is a group of the type 3) from this Theorem.

Let $G \cong 2G_2(3^{2n+1})$. Since the group $2G_2(3)$ is nonsimple, then $n > 0$. In this case (see [8]) $G$ has a subgroup $H$ such that $H = (V_4 \times D) \rtimes \langle b \rangle$, where
For $a$ is an element of order $2^{2n+1}q^n$ from $D$, then the subgroup $A = V_4 \times \langle a \rangle$ is nonprimary and $|N_H(A) : A \cdot C_H(A)| = 6$, which is impossible.

Now suppose that $G$ is a classical non-twisted group of Lie type rank 1, i.e., $G \cong A_1(q^n) \cong PSL(2, q^n)$. In this case $|H| = \frac{q^n-1}{(2,q^n-1)}$. Because of this $a^{q^n-1}$ is either a prime, or a product of two primes, i.e., $G$ is a group of the type 1 from this Theorem.

Now using the survey [10] we can show that $G$ cannot be a sporadic simple group. To demonstrate this, it’s sufficient to show that any sporadic simple group contains a subgroup, which is not NP-group. Let $G_p$ denote a Sylow $p$-subgroup of $G$ for a prime $p$.

1) In the group $M_{11}$ the subgroup $G_3$ is self-centralizing and its normaliser has a form $N(G_3) = G_3 \times K$, where $K$ is isomorphic to the semi-dihedral group of order 16, again contrary with Lemma 1.2.

2) $M_{12}$, $M_{23}$, $M_{24}$, $Co_3$, $Suz$ and $McL$ contain $M_{11}$.

3) $M_{22}$ and $M_{24}$ contain $A_7$, $F_{22}$ contains $S_{10}$, and $F_{23}$ and $F_{24}$ contain $S_{12}$.

4) The group $O'N$ contains $J_1$, and in the group $J_1$ the subgroup $N(G_3)$ is a direct product of two dihedral groups of orders 6 and 10. If $A$ is a subgroup from $N(G_3)$ of order 15, then $|N(A) : A \cdot C(A)|$ is divisible by 4.

5) In the group $J_2$ we have $N(G_3) = G_3 \times \langle a \rangle$, where $C(G_3) = G_3$ and $|a| = 8$.

6) In the groups $J_3$ and $He$ the subgroup $N(G_{17})$ is a Frobenius group of order 17 · 8; in $J_4$ and $Co_2$ the subgroup $N(G_{29})$ is a Frobenius group of order 29 · 28, again contrary to Lemma 1.2 and $Co_1$ and $F_2$ contain $Co_2$.

7) The group $F_3$ contains an involution $\tau$ such that $C(\tau)/O_2(C(\tau)) \cong C_2$.

8) In the groups $L_3$ and $F_3$ the subgroups $N(G_{37})$ and $N(G_{19})$ are Frobenius groups of orders 37 · 18 and 19 · 14, respectively.

9) The group $F_5$ contains $HS$, and in the group $HS$ the subgroup $N(G_3)$ is isomorphic to $S_3 \times S_5$, and if $A_3 \times A_5 \cong A \leq N(G_3)$, then $|N(A) : A \cdot C(A)|$ is divisible by 4.

10) The group $Ru$ contains an involution $\tau$ such that $C(\tau) \cong V_4 \times S_2(8)$, and if $A \cong V_4 \times H$, where $H$ is a subgroup of order 5 from $S_2(8)$, then $|N(A) : A \cdot C(A)|$ is divisible by 4.

Sufficiency. If $A$ is a proper nonprimary subgroup of $G$, then $N(A) < G$. Therefore, it is sufficient to prove, that any maximal subgroup of $G$ is a NP-group.

Suppose first that $G \cong PSL(2, q^n)$, where $q$ is a prime. Since $\frac{q^{2n}-1}{(2,q^n-1)}$ is either a prime or a product of two primes, then, it is not difficult to see, that either $n = 1$ or $q \in \{2, 3\}$ and $n$ is either a prime or the square of a prime (odd, if $q = 3$). From Dickson’s Theorem ([6], Theorem 2.8.27) it follows that the maximal subgroups of $G$ are the groups from the following list: $N(Q) = Q \times \langle a \rangle$, where $Q$ is a Sylow $q$-subgroup of $G$, $|a| = \frac{q^n-1}{(2,q^n-1)}$; the dihedral groups of the orders $2 \cdot \frac{q^n+1}{(2,q^n-1)}$, $S_4$ for $q^n \equiv \pm 1(8)$, $A_4$ for $q^n \equiv \pm 3(8)$, $A_5$ for $q^n \equiv \pm 1(10)$; $PSL(2, q^n)$ for $n = p^2$.

It’s not difficult to check that all these groups are NP-groups.

If $G \cong PSU(3, 2^{2n})$, then since $(2^{2n}-1)$ is a prime, $n$ is a prime too. From [5] it follows that the maximal subgroups of $G$ are the groups of the following...
Theorem 1.3. Let $G$ be a nonsolvable nonsimple $NP$-group. Then one of the following holds:

1) subgroup $F = F^*$ is a nontrivial $p$-group for some prime $p$, and $G/F \cong PSL(2, 4)$;
2) $G \cong Aut(PSL(2, 2^n))$, $n \in \{2, 3\}$;
3) $G = Z(G) \cdot L$, $L \cong PSL(2, q^n)$ or $SL(2, q^n)$, the number $\frac{q^{n-1}}{(2, q^{n-1})}$ is a prime, and if $n = 1$ then either $q \neq \pm 1(8)$ or $Z(G)$ is a 2-group;
4) $G = Z(G) \times L$ and either $L \cong PSL(2, q^n)$, $\frac{q^{n-1}}{(2, q^{n-1})}$ is a product of the two prime numbers and $Z(G)$ is a $q$-group, or $Z(G)$ is a 2-group and $L \cong PSU(2, 2^n)$ is a group from Theorem 1.3;
5) $G = Z(G) \cdot L$, $Z(G)$ is a 3-group and $L$ is isomorphic to the covering group for $PSL(2, 9)$ with $|Z(L)| = 3$.

Proof. Let $G$ be a group satisfying conditions of this Theorem. Let's assume first that $F = F^*$. Then $C(F) \leq F$. If $F$ is a normal subgroup, then $|G : F|$ is a prime and $G$ is a solvable group. Therefore, $F$ is a $p$-group for some prime $p$. Moreover, if $A/F$ is a $p'$-subgroup of $G/F$, then $|N(A) : A|$ divides a prime number.

Let $G/F \cong SL(2, 2^n)$. Then $G_1$ is a non-nilpotent group, and consequently, is nonprimary. Therefore $|G : G_1|$ is a divisor of a prime. Assume that $G_1 = G_1$. Then $G/F$ is a simple $NP$-group. i.e., a group from Theorem 1.3.

Let $G/F \cong PSU(3, 2^n)$. If $p \neq 2$ and $A/F$ is a Sylow 2-subgroup of $G/F$, then $A$ is nonprimary, and $|N(A) : A \cdot C(A)| = \frac{2^n-1}{(3, 2^n-1)}$ is not a prime. Therefore $p = 2$. Then (4, p.166), for subgroup $H/F$ of order $\frac{2^n-1}{(3, 2^n-1)}$ from $N_{G/F}(A/F)$ the equality $C_{G/F}(H/F) = H/F \times L/F$, where $L/F \cong PSL(2, 2^n)$, is true. Therefore, for the nonprimary subgroup $H$, the index $|N(H) : H \cdot C(H)|$ divides by $|L/F|$, which is impossible.

In the case $G/F \cong Sz(8)$, a Sylow 2-subgroup of $G/F$ has the order $2^6$. Hence $p = 2$. If $A/F$ is a subgroup of order 5 from $G/F$, then $|N(A) : A| = 4$, which is impossible. If $G/F \cong Sz(2^2)$, then by analogy $p = 2$ and if $A/F$ is a subgroup of order 25, then $|N(A) : A| = 4$.

Therefore, $G/F \cong PSL(2, q^n)$. If $q \neq p$ and $Q/F$ is a Sylow $q$-subgroup of $G/F$, then $Q$ is nonprimary and the primarity of the number $|N_{G/F}(Q/F) : Q/F|$ is impossible.
implies that \( \frac{q^n - 1}{(2, q^n - 1)} \) is a prime. If \( aF \) is an element of order \( q \) from \( Q/F \) then the index \( |N(a, F) : \langle a, F \rangle| \) divides a prime number and, therefore, \( n \leq 2 \).

If \( n = 2 \) then from the primarity of \( \frac{q^n - 1}{(2, q^n - 1)} \) we get that \( q = 2 \), i.e. \( G/F \cong PSL(2, 4) \). Let \( n = 1 \). Since the groups \( PSL(2, 2) \) and \( PSL(2, 3) \) are solvable, and \( PSL(2, 5) \cong PSL(2, 4) \) then we can suppose that \( q > 5 \). Let \( A/F \) is a subgroup of the prime order \( r \), where \( r \) divides \( \frac{q^n - 1}{2} \). If \( r \neq p \) then the primarity of \( |N(A) : A| = 2 \cdot \frac{q^n - 1}{2r} \) implies \( r = \frac{q^n - 1}{2} \). But the numbers \( \frac{q^n - 1}{2} \) and \( \frac{q^n + 1}{2} \) are primes at the same time only when \( q = 5 \). Suppose now that \( r = p \). Then by the arbitrariness of \( r \), the equation \( \frac{q^n - 1}{2} = p^k \) is solvable. Since \( q > 5 \) then the prime number \( \frac{q^n - 1}{2} \) is odd. But then \( q + 1 \) is divisible by 4. i.e. \( p = 2 \). Since one of the numbers, either \( k \) or \( k + 1 \), is even, then the numbers \( q = 2^{k+1} - 1 \) and \( \frac{q^n - 1}{2} = 2k - 1 \) cannot both be prime at the same time.

Assume now that \( q = p \) and \( aF \) is an element of prime order from a subgroup of order \( \frac{q^n \pm 1}{2(q^n - 1)} \) from \( G/F \). Because \( N_{G/F}(\langle aF \rangle) \) is isomorphic to the dihedral group of order \( \frac{q^n \pm 1}{2(q^n - 1)} \cdot 2 \), and \( |N(\langle a, F \rangle) : \langle a, F \rangle| = \) a prime, then the numbers \( \frac{q^n \pm 1}{2(q^n - 1)} \) are primes. If \( q \) is odd, then \( q^n = 5 \). But \( PSL(2, 5) \cong PSL(2, 4) \). If \( q = 2 \), then because \( (2^n - 1) \) is a prime it follows that \( n \) is a prime. But then in the case \( n > 2 \) the number \( 2^n + 1 \) is not prime. Therefore, \( G/F \cong PSL(2, 4) \).

Suppose now that \( G_1 < G \). Then, by using what’s already been proved, \( G_1/F \cong PSL(2, 4) \) and \( G/F = (G_1/F) \rtimes \langle aF \rangle \), where \( aF \) is an automorphism of the group \( G_1/F \). Let \( A/F \) be a subgroup of order 5 from \( G_1/F \). By Frattini’s argument we can assume that \( aF \in N_{G/F}(A/F) \). But then \( |N(A) : A \cdot C(A)| \) is divisible by 4.

Therefore, if \( F = F^* \), then by the theorem conditions, \( G \) is of type 1). Because of this, we’ll further assume that \( F < F^* \). Then \( F^* = F \cdot L \), when \( L \) is the layer of the group \( G \). By Lemma 1.1, the subgroup \( F \) is abelian and \( F^* \) is a simple group, i.e., a group from Theorem 1.3. Moreover, one of the following holds: \( F = 1 \), \( G = F^* \) or \( 1 < F < F^* < G \).

In the first case \( F^* \) is a group from Theorem 1.3 and \( F^* < G \leq \text{Aut}(F^*) \). From the definition of the \( NP \)-group it follows that \( |G/F^*| \) is a prime. The structure of the automorphism groups of Lie type groups (e.g. [4], theorem 4.238) implies that in our case \( G = F^* \rtimes \langle a \rangle \), \( a \) is a prime order automorphism of group \( F^* \). Set \( |a| = p \).

First assume that \( F^* \cong PSL(2, q^n) \). Let \( Q \) be a Sylow \( q \)-subgroup of \( F^* \) and \( B = Q \rtimes H \) be a Borel subgroup of group \( F^* \). By Frattini’s argument we can assume that \( a \in N(Q) \). But then \( a \in N(N_{R^*}(Q)) = N(B) \). Since \( C(Q) \leq Q \) and \( |N(Q) : Q| = |H| : p \), then, by Lemma 1.2, the number \( |H| = \frac{q^n - 1}{(2, q^n - 1)} \) must be a prime number. But then, as it was noted in the proof of Theorem 1.3, either \( q \in \{2, 3\} \), or \( n = 1 \). By analogy, for a subgroup \( A \) of order \( \frac{q^n + 1}{2(q^n - 1)} \) from \( F^* \) the equality \( |N(A) : A \cdot C(A)| = 2p \) implies that subgroup \( A \) must be a primary group.

Let \( q = 2 \). The primarity of the number \( (2^n - 1) \) implies that \( n \) is a prime. If \( n > 2 \), then \( 2^n + 1 \) is divisible by 3 and, consequently, \( 2^n + 1 = 3^k \) for a number \( k \). Let \( k > 2 \). If \( k = 2r \) is even, then \( 2^n = 3^k - 1 = (3^r - 1)(3^r + 1) \), which is impossible. However, if \( k = 2r + 1 \), then \( 3^k - 1 = 2(1 + 3^2 + \cdots + 3^{2r}) \neq 2^n \).
where the second factor is odd. Therefor, if \( q = 2 \), then the group \( F^* \) is isomorphic to one of the groups \( PSL(2,4) \) or \( PSL(2,8) \).

If \( q = 3 \) then the primarity of the number \( \frac{3^n-1}{2} \) implies that \( n \) is an odd prime. However, from that fact that \( \frac{3^n+1}{2} \) is even and prime it follows that \( \frac{3^n+1}{2} = 2^k \), i.e., \( 3^n = 2^{k+1} - 1 \) for a number \( k \). Since the number \( \frac{3^n-1}{2} = 2^k - 1 \) is prime, then \( k \) is an odd prime. But then \( k+1 = 2r \) and \( 3^n = (2^r - 1)(2^r + 1) \), which is impossible for \( r > 1 \). However if \( r = 1 \), then \( k = 1 \). But then \( n = 1 \) as well, which contradicts the primarity of the group \( F^* \).

Finally, let \( q \) and \( \frac{q+1}{2} \) be primes. If \( q = 5 \), then \( F^* \cong PSL(2,4) \). However if \( q > 5 \), then \( \frac{q-1}{2} \) is odd. Because \( \frac{q+1}{2} \) is primary, we obtain that \( \frac{q+1}{2} = 2^k \), i.e. \( q = 2^{k+1} - 1 \). But then \( \frac{q+1}{2} = 2^k - 1 \). Since one of the numbers \( k \), \( k+1 \) is even, and \( k > 2 \), then the numbers \( (2^k - 1) \) and \( (2^{k+1} - 1) \) can’t both be prime simultaneously.

Suppose now that \( F^* \cong PSU(3,2^{2n}) \). If \( p \neq 2 \) and \( A \) is a Sylow 2-subgroup of \( F^* \), then \( |N(A) : A \cdot C(A)| = p \cdot (2^n - 1) \cdot \frac{2^n+1}{(3^2+1)} \), which is impossible. However, if \( p = 2 \) and \( H \) is a Cartan subgroup of \( F^* \), then \( H \) is nonprimary and \( |N(H) : H \cdot C(H)| = 4 \).

If \( F^* \cong Sz(2^3) \) or \( Sz(2^5) \) and \( A \) is a subgroup of order 5 or 25 of \( F^* \), respectively, then \( |N(A) : A| = 4p \), which contradicts Lemma 1.2.

Therefore, if \( F = 1 \), then \( G \) is of type 2) from this Theorem.

Consider the case when \( G = F^* \), i.e., \( G = F \cdot L \), where \( L \) is the layer of the group \( G \). By Lemma 1.1, the subgroup \( F \) is abelian, i.e., \( F = Z(G) \), and \( L \) is a quasi simple group. Since the group \( G \) isn’t simple, then \( F \neq 1 \). If \( F \) is nonprimary, then the index \( |N_L(A) : A \cdot C_L(A)| \) divides a prime for any subgroup \( A \subseteq L \). By theorem 4 from [2] \( L \cong PSL(2, q^n) \) or \( SL(2, q^n) \), the number \( \frac{q^n-1}{1} \) is a prime and if \( n = 1 \), then \( q \neq 1(8) \), i.e., \( G \) is of type 3) from this Theorem.

Now suppose that \( F \) is a \( p \)-group for a prime \( p \). Since the Schur multiplier of group \( Sz(2^3) \) is trivial then either \( L \) is a group from Theorem 1.3 or \( L \) is isomorphic to a covering of group \( PSL(2, q^n) \), \( Sz(8) \) or \( PSU(3,2^{2n}) \).

Let \( L/Z(L) \cong Sz(8) \). Then \( L/Z(L) \) contains the subgroups \( A_1/Z(L) \) and \( A_2/Z(L) \) of order 5 and 13, respectively, such that \( |N_L(A_1) : A_1 \cdot C_L(A_1)| = 4 \). Since \( p \) isn’t at least one of the numbers 5 or 13, then supposing \( A = F \cdot A_1 \), we get a contradiction with the definition of \( NP \)-group. If \( L \cong Sz(2^9) \) then subgroups of order 25 and 41 should be taken as subgroups \( A_1 \) and \( A_2 \) in the group \( G \).

Therefore, we can assume that \( L/Z(L) \cong PSL(2, q^n) \) or \( PSU(3,2^{2n}) \).

First, assume that \( Z(L) = 1 \), i.e., \( G = Z(G) \times L \). If \( L \cong PSL(2, q^n) \) and \( p \neq q \), then the number \( \frac{q^n-1}{(2q^n-1)} \) should be prime. Moreover, if \( n = 1 \) and \( q \equiv \pm 1(8) \), then \( L/Z(L) \) contains a subgroup \( H/Z(L) \cong S_4 \). If \( V/Z(L) \) is a four-group from \( H/Z(L) \), then the equality \( |N_{H/Z(L)}(V/Z(L)) : V/Z(L)| = 6 \) implies that in this case subgroup \( V \) is primary, i.e., \( p = 2 \). However, if \( p = q \), then the number \( \frac{q^n-1}{(2q^n-1)} \) could be the product of two primes. But, if \( q^n \equiv \pm 1(8) \), then checking a four-group \( V/Z(L) \) again, we get that \( p = 2 \). But then \( q^n = 2^n \neq \pm 1(8) \). If, however \( L/Z(L) \cong PSU(3,2^{2n}) \) and \( p \neq 2 \), then for a Sylow 2-subgroup \( A \) of \( L \), the subgroup \( A \cdot Z(L) \) is nonprimary and again we get a contradiction with the definition of \( NP \)-group.
Now suppose that $Z(L) \neq 1$. Since the Schur multiplier is trivial for groups $PSL(2,2^n)$ when $n > 2$, we can assume that in the case of $L/Z(L) \cong PSL(2,q^n)$ the number $q$ is odd. Then the order of the Schur multiplier is equal to 2 (i.e. $L \cong SL(2,q^n)$) for $q^n \neq 9$ and 6 for $q^n = 9$. Consider the second case. If $[Z(L)]$ is divisible by 2 and $Q/Z(L)$ is a Sylow 3-subgroup of the group $L/Z(L)$, then the subgroup $Q$ is nonprimary and $|N(Q) : Q \cdot C(Q)| = 4$, which is impossible. Hence, when $q^n = 9$ the order of $Z(L)$ is equal to 3. In the case of $L/Z(L) \cong PSL(3,2^n)$ the Schur multiplier order is equal to 3, and if $A/Z(L)$ is a Sylow 2-subgroup of $L/Z(L)$, then subgroup $A$ is nonprimary and $|N_L(A) : A \cdot C_L(A)|$ is not a prime.

Therefore, if $G = F^*$ then $G$ is a group of type 3) or 5) from this Theorem. Finally, consider the case when $1 < F \leq F^* < G$. Then, by using what’s already been proved, $F^*$ is a group of type 3) or 4), while $G/F$ is a group of type 2) from this Theorem. Let $G = F^* \cdot \langle a \rangle, a^p \in F^*$. If $A/F$ is a Sylow $q$-subgroup from $F^*/F$, then the fact that $|N(A) : A \cdot C(A)|$ is divisible by $p$ implies that subgroup $A/F$ is a Cartan subgroup of group $F^*/F$, which is impossible. Hence, for the nonprimary subgroup $H$, the index $|N(H) : H \cdot C(H)|$ is divisible by $2p$, which is impossible.

\[\square\]

1.5. Note. It isn’t difficult to see that the groups type 2) and 5) of Theorem 1.4 are NP-groups. For type 1) groups, the proof of the sufficiency requires the fulfillment of a number of additional restrictions. Let’s note some of them.

Let $t$ be a $p'$-element from $G$, $A$ be a $t$-invariant subgroup from $F$ and $H = F \cdot \langle t \rangle$. Then the index $|N_H(A \cdot \langle t \rangle) : (A \cdot \langle t \rangle) \cdot C_H(A \cdot \langle t \rangle)|$ divides $p$. Looking at the intersections of these subgroups with $F$ and taking into account that $N_F(A \cdot \langle t \rangle) = A \cdot (N_F(A) \cap C(t))$, we get that

\[|A \cdot (N_F(A) \cap C(t)) : A \cdot (C_H(A) \cap C(t))| = |N_F(A) \cap C(t) : (C_H(A) \cap C(t)) \cdot (A \cap C(t))|,
\]

i.e., $|C_{N_F(A)}(t) : C_A(t) \cdot C_{CF(A)}(t)|$ divides $p$.

Let $N_{G/F}(\langle tF \rangle) = \langle tF \rangle \cdot \langle hF \rangle$ and $A$ be a $\langle t, h \rangle$-invariant subgroup from $F$. Since $h \in N(A \cdot \langle t \rangle)$, then in the same notation $N_H(A \cdot \langle t \rangle) = (A \cdot \langle t \rangle) \cdot C_H(t)$. But then $C_{N_F(A)}(t) = C_A(t) \cdot C_{CF(A)}(t)$. Since the subgroup $N_F(A)$ is also $\langle t, h \rangle$-invariant, then

\[C_{N_F(N_F(A))}(t) = (N_F(A) \cap C(t)) \cdot C_{CF(N_F(A))}(t) = C_A(t) \cdot C_{CF(A)}(t).
\]

Continuing this process and taking into account that $F$ satisfies the normaliser conditions, we get the equality $C_F(t) = C_A(t) \cdot C_{CF(A)}(t)$.

Supposing that in this equation $A = [F,a]$ and taking into account that $F = [F,a] \cdot C_F(a)$, we get that $C_F(a) = C_{[F,a]}(a) \cdot C_{CF([F,a])}(a)$, i.e., $F = [F,a] \cdot C_F([F,a])$.

By analogy we can prove, that if $p \neq 2$ and $\langle aF \rangle \cdot \langle bF \rangle \cdot \langle cF \rangle$ is a subgroup of order 12 from $G/F$ and subgroup $A \leq F$ is $\langle a, b, c \rangle$-invariant, then $C_F(\langle a, b \rangle) = C_A(\langle a, b \rangle) \cdot C_{CF(A)}(\langle a, b \rangle)$ and $F = [F,\langle a, b \rangle] \cdot C_F([F,\langle a, b \rangle])$.

Note that all these properties hold if subgroup $F$ is abelian, i.e., in this case $G$ is a NP-group.
References